Dynamical scaling of the surface of finite-density ballistic aggregation

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We study the roughness of aggregates that are formed by ballistic deposition with nonzero flux density of incoming particles. The flux density is controlled by a parameter $0 \leq p \leq 1$. The scaling behavior of the interfacial width ξ does not depend on p and is the same as ξ found in a variety of other, related models. For short times $\xi \propto t^{1/3}$ for a one-dimensional substrate and $\xi \propto t^{0.22}$ for two-dimensional ones. The latter is not consistent with various theoretical predictions. Rough numerical estimates for the long-time exponent are also presented. In addition, we derive a relation between the width of the geometrical boundary of the aggregate and the width of the active region of growth. This relation, true for models in which the active zone is asymptotically correlated to the surface of the aggregate, is verified by our simulations.

I. INTRODUCTION

It has long been recognized that the roughness of thin films is an important factor in physical phenomena such as adsorption,[1-3] catalysis,[2] and the dissolution of a fractal object.[4] Optical phenomena also depend very much on the surface structure,[5,6] as do the wetting properties of a surface.[7]

From a theoretical point of view, the structure of a growing interface is interesting in many respects. It is a problem which involves both random dynamics and geometry, features that are common to many nonequilibrium phenomena. The interface is the region where growth occurs in real systems and its structure will ultimately determine the structure of the whole deposit.[8] It is not yet clear, however, how this actually occurs since similar structures, whose surface properties belong to the same "universality class," may not have the same bulk properties. For example, it is believed that many aspects of the interfacial dynamics are the same for a class of models that include the Eden model and ballistic aggregation.[9-14]

In this paper we will discuss the interfacial roughness for aggregates that are obtained by finite-density ballistic deposition. This model, to be described in detail below, is such that at any time step, a finite fraction of the perimeter sites are occupied with probability p. An important measure of the roughness is related to the height autocorrelation function[5] $G(u)=\langle (h_A-\bar{h})(h_B-\bar{h})\rangle$, where h_A and h_B are the heights at A and B which are separated by the distance u, and $\bar{h}$ is the mean height. In the limit where u goes to zero, $G(u)$ gives ξ^2, the square of the fluctuations for the height ξ which we may take as a measure of the interface width. We will consider, in particular, the following questions:

(i) How does the width scale with time?

(ii) What is the effect of changing the random flux of the incoming particles? (One might expect that lowering p increases the amount of noise, and therefore changes the scaling behavior and/or the morphology of the interface.)

(iii) What is the relationship between the active zone (the zone that captures the incoming particles) and the geometrical zone (the boundary of the aggregate)?

The behavior for the finite systems at long time is also treated briefly.

It has been conjectured by Family and Vicsek[9] that the width of models such as ours scales like $\xi \sim L^{\alpha} f(\bar{h}/L^{\gamma})$, where L is the linear size of the base and $f(x)$ goes like $x^{\nu}(\gamma\nu=\alpha)$ for small x. There have been conflicting values reported for the scaling exponents in the case in which the base is two dimensional. On the one hand, the Kardar *et al.*[15] renormalization-group analysis of continuous models gives $\alpha=\frac{1}{2},\nu=\frac{1}{3}$. Based on this and simulation results they conjectured the same values for the exponent in two dimensions. On the other hand, a treatment of the single-step model of Meakin *et al.*[10] mapped onto the six-vertex spin model gives $\alpha=0,\nu=0$. In addition, the latter authors pointed out that their simulation results correspond to none of the above. Our results, close to the results of the simulations of Meakin *et al.*, are $\alpha=0.31,\nu=0.22$. This suggests a kind of universality in the sense that our scaling exponents are robust with respect to the density control parameter p.

In Sec. II we describe the finite-density ballistic model and give a brief review of the results for other models and the analytical calculations that have been done. In Sec. III we write the stochastic equations that describe the growth. Under the assumption that the growth velocity is constant, we derive a simple relation between the width of the active zone and the width of the geometrical surface. In Sec. IV we present the results of our simulations. In particular, we verify the relation between the geometrical width and the active width derived in Sec. III, and we study the short-time ($\xi \sim t^{\nu}$) and the long-time ($\xi \sim L^{\alpha}$) scaling behavior of the interface. We conclude our paper with a review of our results and a discussion of the problems that remain open.

