Random bias as an example of global dynamical disorder in continuous-time random-walk theories

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We study a random-bias model defined by a random walk with stochastic transition probabilities modeled by a dichotomic Markov noise. Conditions under which the model can be described by a Markovian chain with internal states are given. The effective Green's function of the problem is calculated using the resolvent-matrix method. The moments of this distribution and the non-Markovian master equation which it satisfies are analyzed. Nondiffusive behavior is found during a transient regime. The model of random bias is extended to a situation with static disorder modeled by a waiting-time density with a long-time tail. It is found that the random bias leads to a change of the coefficient in the anomalous diffusion law.

I. INTRODUCTION

A simple model for transport in a disordered system consists of a random walk on a one-dimensional lattice with random hopping rates.[1] This gives a good understanding of conductivity experiments on amorphous media.[2] Several methods such as the effective medium approximation,[3] renormalization-group calculations,[4] and the continuous-time random-walk (CTRW) theory[5] (Hartree approximation) have been developed to study these problems.

Random walk in a medium with static disorder leads to anomalous diffusion in some circumstances, for example, in the presence of an underlying fractal geometry. In the context of CTRW theory, anomalous diffusion is associated with a waiting-time distribution with a long-time tail.[2,3,6] More recently, models of random walk with global dynamical disorder have been considered.[7,8] In this paper we address the problem of the effect of global dynamical disorder in anomalous diffusion in the framework of CTRW theories.

More specifically, we study a random-bias problem associated with a random walk in a dichotomic random uniform external field. The problem can be formulated in terms of a random walk with internal states[9] and it is solved using the *resolvent-matrix method*[10] to calculate the effective Green's function of the random walker. In this approach we profit from the possibility of introducing a waiting-time distribution modeling static disorder to consider the simultaneous[11] effect of static and global dynamical disorder in diffusion problems.

The stochastic master equation

As a model of random bias we consider the following one-step master equation *functional* of an external noise $\alpha(t)$:

$$\frac{\partial}{\partial t}Q(s,t,[\alpha(t)])=\{[r_0-\alpha(t)r_1](\mathbb{E}^{-1}-\mathbb{I})+[r_0+\alpha(t)r_1](\mathbb{E}-\mathbb{I})\}Q(s,t,[\alpha(t)]) , \quad (1.1)$$

where the operators $\mathbb{E}$, $\mathbb{E}^{-1}$, and $\mathbb{I}$ are defined by their effect on an arbitrary function $f(s)$ as

$$\mathbb{E}f(s)=f(s+1) ,\quad \mathbb{E}^{-1}f(s)=f(s-1) ,\quad \mathbb{I}f(s)=f(s) . \quad (1.2)$$

Equation (1.1) models random walk in a medium with global dynamical disorder in which the transition probabilities per unit time for jumps forward and backward have a stochastic part with the same value except for the opposite signs.

The solution of Eq. (1.1) depends on the nature of the stochastic process $\alpha(t)$. Here we assume that $\alpha(t)$ is a two-level Markov process[12] which takes values $\pm\Delta$ and has correlation time ν^{-1}

$$\langle\alpha(t)\rangle=0,\quad \langle\alpha(t)\alpha(t')\rangle=\Delta^2e^{-\nu|t-t'|} . \quad (1.3)$$

Under these assumptions the *effective* Green's function

(i.e., the walker propagator averaged over the external noise)

$$P(s,t)=\langle Q(s,t,[\alpha(t)])\rangle_{\alpha(t)} \tag{1.4}$$

will obey an integrodifferential equation which must be solved in order to understand the problem. An alternative point of view can be taken for the problem stated in (1.1), introducing a master equation with internal states (i.e., composite Markov processes[13]) where the internal states represent the values of the external noise. The joint probability distribution $P(s,\beta,t)$ is the probability for the walker to be at site s and the noise in the internal state β at time t. The effective Green's function can be obtained as

$$P(s,t)=\sum_{\beta} P(s,\beta,t) \ . \tag{1.5}$$

This is the main idea of the present work: to represent the random bias as a problem with internal states and to solve it by means of the resolvent-matrix technique.[10]

