Decay rates in bistable Landau potentials driven by weakly colored Gaussian noise

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We investigate a bistable Fokker-Planck equation describing the overdamped motion of particles in a Landau potential driven by weakly colored Gaussian noise. Our focus is on the noise-intensity dependence of the decay rates in the limit of small correlation times of the noise. In linear order of the external-noise correlation time τ the decay rates can be expressed in terms of the eigenvalues of a white-noise Fokker-Planck equation. Numerical results for the first nonvanishing eigenvalue, i.e., the inverse mean first-passage time, are presented for arbitrary noise intensity. In the limit of very low noise intensity an analytical expression is derived and in the strong-noise limit an asymptotic form is given for this eigenvalue.

Recently a great deal of interest has been devoted to the understanding of (nonlinear) dynamical systems perturbed by noise. Recent experiments and numerical calculations on dye lasers,[1] the laser gyroscope,[2] and bistable stochastic systems[3,4] confirmed the need for modeling noisy disturbances by colored noise, i.e., stochastically fluctuating forces with (at least one) finite correlation time τ.

The archetypal bistable situation is the overdamped motion in the Landau potential

$$f(x)=-ax^2/2+bx^4/4 \tag{1}$$

driven by exponentially correlated Gaussian noise;[4–10] i.e.,

$$\begin{gathered}\dot{x}=ax-bx^3+\epsilon(t)\;,\\ \langle\epsilon(t)\rangle=0,\ \langle\epsilon(t)\epsilon(s)\rangle=(D/\tau)\exp(-\mid t-s\mid/\tau)\;.\end{gathered} \tag{2}$$

Here τ denotes the correlation time and D is the noise intensity. In the weak-noise regime (small D) a typical quantity of interest in such a system is the mean first-passage time (MFPT); i.e., the time measure of a (thermally) activated escape of a particle from one metastable potential well to a neighboring one.[11]

The early work on the dynamical system (2) concentrated on the small-τ behavior,[5,6] whereas recent progress has been made in the moderate- to large-τ regime in various bistable and monostable stochastic systems.[1–4,8,12] However, there has been renewed interest on the small-τ (Markovian) limit of Eq. (2).[7–9,13] Though the limit of weak-noise color has been investigated extensively over the years, there still exist a variety of conflicting predictions which have been sorted out in Refs. 7 and 10. In particular, the conflict between various theoretical predictions in the moderate- to large-τ regime has been tied down to some extent in Ref. 4 for a bistable periodic potential, but there still exists some need for exact numerical or analytical results for the MFPT in the small-τ regime.[7,10] Since the equivalence of the inverse MFPT and the first nonzero eigenvalue λ_1 of the corresponding bistable Fokker-Planck equation (FPE) is well established in the weak-noise limit,[10,14] our focus is on the small-τ behavior of $\lambda_1(\tau)$, see also Refs. 4(a) and 10.

Following the τ expansion of Ref. 5 the correct prefactor for the MFPT and the lowest eigenvalue have been deduced in Ref. 6. (This corresponds to evaluating the correct small-τ dependence of the Arrhenius law.) Referring to the dynamics of Eq. (2), this result may be summarized in leading order in τ as follows:

$$\lambda_1(a,b,D,\tau)=(1-\beta_1 a\tau)\lambda_1(a,b,D,\tau=0)+O(\tau^2)\;. \tag{3}$$

Employing the method of steepest descent β_1 has been calculated to be $\beta_1=1.5$ (Ref. 6) in the asymptotic limit of vanishing noise intensity. Because analog and digital simulations cannot deal with the limit of arbitrary small noise intensity a finite D has to be taken into account and the question arises quite naturally what the value of β_1 in Eq. (3) is for (very) small but finite D. In this communication particular emphasis is given to the low-noise-intensity regime of Eq. (3) and an analytic expression for β_1 valid for small D is derived. Moreover, numerical results for arbitrary D and an asymptotic expansion for large D are presented for the coefficient β_1 in Eq. (3).