II. FINITE-DENSITY BALLISTIC DEPOSITION AND OTHER MODELS

A. Model

Finite density ballistic aggregation[16] is a generalization of the original ballistic deposition model of Vold and Sutherland[11,12,10] that was introduced to describe colloidal aggregation. The Vold-Sutherland model is also relevant for the description of vapor deposition of thin films.[17] In fact, simulations of the ordinary ballistic deposition model reproduce the familiar columnar structure[17] that is seen in vapor-deposited thin films grown on cold substrates. In this simplified model a particle falling from a random point in a straight line sticks as soon as it meets an active site, i.e., a nearest neighbor of a particle of the deposit. This is an idealization of low-temperature deposition when relaxation can be neglected. One and only one particle can be added at a time, and the film is grown on a flat substrate by adding many particle sequentially.

We will study a generalization of this model in which many particles are added simultaneously to the structure at each time step. The growth algorithm is as follows. At each time step all the columns of the lattice normal to the substrate along which a particle may fall are sampled and a particle is allowed to fall independently and randomly in each column with a probability p. These particles then fall in their columns and stick to the cluster when they meet an active site. The active sites at time N do not include nearest neighbors of the other particles that are dropped at time N. This parameter p mimics an incoming flux that we can vary at will, and allows us to "tune" the fluctuations in the dynamics of the process. Our numerical results indicate that the scaling exponents are apparently independent of p. However, other aspects of the growth do depend on p. In the limit where the number of incoming particles is small, i.e., p, goes to zero, we recover the ordinary, asynchronous ballistic deposition model. This is confirmed by our simulations. The results of our model for small p agree very well with the results for the ordinary ballistic deposition model. This is shown in Fig. 5 and will be discussed further below. In addition to its use in the current study, our model of finite-density ballistic aggregation has some other interesting features. As p is varied, the bulk morphology of the deposits varies in an intriguing way. The model has also been shown to manifest quasiperiodic growth oscillations whose frequencies depend on p.[16] Before we proceed to the analysis of the interface, let us review the scaling hypothesis[9] for deposition of aggregates and summarize the results that have been obtained by other authors for related models.

B. Interface results for related models

The scaling behavior of the interface has been studied for a number of models of growing dense structures. These models include the Eden model,[13,14] ordinary ballistic deposition,[9,10] and the single-step model of Meakin *et al.*[10] In all these models, the interface width has been found to scale as $\bar{h}^{\nu}$ for short time and to saturate at a value which increases with the linear dimension L of the substrate as L^{α}.

It appears that the values of the exponents fall into universality classes. The results to date, suggest furthermore that the universality classes are determined only by the dimensionality of the system.

It is generally believed that the short- and long-time scaling behavior can both be summarized in a scaling form[9]

$$\xi \sim L^{\alpha} f(\bar{h}/L^{\gamma}) , \tag{1}$$

with $f(x)$ being constant for large x and $f(x) \propto x^{\nu}$ with $\nu=\alpha/\gamma$ for small x. The existence of this scaling form has been explicitly demonstrated, however, only for the case of ballistic deposition in one dimension.[9] (The dimension we use here and in the following is the dimension of the substrate. The reader is cautioned that other authors refer to the dimensionality of the whole aggregate.)

In one dimension the numerical results for all the models examined are consistent with the values of $\alpha=\frac{1}{2}$ and $\nu=\frac{1}{3}$. For example, for ordinary ballistic deposition the simulations of Family and Vicsek[9] give $\nu=0.30\pm0.02$ and $\alpha=0.42\pm0.03$ for the active zone. The ν exponent was determined by studying growth on a square lattice with a base as wide as $L=2000$ with $\bar{h}$ between 20 and 1000. A larger-scale simulation ($L=2^{18}, \bar{h}=5000$ for the short-time exponent) was done on a square lattice by Meakin *et al.*[10] Their results are $\alpha \simeq 0.47$ and $\nu=0.331 \simeq 0.006$. In the same paper they studied a closely related model, the single-step model. Their results are $\nu=0.332\pm0.003$ ($L=2^{18}$) and $\alpha=0.5000 \pm 0.015$ obtained from square lattice with bases of size $64 \leq L \leq 1024$.

Similar simulations have been done for the Eden model by Meakin, Jullien, and Botet (their model C).[13] They studied deposition along the axis and along the diagonals (staggered lattice) of a square lattice. The size of their deposits were $10^8-2\times10^9$ sites with a base of $32 \leq L \leq 1024$ units. They extracted an exponent α very close to 0.5 (from 0.498 to 0.505 along the axis and from 0.492 to 0.499 along the diagonal). Using a base L up to 8192, they found $\nu=0.307\pm0.007$. The supercomputer simulations of Zabolitzky and Stauffer[14] on the Eden model grown from a linear substrate gave $\nu \simeq \frac{1}{3}$ with L up to 2^{20}. However, as they pointed out, this might be only a lower bound; the effective exponent showed a definite increase toward the end of their simulations. They also found the exponent α to be 0.511 ± 0.025.