For the problem described by Eq. (1.1), we introduce a Markovian chain with internal states; $P_n(s,\beta)$ is the joint probability of the walker to be at site s with internal state β in the nth-step walk. It satisfies the following recurrence relation:

$$P_n(s,\beta)=\sum_{s'\beta'} \Psi(s-s',\beta,\beta')P_{n-1}(s',\beta') \ , \tag{1.6a}$$

where

$$\sum_{s,\beta} \Psi(s-s',\beta,\beta')=1 \ . \tag{1.6b}$$

Ψ is the matrix which defines the model,

$$\Psi(s-s',\beta,\beta')=\delta_{\beta,\beta'}[\delta_{s-1,s'}(r-\beta)+\delta_{s+1,s'}(r+\beta)] + \delta_{\beta,-\beta'}\delta_{s,s'}|\beta| \ , \tag{1.7}$$

the parameter r is the single-step transition probability for the walker in absence of noise, and the internal state runs over the values $\beta=\pm\Delta$. The Markovian chain [(1.6)–(1.7)] can be transformed into a multistate CTRW (Refs. 14 and 15) introducing a waiting-time density $\psi(t)$. It is shown in the Appendix that this continuous-time version of (1.6) and (1.7) is equivalent to the original random-bias problem (1.1) only for a particular choice of the parameters, namely, $r_0=r\lambda$, $r_1=\lambda$, $v=2\Delta\lambda$ with $\psi(t)=\lambda e^{-\lambda t}$.

The paper is organized as follows. In Sec. II we solve the problem posed by (1.6) and (1.7) using the *resolvent-matrix method*. We construct the associated CTRW and we calculate the corresponding diffusion coefficient and spectrum of the velocity autocorrelation function. In Sec. III we derive the equation for the effective Green's function of the CTRW problem calculated in Sec. II. The Markovian limit of this equation is discussed. Finally, in Sec. IV we go beyond the random-bias problem defined by (1.1) introducing a static disorder. This is done using a long-time-tail waiting-time density[2] as is usual in the CTRW theory. This gives a model of random bias in the presence of static disorder which allows some understanding of the simultaneous effect of static and global dynamical disorder.

II. EFFECTIVE GREEN'S FUNCTION

We wish to find the properties of the marginal probability distribution

$$P_n(s)=\sum_{\beta=\pm\Delta} P_n(s,\beta) \ , \tag{2.1}$$

where the distribution $P_n(s,\beta)$ is determined by the recurrence relation (1.6). Due to the translational invariance we can transform to the Fourier space [the Fourier representation will be characterized by the argument $(s\to k)$ of the functions], then

$$\Psi(k)=\begin{bmatrix} \Psi_{\Delta,\Delta}(k) & \Psi_{\Delta,-\Delta}(k) \\ \Psi_{-\Delta,\Delta}(k) & \Psi_{-\Delta,-\Delta}(k) \end{bmatrix} , \tag{2.2a}$$

where

$$\Psi_{\beta,\beta'}(k)=\sum_{s} e^{ik(s-s')}\Psi(s-s',\beta,\beta') \ , \tag{2.2b}$$

that is, according to (1.7),

$$\begin{aligned} &\Psi_{\pm\Delta,\pm\Delta}(k)=2r\cos(k)\mp 2i\Delta\sin(k) \ , \\ &\Psi_{\pm\Delta,\mp\Delta}(k)=\Delta \ . \end{aligned} \tag{2.2c}$$

The resolvent matrix $\mathcal{L}$ is the space convolution operator for the non-Markovian recurrence relation of the marginal distribution

$$P_n(s)=\sum_{n'}\sum_{s'} \mathcal{L}_{n,n'}(s-s')P_{n'}(s') \ . \tag{2.3}$$

Using the results of Ref. 10, the generating Green's function of $P_n(s)$ in Fourier space becomes

$$R(k,z)\equiv \sum_{n=0}^{\infty} z^n P_n(s) = \frac{P_0(k)+z[P_1(k)-P_0(k)\mathcal{L}_{21}(k)]}{1-[z^2\mathcal{L}_{20}(k)+\mathcal{L}_{21}(k)]} \ , \tag{2.4}$$

where $\mathcal{L}_{20}(k)$ and $\mathcal{L}_{21}(k)$ are the relevant elements of the resolvent matrix, which are obtained as the solutions of the following set of linear equations:[10]

$$\mathcal{L}_{20}(k)\sum_{\beta} \Psi^0_{\beta,\beta'}(k)+\mathcal{L}_{21}(k)\sum_{\beta} \Psi^1_{\beta,\beta'}(k)=\sum_{\beta} \Psi^2_{\beta,\beta'}(k) \ . \tag{2.5}$$