In the small-τ limit (τ expansion) (Ref. 5) the effective Fokker-Planck-like dynamics of Eq. (2) is given by the short-relaxation-time Fokker-Planck approximation[6] (SRTFPA)

$$\frac{\partial}{\partial t}P=\left[\frac{\partial}{\partial x}f'(x)+\frac{\partial^2}{\partial x^2}D[1-\tau f''(x)]\right]P\;, \tag{4}$$

with the probability distribution $P=P(x,t)$. In the one-variable case a FPE with x-dependent diffusion can always be transformed to a FPE with a constant diffusion coefficient, see for instance Ref. 15, p. 97. In linear order in τ such a transformation of Eq. (4) is achieved by introducing the new variable

$$y=x+\frac{\tau}{2}f'(x)+O(\tau^2)\;. \tag{5}$$

The SRTFPA (4) is thus transformed to

$$\frac{\partial}{\partial t}\tilde{P}=\left[\frac{\partial}{\partial y}F'(y)+D\frac{\partial^2}{\partial y^2}\right]\tilde{P}\ , \tag{6}$$

where the new potential $F(y)$ is given by

$$F(y)=f(y)-\frac{\tau}{2}f''(y)+O(\tau^2) \tag{7a}$$

and $\tilde{P}=\tilde{P}(y,t)$. For the Landau potential (1) the potential (7a) has again the form of the Landau potential with a rescaled parameter $a\rightarrow a+3bD\tau$; i.e.,

$$F(y)=-(a+3bD\tau)y^2/2+by^4/4+aD\tau/2\ . \tag{7b}$$

[The constant term in Eq. (7b) may be omitted because only derivatives of $F(y)$ enter the FPE (6).] Therefore, in leading order in τ all eigenvalues of Eq. (4) are expressed through the eigenvalues of Eq. (6) which is a white-noise FPE:

$$\lambda(a,b,D,\tau)=\lambda(a+3bD\tau,b,D,0)+O(\tau^2)\ . \tag{8}$$

A simple transformation of the variables x, ϵ, and t shows that for the white-noise case the following relations hold:

$$\lambda(a,b,D,0)=a\bar{\lambda}(bD/a^2)=\sqrt{b/D}\,\tilde{\lambda}(a/\sqrt{bD}\,)\ , \tag{9}$$

where $\bar{\lambda}(\bar{D})=\bar{\lambda}(bD/a^2)$ and $\tilde{\lambda}(\tilde{a})=\tilde{\lambda}(a/\sqrt{bD}\,)$ are defined by

$$\bar{\lambda}(\bar{D})\equiv\lambda(1,1,\bar{D},0),\quad \tilde{\lambda}(\tilde{a})\equiv\lambda(\tilde{a},1,1,0)\ . \tag{10}$$

Therefore we have

$$\begin{aligned}\lambda(a+3bD\tau,b,D,0)&=(a+3bD\tau)\bar{\lambda}[bD/(a+3bD\tau)^2]\\&=\sqrt{b/D}\,\tilde{\lambda}[(a+3bD\tau)/\sqrt{bD}\,]\ .\end{aligned} \tag{11}$$

From Eqs. (8) and (11) we readily derive our main result. That is the β coefficient valid for any arbitrary eigenvalue λ in Eq. (3):

$$\begin{aligned}\beta(\bar{D})&=6\bar{D}^2\frac{d}{d\bar{D}}\ln[\bar{\lambda}(\bar{D})]-3\bar{D}\\&=-3\sqrt{\bar{D}}\frac{d}{d\tilde{a}}\ln[\tilde{\lambda}(\tilde{a})]\Big|_{\tilde{a}=1/\sqrt{\bar{D}}}\ .\end{aligned} \tag{12}$$

Thus we have expressed the β coefficient in Eq. (3) for any eigenvalue λ by the logarithmic derivative of the corresponding white-noise eigenvalue in the Landau potential (1) with the normalization $a=b=1$ or $b=D=1$, respectively.