Two different analytical calculations of the scaling exponent in one dimension have been done. The single-step model was mapped onto a spin model,[10] giving $\alpha=\frac{1}{2}$ for long time scaling behavior. This result follows from the fact that the spins become uncorrelated when the steady state is reached. Using a different approach, Kardar, Parisi, and Zhang[15] argued that growth of the interface (in the moving frame) can be described in the continuum limit by the equation

$$\frac{\partial h}{\partial t}=a\nabla^2 h+b(\nabla h)^2+\eta(x,t)\ . \tag{2}$$

This is a nonlinear Langevin equation where the $\eta(x,t)$ is uncorrelated Gaussian noise. Kardar *et al.* claim that higher-order nonlinear terms as well as the exact form of the noise are irrelevant for the scaling behavior. Using dynamic renormalization-group techniques they found $\alpha=\frac{1}{2}$ and $\nu=\frac{1}{3}$ exactly.

Thus, in one dimension, all results, both numerical and theoretical, are consistent with the values $\alpha=\frac{1}{2},\nu=\frac{1}{3}$ for a number of different but related models. The situation in two (and higher) dimensions is less clear. While the continuum equations of Kardar *et al.* may still be applicable, there are now two stable fixed points, one at $b=0$ and the other at $b=\infty$. Assuming that the strong coupling fixed point is the relevant one (the fixed point at $b=0$ would yield the trivial exponents, with logarithmic corrections to scaling in two dimensions), the exponents are not computable in perturbation theory. (However, via a mapping to a polymer problem, Kardar *et al.* numerically measured exponents of $\nu\simeq\frac{1}{3},\alpha\simeq\frac{1}{2}$.) It has also been argued by Meakin *et al.*[10] that the continuum equations lead to a scaling relation

$$\frac{2}{\alpha}=1+\frac{1}{\nu} \tag{3}$$

in any dimension.

A model related, but not identical, to the single-step model is the six-vertex model. Under the assumption that the long-time behavior of the single-step model reduces to the equilibrium ensemble of the six-vertex model with equal vertex energies (the analogous result being known to hold in one dimension), Meakin *et al.* derived the result that the height autocorrelation function $G(u)\sim\ln u$ for large u. This implies that $\alpha=0$ and, assuming the validity of the scaling relation (3), also that $\nu=0$. Care must be taken with this analogy, however, because there are important differences between the allowed configurations and their weights in two dimensions of the single-step and the vertex models.

None of the above seem to agree with the simulation results in two dimensions that indicate $\alpha=0.333$ for the ballistic (regular) model,[10] and $\alpha=0.363$ for the single-step model.[10] The short-time exponent ν for both models seems to be close to $\frac{1}{4}$. Unfortunately, in that case the data is not conclusive and it is not obvious that the asymptotic regime value ($L\to\infty$) has been obtained. To be in that regime, we need $h\ll L^{\gamma}$; otherwise we may expect significant crossover effects between the long- and short-time regimes. these constraints are easily dealt with in one dimension (although, as stated earlier, Zabolitzky and Stauffer[14] pointed out that the value $\nu=\frac{1}{3}$ for the Eden model might be only a lower bound). On the other hand, in higher dimensions, memory size severely restricts the scale of the simulations. In fact, a large-scale (supercomputer) simulation[14] of the Eden model with a base 8192×8192 did not reach the true asymptotic regime ($L\to\infty$), with ν starting to increase.

III. FORMALISM

In this section we describe some formal properties of our deposition process. The geometric surface, defined as the highest occupied site for each column, is parametrized by the height $h_N(x)$ where x is the position of the column and N is the time. $h_N(x)$ is a single-valued function. The active zone is parametrized by $a_N(x)$ and is the highest site of each column that has a nearest neighbor which is occupied.[10] The width ξ_N and δ_N of the geometric surface and of the active zone, respectively, are thier standard deviations. The dynamics of the synchronous ballistic deposition model follows the following two equations:

$$a_N(x)=\max\{h_N(x)+1,\{h_N(x')\}\}\ , \tag{4}$$

$$h_{N+1}=q_N(x)a_N(x)+[1-q_N(x)]h_N(x)\ . \tag{5}$$

$h_N(x')$ are the heights of the columns which are nearest neighbors to the one at x. [More generally, if the sticking range extended over more than nearest neighbors, $h_N(x')$ would include all those columns that directly "interact" with particles in the column at x.] $a_N(x)$ is the height of the active site for the column located at x, and $q_N(x)$ is a random variable, independent for all x and N. It takes a value of 1 with probability p and 0 with probability $(1-p)$. Although in our case the q_N's are uncorrelated, generalizations can be made to take into account memory effects.