$P_0(k)$ and $P_1(k)$ are a set of parameters determined by the initial preparation.[10]

From (1.6) and (2.1) we obtain

$$\begin{aligned} &P_0(k)=\sum_{\beta} P_0(k,\beta) \ , \\ &P_1(k)=\sum_{\beta,\beta'} \Psi_{\beta,\beta'}(k)P_0(k,\beta') \ . \end{aligned} \tag{2.6}$$

For our model the $\Psi(k)$ matrix is given by Eq. (2.2), so supposing the walker initially at the origin we get

$$\begin{aligned} &P_0(k)=c_1+c_2=1 \ , \\ &P_1(k)=2r\cos(k)+(c_2-c_1)2i\Delta\sin(k)+\Delta \ , \end{aligned} \tag{2.7}$$

where $c_1 = P_0(k,\Delta)$ and $c_2 = P_0(k,-\Delta)$ characterize the initial preparation (i.e., equilibrium initial preparation of the noise corresponds to the case: $c_1 = c_2 = \frac{1}{2}$).

By resolving (2.5) one gets

$$\mathcal{L}_{21}(k) = 4r\cos(k) \,, \qquad \mathcal{L}_{20}(k) = 4(\Delta^2 - r^2)\cos(k) - 3\Delta^2 \,. \tag{2.8}$$

Note that the normalization condition, Eq. (1.6b), implies that

$$2r + \Delta = 1 \,. \tag{2.9}$$

This requirement also appears as the normalization condition for the marginal[10] probability distribution $P_n(k)$ given by

$$\sum_{n'} \mathcal{L}_{n,n'}(k)\,|_{k=0} = 1 \,. \tag{2.10}$$

Due to the fact that the $(r-\beta)$ is a transition probability [see Eq. (1.7)], we conclude that the Δ parameter is bounded to the values

$$\Delta \in [0, \tfrac{1}{3}] \,. \tag{2.11}$$

The generating Green's function (2.4) completely solves our problem because the effective Green's function (in discrete time) is

$$P_n(k) = \frac{1}{n!} \partial_z^n R(k,z)\,|_{z=0} \,. \tag{2.12}$$

Having solved the Markov chain problem posed by Eqs. (1.6)–(1.7), we can now introduce a continuous-time description by using the CTRW hierarchy.[16] The effective Green's function in the continuous-time description [the Laplace representation will be characterized by the argument $(t \to u)$ of the functions] is

$$P(k,u) = \left[\frac{1-z}{u} R(k,z) \right]_{z=\psi(u)} \,, \tag{2.13}$$

where $\psi(u) = \int_0^\infty e^{-ut}\psi(t)dt$ is the Laplace transform of the waiting-time density, and $P(k,u)$ is the Laplace transform of $P(k,t)$. It is well known that the exponential waiting-time model $[\psi(t) = \lambda e^{-\lambda t}]$ gives the natural extension from the discrete to the continuous-time description.[1,2,5,6,9,14–18] The effective Green's function can then be obtained from Eqs. (2.4) and (2.13) using $\psi(u) = \lambda/(\lambda+u)$,

$$P(k,u) = \frac{1}{\lambda+u} \frac{P_0(k) + [\lambda/(\lambda+u)][P_1(k) - P_0(k)\mathcal{L}_{21}(k)]}{1 - [\lambda/(\lambda+u)]^2 \mathcal{L}_{20}(k) - [\lambda/(\lambda+u)]\mathcal{L}_{21}(k)} \,. \tag{2.14}$$

This expression permits the calculation of all the moments of the effective distribution.