In the case of small noise intensity $\bar{D}$ the Kramers rate for the first eigenvalue $\bar{\lambda}_1(\bar{D})$ may be used in Eq. (12) and we obtain in leading order in $\bar{D}$

$$\beta_1=1.5-3\bar{D}\ . \tag{13}$$

However, since refined versions of Kramers original result have been calculated we might also employ an improved Kramers rate, see, e.g., Ref. 15, Eq. (5.112),

$$\bar{\lambda}_1(\bar{D})=\frac{\sqrt{2}}{\pi}[1-\tfrac{3}{2}\bar{D}+O(\bar{D}^2)]\exp\left[-\frac{1}{4\bar{D}}\right]\ , \tag{14}$$

which by insertion into Eq. (12) leads to

$$\beta_1=1.5-3\bar{D}(1+3\bar{D})+O(\bar{D}^3)\ . \tag{15}$$

In the regime of moderate to large noise intensity it seems to be more appropriate to use the second normalization in Eq. (10). For very large-noise intensity (small $\tilde{a}$) a Taylor expansion of $\ln[\tilde{\lambda}(\tilde{a})]$ around $\tilde{a}=0$ thus leads to the asymptotic expression

$$\beta_1(\bar{D})=1.412(\bar{D})^{1/2}+0.396+O(\bar{D}^{-1/2})\ . \tag{16}$$

On the other hand, $\ln[\bar{\lambda}_1(\bar{D})]$ (or $\ln[\tilde{\lambda}_1(\tilde{a})]$) and its derivative may be obtained for arbitrary values of the noise intensity by computing the smallest nonvanishing eigenvalue $\bar{\lambda}_1(\bar{D})$ [or $\tilde{\lambda}_1(\tilde{a})$] of the white-noise FPE (6) by means of the matrix-continued-fraction (MCF) method.[15] Because of the simplicity of the problem only 2*2 matrices are involved in this case. [We note that in Ref. 16 one of us (H.R.) has already calculated $\tilde{\lambda}_1(\tilde{a})$ by employing the numerical algorithm just mentioned.] We thus obtain the β_1 coefficient by numerically differentiating $\ln[\bar{\lambda}_1(\bar{D})]$ (or $\ln[\tilde{\lambda}_1(\tilde{a})]$). Using a 15-digit arithmetic β_1 can be calculated down to $\bar{D}=0.015$.

The result for β_1 as a function of the noise intensity $\bar{D}$ is displayed in Fig. 1. The inset of Fig. 1 demonstrates that β_1 always remains positive. It first decreases with increasing $\bar{D}$, passes through a minimum value $\beta_1^{\min}=1.245$ at $\bar{D}=0.119$, and then increases to infinity. Moreover, there exists a certain $\bar{D}$ value ($\bar{D}\approx0.456$) for which β_1 again takes on the steepest descent ($\bar{D}\rightarrow0$) value, $\beta_1=1.5$. This nonmonotonic behavior of β_1 in the range $0<\bar{D}<0.456$ may lead to confusion in analog or digital simulation experiments. Second, we recognize that the linear approximation (13) for β_1 is in good agreement with the numerically exact result for $\bar{D}\leq0.02$, which is

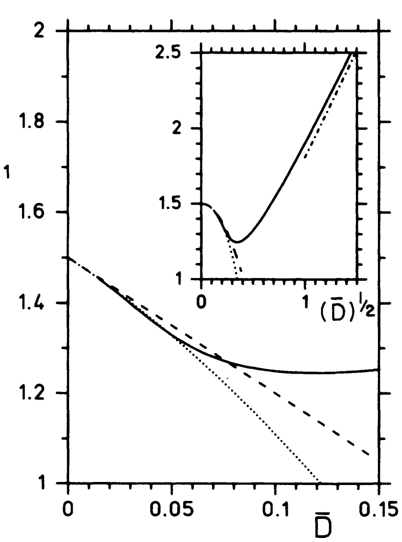

FIG. 1. Coefficient β_1 of Eq. (3) vs normalized noise intensity $\bar{D}=bDa^{-2}$ and, in the inset, vs $(\bar{D})^{1/2}$. Solid line: MCF result. Dashed line: Kramers rate result (13). Dotted line: improved Kramers rate result (15). Dotted-dashed line: asymptotic expansion (16) for large $\bar{D}$.