The first equation implies nontrivial correlations among different columns. Despite its apparent simplicity, it is a difficult equation to deal with because of the highly nonlinear Max operator. Various linear approximations of it have been tried without any satisfactory result. In the sequel, we will suppose that the dynamics embodied in this equation implies that the active zone (i.e., the a_N's) have the same growth velocity as the geometric surface (i.e., h_N's). This assumption is supported by the results of our simulations (see Sec. IV).

Now consider the second equation. Since $q_N=0$ or 1, we have

$$\langle q_N^m\rangle=\langle q_N\rangle=p\ , \tag{6}$$

where $\langle\ \rangle$ may be understood to be either an average over an ensemble of clusters, or an average over the random variable in a single cluster. It is the former that we shall have in mind in most of the remainder of this section. In any case, if we average over a collection of clusters, translation invariance will require $\langle h_N(x)\rangle=\langle h_N(y)\rangle$ for any x and y.

Because $q_N=0$ or 1, for any function ψ

$$\psi(h_{N+1})=q_N\psi(a_N)+(1-q_N)\psi(h_N)\ . \tag{7}$$

In particular,

$$e^{ish_{N+1}}=q_N e^{isa_N}+(1-q_N)e^{ish_N}\ , \tag{8}$$

with

$$0<s<2\pi\ .$$

We can then obtain the characteristic functions

$$\langle e^{ish_{N+1}}\rangle = p\langle e^{isa_N}\rangle + (1-p)\langle e^{ish_N}\rangle \ . \tag{9}$$

The characteristic functions are the generating functionals for the moments of h_N and a_N. As usual, it is more convenient to consider the cumulants of the moments. The first cumulant is the mean, which is related to the growth velocity. The second cumulant is the square of the standard deviation and gives us the roughness. The third cumulant is a measure of the deviation from Gaussian behavior since all cumulants after the second are zero for a Gaussian.

The generating functionals for the cumulants are

$$u_N(s) = \ln\langle e^{ish_N}\rangle \ , \tag{10}$$

$$w_N(s) = \ln\langle e^{isa_N}\rangle \ , \tag{11}$$

with

$$u_N(0) = w_N(0) = 0 \ . \tag{12}$$

Therefore

$$u_{N+1}(s) = \ln[p\langle e^{isa_N}\rangle + (1-p)\langle e^{ish_N}\rangle] \ . \tag{13}$$

For $s=0$ a first differentiation gives

$$\langle h_{N+1}\rangle = p\langle a_N\rangle + (1-p)\langle h_N\rangle \tag{14}$$

or

$$\langle h_{N+1}\rangle - \langle h_N\rangle = p(\langle a_N\rangle - \langle h_N\rangle) \ . \tag{15}$$

On the left-hand side of (15), we have the growth velocity of the geometric interface V_N, and,

$$V_N - V_{N-1} = p[(\langle a_N\rangle - \langle a_{N-1}\rangle) - (\langle h_N\rangle - \langle h_{N-1}\rangle)] \ . \tag{16}$$

As we have mentioned, from Eq. (4), it is reasonable to assume that the active zone is locked to the geometric surface. Thus

$$V_N - V_{N-1} = 0 \tag{17}$$

and the average growth velocity is a constant.

The second cumulant follows ($r = 1-p$):

$$\begin{aligned} u''_{N+1}(s) = & \frac{1}{(pe^{w_N} + re^{u_N})^2} \\ & \times [pw''_N e^{w_N} + ru''_N e^{u_N} + p(w'_n)^2 e^{w_N} \\ & + r(u'_N)^2 e^{u_N}(pe^{w_N} + re^{u_N}) \\ & - (pw'_N e^{w_N} + ru'_N e^{u_N})^2] \ . \end{aligned} \tag{18}$$

If ξ is the standard deviation for the geometric surface and δ the one for the active zone, then putting $s=0$ yields

$$\begin{aligned} \xi^2_{N+1} = & p\delta^2_N + r\xi^2_N + p\langle a_N\rangle^2 + r\langle h_N\rangle^2 \\ & - (p\langle a_N\rangle + r\langle h_N\rangle)^2 \ . \end{aligned} \tag{19}$$

This is the evolution equation for the roughness. After some algebra we find

$$\xi^2_N - \delta^2_N = \frac{V^2}{p^2} - \frac{V^2}{p} - \frac{1}{p}(\xi^2_{N+1} - \xi^2_N) \tag{20}$$

and V is the average growth velocity.