The first moment is

$$\langle s(t) \rangle = L_u^{-1} \left[\frac{1}{i} \frac{\partial}{\partial k} P(k,u)\,|_{k=0} \right] = -(c_1 - c_2)(1 - e^{-2\lambda\Delta t}) \tag{2.15}$$

(here L_u^{-1} represents the inverse Laplace transform). Note that for equilibrium initial preparation of the noise $(c_1 = c_2 = \frac{1}{2})$, $\langle s(t) \rangle$ vanishes identically. The characteristic time $[\tau_N = 1/(2\lambda\Delta)]$ of the transient decay of $\langle s(t) \rangle$ coincides with the correlation time of the external noise. We also note that if the transition probability of the external noise goes to zero $(\Delta \to 0)$, the first moment vanishes.

The second moment is

$$\langle s^2(t) \rangle = L_u^{-1} \left[\frac{1}{i^2} \frac{\partial^2}{\partial k^2} P(k,u)\,|_{k=0} \right] \,. \tag{2.16}$$

Using the effective Green's function (2.14), we obtain

$$\langle s^2(t) \rangle = (1-\Delta)\lambda t e^{-2\lambda\Delta t} + \left[\frac{1+\Delta}{\Delta} \right] [1 - (1 + 2\Delta\lambda t)e^{-2\lambda\Delta t}] + (1+3\Delta) \left[\lambda t - \frac{1}{\Delta} + \left[\lambda t + \frac{1}{\Delta} \right] e^{-2\lambda\Delta t} \right] \,. \tag{2.17}$$

From (2.17) the diffusion coefficient, the velocity autocorrelation function, and its spectrum can be calculated. Noting that

$$\lim_{t\to\infty} \langle s^2(t) \rangle \to (1+3\Delta)\lambda t \,, \tag{2.18}$$

we obtain for the diffusion coefficient

$$D = \tfrac{1}{2}\lambda(1+3\Delta) \,. \tag{2.19}$$

Since unitary lattice parameter has been used in our model (1.7), Eq. (2.19) gives the effective diffusion coefficient per unit length. Note that for $\Delta = 0$ Eq. (2.17) reproduces a pure diffusion behavior but for $\Delta \neq 0$ a nondiffusive regime occurs during the transient.

The spectrum of the velocity autocorrelation function is

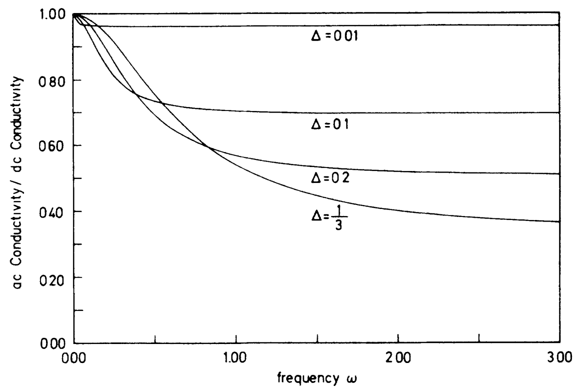

FIG. 1. Numerical evaluation of $\kappa(\omega)/\kappa(0)$ as given by Eq. (2.20) for three values of the noise amplitude Δ. The frequency is in units of $\lambda(=1)$.

$$\kappa(\omega)=\mathrm{Re}\left[\frac{u^2}{2}\langle s^2(u)\rangle\right]_{u=i\omega}$$
$$=\frac{\frac{1}{2}\lambda(1-\Delta)\omega^4+4\lambda^3\Delta^2(1+\Delta)\omega^2+8\lambda^5\Delta^4(1+3\Delta)}{(\omega^2+4\lambda^2\Delta^2)^2} . \tag{2.20}$$

In Fig. 1 we show $\kappa(\omega)/\kappa(0)$ as a function of frequency. The effect of the external noise is to reduce the ac conductivity relative to the dc conductivity. The constant asymptotic limit at high frequency is due to the pure jump-diffusion description used in the present model.[17] The transient behavior of the mobility is due to the correlation time of the external noise. The frequency where $\kappa(\omega)/\kappa(0)$ reduces itself to half value is of the order of $\omega \cong 2\Delta\lambda$ as was expected.