valid within the SRTFPA (4). However, the improved Kramers rate result (15) for β_1 is a better approximation in the range $\bar{D} \leq 0.05$. Thus the values $\bar{D}=0.02$ or 0.05 clearly constitute the weak-noise limit of our model if the original or the improved Kramers rate is used, respectively. In the strong-noise limit the asymptotic expression (16) also approximates the numerics quite well. Finally, we want to compare our results for β_1 with the two values reported in Ref. 10, where the full two-dimensional Markovian problem corresponding to the one-dimensional non-Markovian dynamics (2) has been solved numerically. For $\bar{D}=0.05$ both results agree, $\beta_1=1.33$, while for $\bar{D}=0.03$ the β_1 value of Ref. 10, $\beta_1=1.44$, slightly exceeds our result, $\beta_1=1.40$.

In conclusion we have derived a formula for the eigenvalues of the SRTFPA (small-τ regime) for the Landau potential valid for arbitrary noise intensity $\bar{D}$, see Eq. (12). Analytical approximations (improved Kramers rate) are in good agreement with the numerical results for weak-noise intensity. In view of results present elsewhere we might conclude that the SRTFPA of Refs. 5 and 6 yields quantitative agreement for the escape rate[10] and the stationary distribution function[3(b)] provided the relaxation time of the noise is small enough. (It should be mentioned that the relevant τ range of the SRTFPA shrinks with decreasing noise intensity D.[10]) Moreover, it was pointed out that for small noise color the numerical effort can be substantially diminished. In the small-τ limit it seems not to be necessary to treat the full two-dimensional Markovian problem corresponding to the one-dimensional non-Markovian dynamics (2), instead one only has to solve the white-noise FPE (6). For intermediate- to large-τ values, however, the MCF method provides the only approach for numerically solving the full two-dimensional problem.[3(a),4,10]

Some final remarks seem to be appropriate. First of all, we stress that our approach and the application of Eq. (12) is clearly restricted to the small-τ limit, since it relies on the SRTFPA (4). (A similar calculation may also be performed for the periodic bistable potential which has been investigated in Ref. 3.) Thus Eq. (12) does not apply in the moderate- to large-τ regime, where the decoupling ansatz[3(b)] was proved to give *qualitatively* correct results for the escape rate and the eigenvalue $\lambda_1(\tau)$, respectively, in a bistable periodic potential[3(a)] and in the Landau potential (1).[10] In particular the aim in Ref. 4(a) was to confirm the exponential τ/D behavior of the MFPT for moderate- to large-τ in the weak-noise limit (small D). On the other hand, it is well known that the decoupling ansatz does not reproduce the correct small-τ behavior of $\lambda_1(\tau)$, see, e.g., Ref. 7. This may also be seen by expanding the approximate law $\lambda_1(\tau)\sim\exp\{-\kappa\tau\}$, which has been employed in Ref. 3(a). Finally, we note that the analytical prescriptions of the SRTFPA and the decoupling ansatz for the decay rates should not be mixed with the corresponding predictions for the stationary distribution function $P_{st}(x)$. It has been shown very recently that the decoupling ansatz reproduces $P_{st}(x)$, e.g., in the ring laser gyroscope[2(b)] and in a periodic bistable potential[3(b)] very nicely over a (much) larger-τ range than the SRTFPA provided the noise intensity D is small enough.

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