The left-hand side is a quantity we will call Δ_N. The term $(\xi^2_{N+1} - \xi^2_N)$ is the derivative of the squared width. Since (experimentally) the exponent $2\nu < 1$, this term goes to zero as $t^{2\nu-1}$. Therefore, for large N, Δ_N approaches a constant Δ, and

$$\Delta = V^2\left[\frac{1}{p^2} - \frac{1}{p}\right] \ . \tag{21}$$

As we shall show in Sec. IV, Eq. (21) is supported by our simulations. In the limit where p goes to zero, according to the simulations for small p and the simulations of the ballistic deposition model, Δ is finite (i.e., $V \sim p$). Note that $V \sim p$ follows at once from the definition of p and the fact that the ξ and δ grow more slowly than linearly. If the active zone and geometric interface grow with the same velocity, then Δ_N is a constant. We believe that this is a feature shared by a large class of models. We also assumed that the roughness exponent is no larger than 1 (for the squared width). If we consider that for independent columns, the exponent is 1 and if we believe that the coupling between columns tends to reduce the width, then the exponent may be expected to be no larger than 1. This result agrees with the simulation results and implies that asymptotically the active zone and the geometric surface grow with the same power law.

IV. SIMULATION RESULTS

A. Description of our simulations

We performed simulations of finite-density ballistic aggregation for different values of the parameter p. We also

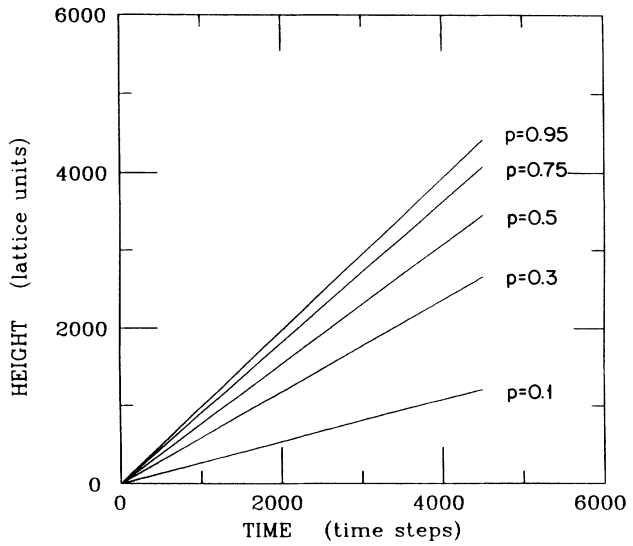

FIG. 1. Height of the deposit plotted against the growth time for different probabilities. The growth velocity is constant. The size of the base is 820×820.

simulated the ordinary ballistic deposition model (i.e., the low-density limit of our model) for comparison. We studied the geometric surface and the active zone simultaneously. Most of our simulations were done for a two-dimensional base but we did review briefly the one-dimensional case.

In two dimensions, we studied the nonequilibrium behavior ($h \ll L^{\gamma}$) of deposits on square lattice bases as large as 820×820. These were grown for 4500 steps. In addition, for the study of the steady state we grew sets of ten runs of deposists on bases 50×50, 100×100, 200×200, 300×300, and 400×400 large and for as long as 6500 time steps.

We first consider the mean height $\bar{h}_N$ (as well as $\bar{a}_N$). These were obtained by averaging $h_N(x)$ [and $a_N(x)$] over all columns. This averaging is expected to be the same as averaging over clusters and we shall use it in the analysis of our simulations. After some initial transients $\bar{h}_N$ and $\bar{a}_N$ are linear in time (Fig. 1), implying that the growth velocity is a constant. This result is true for both one and two dimensions. Since the flux of particles is constant, equal to pL^d, this means that the average density of the bulk is uniform and that the deposit has a trivial dimension[20] equal to the dimension of space.

We also studied the widths, ξ_N and δ_N. Decreasing p increases the widths of the interface and qualitatively decreases the smoothness of the curve (see Fig. 2). These, when plotted against time (Fig. 3), show a smooth increase, except for very late times when the curve becomes noisier and finally fluctuates around some average value.

We will examine first how ξ and δ compare with each other, then discuss in more detail their behavior with time.