III. NON-MARKOVIAN MASTER EQUATION FOR THE EFFECTIVE GREEN'S FUNCTION

We now study the equation which governs the evolution of the effective Green's function of the walker. The Green's function of the walker in the model defined by (1.1) is a functional of the external noise $\alpha(t)$ and the *effective Green's function* is the average over the different realizations of the stochastic process $\alpha(t)$ [see Eq. (1.4)]. We have described this situation by using Eq. (1.5) and a Markov chain with internal states and later going to a continuous-time description (the equivalence of the two procedures is proved in the Appendix). As a consequence, the Green's function, of the original problem (1.1), averaged over the external noise, is given by Eq. (2.14).

To derive the equation for $P(k,t)$ we rewrite (2.13) in the form

$$P(k,t)=L_u^{-1}\left[\frac{1-\psi(u)}{u}R(k,z=\psi(u))\right]$$
$$=\int_0^t \Phi(t-\tau)\sum_{n=0}^{\infty}\psi^{(n)}(\tau)P_n(k)d\tau , \tag{3.1}$$

where $\Phi(t)$ is the probability that the walker remains fixed during the time interval $(0,t)$ and $\psi^{(n)}(t)$ is the probability density for the time at which the nth step occurs.[16] If an exponential waiting-time density $[\psi(t)=\lambda e^{-\lambda t}]$ is used, we obtain a relation between the marginal initial conditions $\partial_t^n P(s,t)|_{t=0}$ and the set of parameters $P_0(s)$ and $P_1(s)$,

$$\lim_{t\to 0} P(s,t)=P_0(s) ,$$
$$\lim_{t\to 0}\partial_t P(s,t)=\lambda[P_1(s)-P_0(s)] . \tag{3.2}$$

Using Eq. (2.14), and a few algebraic manipulations, we obtain

$$u^2P(k,u)-uP_0(k)-\lambda[P_1(k)-P_0(k)]+\lambda[2-\mathcal{L}_{21}(k)]uP(k,u)$$
$$=\lambda[2-\mathcal{L}_{21}(k)]P_0(k)-\lambda^2[1-\mathcal{L}_{21}(k)-\mathcal{L}_{20}(k)]P(k,u) . \tag{3.3}$$

Laplace inversion of (3.3) using (3.2) gives

$$\frac{\partial^2}{\partial t^2}P(k,u)+\lambda[2-\mathcal{L}_{21}(k)]\frac{\partial}{\partial t}P(k,t)=-\lambda^2[1-\mathcal{L}_{21}(k)-\mathcal{L}_{20}(k)]P(k,t) . \tag{3.4}$$

This is a non-Markovian evolution equation for $P(k,t)$. We stress that this equation is an *exact result* valid for general $\mathcal{L}_{21}(k)$ and $\mathcal{L}_{20}(k)$ (i.e., a general Markovian chain with internal states where β runs over two different values[10]).

Equation (3.4) can be written in the usual form of a generalized master equation as follows:

$$\frac{\partial}{\partial t}P(k,t)=\int_0^t M(k,t-\tau)P(k,\tau)d\tau , \tag{3.5}$$

where

$$M(k,u)=L_u(M(k,t))=u-\frac{[u^2+A(k)u+B(k)]P(k,t=0)}{uP(k,t=0)+[\partial P(k,t=0)/\partial t]+A(k)P(k,t=0)} , \tag{3.6}$$

$$A(k)\equiv\lambda[2-\mathcal{L}_{21}(k)] ,$$
$$B(k)\equiv\lambda^2[1-\mathcal{L}_{21}(k)-\mathcal{L}_{20}(k)] . \tag{3.7}$$

From this equation it is easy to see that the normalization condition is satisfied because $\sum_{n'}\mathcal{L}_{n,n'}(k=0)=1$. In real space the effective Green's function satisfies the following evolution equation:

$$\frac{\partial}{\partial t}P(s,t)=\int_0^t \sum_{s'} M(s-s',t-\tau)P(s',\tau)d\tau \ , \tag{3.8}$$

where

$$M(s-s',t)=\frac{1}{N}\sum_k e^{-ik(s-s')}\left\{\left[-B(k)-\frac{\partial}{\partial t}P(k,t=0)A(k)-\left[\frac{\partial}{\partial t}P(k,t=0)\right]^2\right]e^{-t/\tau_c(k)}+\frac{\partial P(k,t=0)}{\partial t}\delta(t)\right\} \tag{3.9a}$$

and

$$\tau_c(k)=\left[A(k)+\frac{\partial P(k,t=0)}{\partial t}\right]^{-1} . \tag{3.9b}$$