B. Geometric surface versus active zone

When we compare ξ_N to δ_N, we notice that after the first initial steps ξ_N is always larger than δ_N. As seen in Fig. 4(a), this is true for all values of p, including the ordinary ballistic deposition model ($p\to0$). Consider $\Delta_N=\xi_N^2-\delta_N^2$. Our data shows that after an initial transient this quantity fluctuates around a constant. This has been observed for both one-dimensional and two-dimensional bases. It is also true for any value of the probability p [see Figs. 4(a) and 5(a)] including the limiting case of regular ballistic deposition (or $p\sim1/L^d$). However, Δ_N decreases when p increases, and it naturally becomes zero when p goes to 1. But the duration of the initial transient does not seem to depend on p. If we vary the size of the base, the quantity Δ_N does not appear to change appreciably [Fig 4(b)], although for very small samples the fluctuations of Δ_N are larger and the transient is less visible. These results are consistent with the assumption made in Sec. III that the geometric and active zones grow with the same velocity. Δ_N is an important dynamical quantity in that it measures the screening. In the model discussed here Δ_N is nonzero and time independent. On the other hand, in the Eden model Δ_N is zero. It seems clear that the value of Δ_N is important in determining properties of the bulk, for example, the porosity. Its functional form might also be closely related to the question of whether or not the bulk is fractal.[18,19]

From a practical point of view, the fact that in our case Δ_N depends asymptotically neither on time nor on size implies that the scaling behavior, if any, of the geometric zone and the active zone are asymptotically (in both limits) identical. We therefore need only study either the geometric width (which we shall do in one dimension), or the active zone width (which we shall do in two dimensions).

C. Dynamical exponents

Following Family and Vicsek,[9] the smooth initial part of the plot of ξ_N or δ_N is size independent and the width

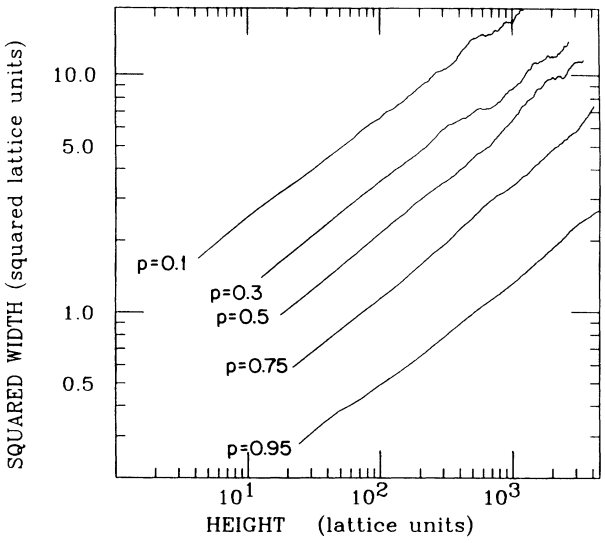

FIG. 2. Short-time behavior. The active zone squared width is plotted, and the size of the base is 820×820.

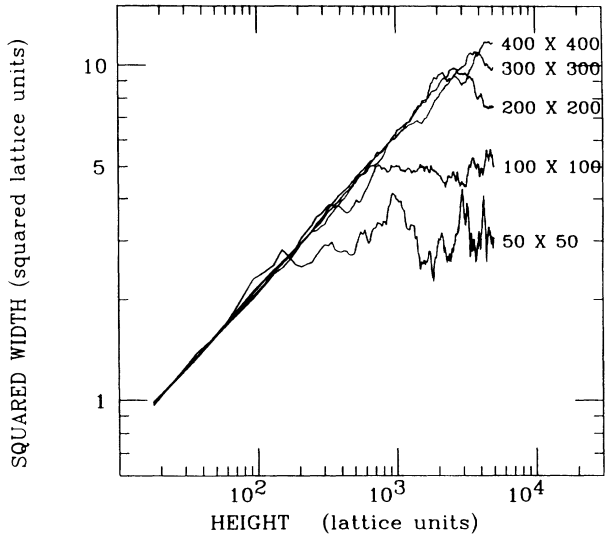

FIG. 3. Long-time behavior. The active zone squared width is plotted, and the growth probability is 0.5.

should behave like t^{ν}. We studied this regime, bearing in mind the limitations of the computer simulations discussed above.

In one dimension, when we analyze the geometric width versus time, we observe that the exponent is $\nu \approx 1/3$ for all values of the probability p. This is comforting since it is consistent with the universality of the exponent claimed by Kardar *et al.*,[15] and with numerical results for other models[10,9,13,14]

In two dimensions the active zone was considered. Our results for the dynamic exponent come from simulations with a base of 1000×1000 for p equal zero and 820×820 for p different from zero. The asymptotic power-law regime is attained for the active width sooner than for the geometric width. The plot of the geometric zone width shows an initial curvature in a log-log plot versus time but finally aligns itself with the active width plot. This is consistent with the relation between the active and geometric width that we discussed above.