Equations (3.5) and (3.8) (in Fourier and real space, respectively) give a complete description of the effective Green's function. They are equivalent to Eq. (3.4). The time-dependent transition probabilities $M(k,t)$ (in Fourier space) characterize the generalized master equation of the marginal problem. Equation (3.8) is the evolution equation for the joint description of two types of fluctuations, one describing the random walk by a discrete Markovian master equation, and the other the external noise, which was introduced at the level of the master equation by replacing the constant coefficients by random processes [Eq. (1.1)]. It is obvious that the generalized master equation (3.8), in general, involves nonvanishing transition probabilities for any step size. This is an important difference with the one-step *stochastic* master equation of Eq. (1.1). For illustrative purposes we consider here the long-time regimen of Eq. (3.8). In this regime we explicitly show that the transition probabilities for the effective Green's function involve spatial hopping for any step size. We observe in (3.8) that the non-Markovian effect is a causal convolution of $P(k,t)$ with $\exp[-(t-\tau)/\tau_c(k)]$. The characteristic time $\tau_c(k)$ can be used to define a transient, non-Markovian regime.

Considering for times $t \gg \tau_c(k)$ we can approximate the generalized master equation by a Markovian one as follows:

$$\int_0^t d\tau\, P(k,\tau)\exp[-(t-\tau)/\tau_c(k)] \cong \tau_c(k)P(k,\tau) \ ; \tag{3.10}$$

then Eq. (3.5) reduces to

$$\frac{\partial}{\partial t}P(k,t)\cong M_m(k)P(k,t) \ . \tag{3.11}$$

Using (3.9a) and (3.9b) we obtain for the transition probability in the limit $t \gg \tau_c(k)$ (Markovian regimen)

$$\begin{aligned} M_m(s-s') = {} & \frac{4\Delta^2\lambda}{1+\Delta}\sum_{n=0}^{\infty}\sum_{h=0}^{n}\binom{n}{h}\left[\frac{a_1}{a_2}\right]^h\left[-\frac{2\Delta c}{a_2}\right]^{n-h}2^n \ . \\ & \times\left[(c^2-1)\sum_{m=0}^{h}\sum_{m'=0}^{n-h}\binom{h}{m}\binom{n-h}{m'}(-1)^{m'}\delta(n-2(m+m')-(s-s'))\right. \\ & +\left[\frac{1-c^2}{4}\right]\sum_{m=0}^{h+2}\sum_{m'=0}^{n-h}\binom{h+2}{m}\binom{n-h}{m'}(-1)^{m'}\delta(n-2(m+m'-1)-(s-s')) \\ & \left.-\frac{c}{2}\sum_{m=0}^{h}\sum_{m'=0}^{n-h+1}\binom{h}{m}\binom{n-h+1}{m'}(-1)^{m'}\delta(n+1-2(m+m')-(s-s'))\right] \\ & +\lambda\left[-(1-\Delta)\delta[0-(s-s')]+\left[\frac{1-\Delta}{2}\right][\delta(1-(s-s'))+\delta(-1-(s-s'))]\right. \\ & \left.+c\Delta[\delta(1-(s-s'))-\delta(-1-(s-s'))]\right] , \end{aligned} \tag{3.12}$$

where $c\equiv c_2-c_1$, $a_1\equiv 1-\Delta$, and $a_2\equiv 1+\Delta$. In general, $M_m(s-(s\pm n))$ involves nonvanishing transition probabilities for any n. Note that Eq. (1.1) is a one-step *stochastic* master equation, while the master equation (3.11) for the process averaged over $\alpha(t)$ (effective Green's function) involves transitions to any site.