The two-dimensional dynamic exponent is found to be $\nu \simeq 0.22$ independent of the probability p [see Figs. 2 and 5(b)] and including the limit $p=0$ (ballistic deposition). However, this is inconsistent with the theoretical predictions mentioned earlier.[10,15] ($\nu = \frac{1}{3}$ or $\nu = 0$). We cannot

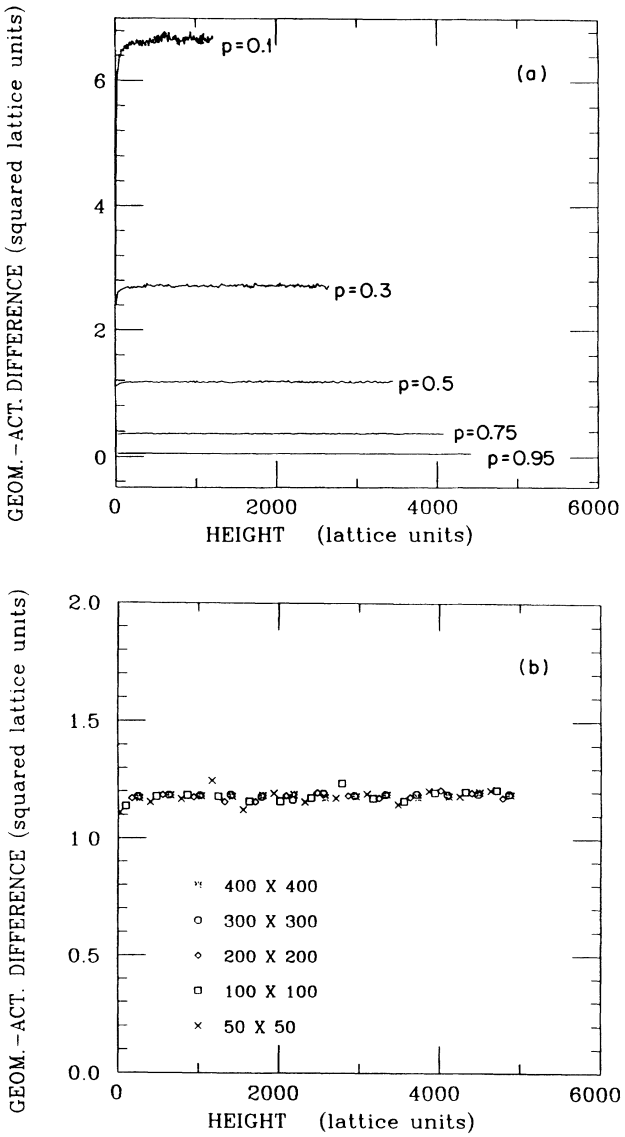

FIG. 4. The difference between the squared widths for the geometric surface and for the active region of growth is a constant, independent of time [as seen in (a)], and of the size of the base [as seen in (b)]. The size of the base is 820×820 for the first plot. The expected values from Eq. (21) are 6.4, 2.7, 1.2, 0.4, and 0.05 for $p = 0.1$, 0.3, 0.5, 0.75, and 0.95, respectively. For the second plot, the growth probability is 0.5, and the plotted curves represent an average of ten runs.

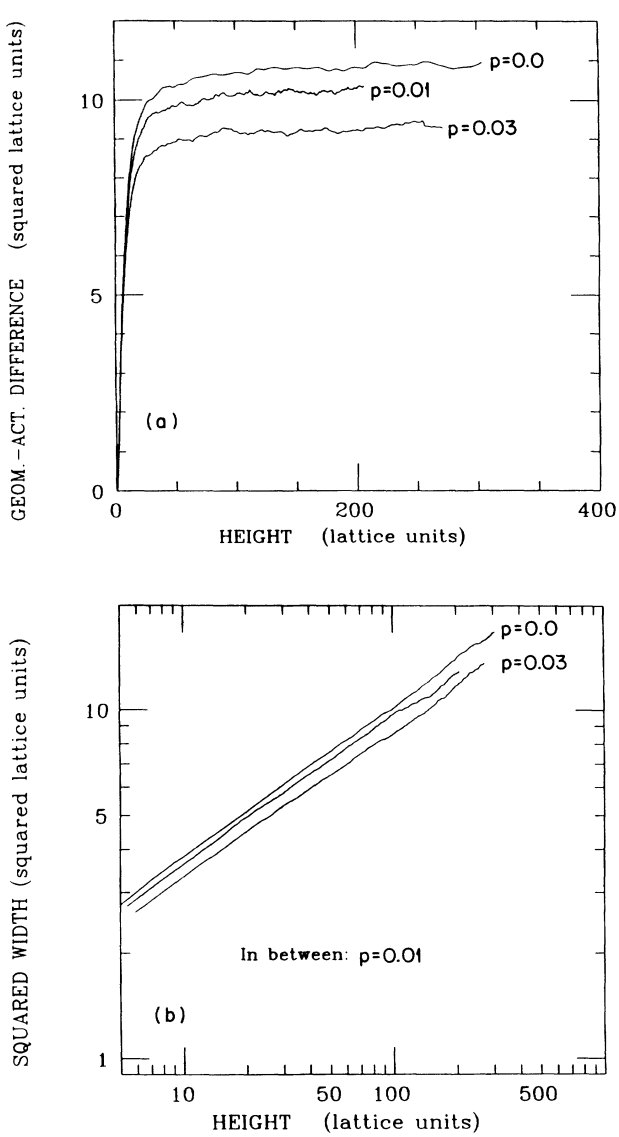

FIG. 5. Small-growth-probability regime. As p becomes smaller we recover the ordinary ballistic deposition model ($p \to 0.$). Plotted against the height of the deposit, we have in (a) the difference between the squared width of the geometrical boundary and the squared width of the active zone, and in (b) the squared width of the active zone. For $p = 0.01$, 0.03 the base is 820×820 wide, and for $p = 0.0$ the base is 1000×1000 wide.

preclude the possibility that we have not yet achieved the true asymptotic regime, though we have pushed the computation to the limits of our computing resources. The p independence of our exponent, however, renders this an unlikely possibility, in our opinion.

D. Finite-size effects

If we wait long enough, the curve of the width versus time first becomes noisy, then fluctuates around a constant value (Fig. 3). We believe that this is a finite-size effect. This is supported by the fact that as L increases, this behavior sets in at a later time. (Remember that the short time curve is common for different sizes; see Fig. 3.) Our data comes from sets of ten runs with square bases of sides $L=50$, 100, 200, 300, and 400. The structure of the crossover and of the plateau is not clear. The large effect of the noise hides any pattern. The curve, however, becomes smoother after averaging.

A rough computation of the long-time exponent gives $\alpha=0.3$. This is not very accurate since the oscillations of the plateau (even upon averaging ten samples) restricts our precision. At any rate, it is not accurate enough to check Meakin *et al.*'s scaling relation[10] or to compare with their simulation values ($\alpha=0.33$ for the $p=0$ ballistic deposition model or $\alpha=0.36$ for the single-step model). On the other hand, it is very different from the theoretical predictions ($\alpha=0$ or $\alpha=\frac{1}{2}$),[10,15] a difference which we regard as significant.

V. SUMMARY AND CONCLUSIONS

The finite-density ballistic aggregation model we have studied is an interesting generalization of the ordinary ballistic deposition.[11,12] The former depends on a parameter p and the latter is recovered as pL^d approaches 1 from above. This parameter allowed us to check the universality of the scaling behavior as p is varied.

The width of the deposit scales with time in the short-time regime and with the size of the base for the long-time regime. In one dimension, we showed using Monte Carlo simulations that the scaling exponents were consistent with those found in a variety of other one-dimensional models, and with theoretical predictions for one-dimensional systems. The short-time exponent is $\nu=\frac{1}{3}$, independent of p, and this is the same as the Eden model,[13,14] the single-step model[10] or the Vold and Sutherland model.[9,10] In two dimensions, our value for the short-time exponent is $\nu\simeq 0.22$. This is again independent of the value of the parameter p. It is also consistent with previous numerical results for the ballistic deposition model[10] and the single-step model.[10] However, it corresponds to none of the theoretical predictions.[10,15]

A rough estimate at $p=0.5$ in two dimensions for the long-time exponent gives $\alpha\simeq 0.3$. This is to be compared with $\alpha\simeq 0.33$ for regular ballistic model[10] and $\alpha\simeq 0.36$ for single-step models.[10] But here also theoretical predictions are not met ($\alpha=0$ or $\alpha=\frac{1}{2}$).[10,15]

Our simulations showed that the width of the active zone (screening zone) is related to the width of the geometrical surface of the deposit by a simple relation. The quantity $\Delta_N=(\xi_N^2-\delta_N^2)$ is asymptotically independent of time or of the size of the deposit. This was confirmed by our analysis that showed the result to be true for any model where the active zone grows at the same speed as the geometrical surface. For this class of models, the growth velocity is asymptotically constant (aside from the effects reported in Ref. 16) and the bulk is compact.

The limitations of computer power show the need for a theory that gives meaningfully accurate results. The existing analytical calculations of the scaling exponents are satisfactory only in one dimension. A more complete theoretical picture is necessary for the description of the interface of growing deposits, which is important in many physical applications.[5] All the current theoretical treatments[10,15] are, we believe unsatisfactory, and reflect the primitive nature of our understanding of even this very simple model of nonequilibrium development.

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