We write Eq. (3.12) up to the contributions to third neighbors as

$$\begin{aligned}\lambda^{-1}M_m(s-s') = {} & \left[-1+\Delta+\frac{(2c^2-1)\Delta^2}{a_2}-\frac{4c^2\Delta^3}{a_2^2}\right]\delta_{s,s'}+\left[\frac{(1-c^2)\Delta^2}{2a_2^2}a_1+\frac{(1-c^2)c\Delta^3}{a_2^2}\right]\delta_{-3,s-s'} \\ & +\left[\frac{(1-c^2)\Delta^2}{a_2}+\frac{c\Delta^2}{a_2^2}a_1+\frac{2c^2\Delta^3}{a_2^2}\right]\delta_{-2,s-s'} \\ & +\left[\frac{1}{2}-\frac{\Delta}{2}-c\Delta+\frac{2c\Delta^2}{a_2}-\frac{(1-c^2)\Delta^2}{2a_2^2}a_1-\frac{3(1-c^2)c\Delta^3}{a_2^2}\right]\delta_{-1,s-s'} \\ & +\left[\frac{1}{2}-\frac{\Delta}{2}+c\Delta-\frac{2c\Delta^2}{a_2}-\frac{(1-c^2)\Delta^2}{2a_2^2}a_1+\frac{3(1-c^2)c\Delta^3}{a_2^2}\right]\delta_{1,s-s'} \\ & +\left[\frac{(1-c^2)\Delta^2}{a_2}-\frac{c\Delta^2}{a_2^2}a_1+\frac{2c^2\Delta^3}{a_2^2}\right]\delta_{2,s-s'}+\left[\frac{(1-c^2)\Delta^2}{2a_2^2}a_1-\frac{(1-c^2)c\Delta^3}{a_2^2}\right]\delta_{3,s-s'} \ . \end{aligned} \quad (3.13)$$

Note that if the external noise is turned off ($\Delta=0$) we reobtain the expected one-step master equation (usual random walk without external noise),

$$\lambda^{-1}M_m(s-s')\,|_{\Delta=0}=\tfrac{1}{2}(\delta_{s,s'+1}+\delta_{s,s'-1})-\delta_{s,s'} \ . \quad (3.14)$$

If we consider the equilibrium initial condition for the noise ($c_1=c_2=\frac{1}{2}$), the expression (3.13) becomes much simpler,

$$\begin{aligned}\lambda^{-1}M_m(s-s')\,|_{c=0} = {} & -(1-\Delta)\delta_{s,s'}+\frac{(1-\Delta)(1+2\Delta)}{2(1+\Delta)^2}(\delta_{1,s-s'}+\delta_{-1,s-s'}) \\ & +\frac{\Delta^2}{1+\Delta}(\delta_{2,s-s'}+\delta_{-2,s-s'})+\frac{\Delta^2(1-\Delta)}{2(1+\Delta)^2}(\delta_{3,s-s'}+\delta_{-3,s-s'}) \ . \end{aligned} \quad (3.15)$$

From this result we can study the relative amplitude of the transition probability for hopping to sites far away from the nearest neighbor. In Fig. 2 we show this factor as a function of the noise parameter Δ. It is seen that the transition probabilities to remote sites diminished with jump size.

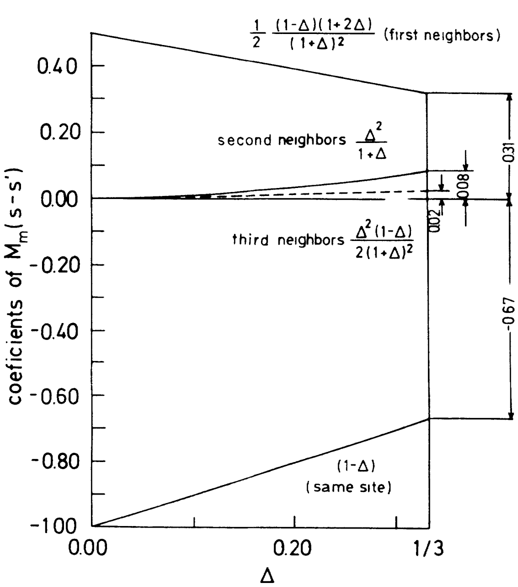

FIG. 2. Probability transition [Eq. (3.15)] to first, second, and third neighbors as a function of the noise amplitude Δ for the "effective" walker in the Markovian regime [$t \gg \tau_c(k)$].

IV. RANDOM BIAS IN A RANDOM MEDIA

The effective Green's function in the continuous-time approach is calculated from Eq. (2.13) where $\psi(u)$ is the Laplace transform of the waiting-time probability density. Waiting-time densities with divergent first moments have been used by Scher and Montroll to study transport processes in amorphous solids.[2] In their model the carriers move by hopping between localized states, created by a random potential arising from the static disorder. Schlesinger and Hughes[6,18] take for the form of $\psi(t)$ at long time $\psi(t)\cong At^{-1-\alpha}$ (with $0<\alpha<1$), so that $\psi(u)\cong 1-(A/\alpha)\Gamma(1-\alpha)u^{\alpha}$. We now consider a CTRW version of the Markov chain model [(1.6)–(1.7)] using such waiting-time density. In this way we introduce the effect of a static disorder in our model. We can then analyze the long-time behavior of our random-bias model in the presence of static disorder. This give some understanding of the effect of simultaneous dynamic and static disorder.

Using the long-tail waiting-time density in Eq. (2.13), the long-time behavior of the second moment of the effective Green's function is

$$\begin{aligned}\lim_{t\to\infty}\langle s^2(t)\rangle &= \lim_{u\to 0}L_u^{-1}\left[\frac{1}{i^2}\frac{\partial^2}{\partial k^2}P(k,u)\,|_{k=0}\right] \\ &\cong \frac{(1+3\Delta^2)\alpha}{2\Delta A\,\Gamma(1-\alpha)}t^{\alpha} \ . \end{aligned} \quad (4.1)$$

Here we have used $c=0$ as the initial condition for the external noise. Equation (4.1) gives the variance of the walker probability distribution, where two averages have been carried out: one over the external noise $\alpha(t)$ [given by Eq. (2.13)] and the other over a random media modeled here by the Scher and Lax long-time-tail waiting-time density.[2,5,6,18]

From (4.1) we conclude that in this model global external noise does not change the anomalous exponent t^{α} which is determined by the average over the random media. External noise (global dynamical disorder) only changes the prefactor.

In the limit of weak external noise ($\Delta \cong 0$), a detailed calculation shows that

$$\lim_{t\to\infty}\lim_{\Delta\to 0}\langle s^2(t)\rangle \cong \frac{1-\alpha+\alpha(\alpha+1)/2}{\Gamma(1+2\alpha)} \times \left[\frac{\alpha}{A\,\Gamma(1-\alpha)}\right]^2 t^{2\alpha} . \tag{4.2}$$

This result can be physically understood because in the limit $\Delta \cong 0$ the time between changes of the external noise $\langle t_N\rangle \cong (\lambda\Delta)^{-1} \to \infty$ (see the Appendix). In this limit the walker feels the presence of a "fixed" bias[18] for which the result (4.2) was expected.

V. CONCLUSION

We have presented a straightforward method to calculate the effective Green's function of a random-bias problem by using the resolvent matrix theory.

The evolution equation for the effective Green's function was given in Sec. III. We studied the Markovian regimen [$t \gg \tau_c(k)$] of this equation and found that the associated transition probability $M_m[s-(s\pm n)]$ involves nonvanishing steps for all n, in contrast with the one-step *stochastic* master equation (1.1). This is a consequence of making the average over the external noise $\alpha(t)$. In Fig. 2 we show a graphic of $M_m[s-(s\pm n)]$ to at most third neighbors, as a function of the noise amplitude Δ.

We have studied the behavior of the moments of the effective Green's function; nondiffusive behavior is found during a transient regime of order $\tau_N = 1/(2\Delta\lambda)$.

Using the fluctuation-dissipation theorem we have obtained the mobility of the random-bias model, showing that the transient behavior has the duration of the correlation time (τ_N) of the external noise $\alpha(t)$. As a consequence of the noise, the ac conductivity is reduced relative to the dc conductivity, for a fixed value of the noise amplitude Δ, as shown in Fig. 1.

In Sec. IV we discussed in the context of the Scher and Lax theory the random-bias problem in a random medium, by using a waiting-time density with a long-time tail. We showed that the presence of the global external noise does not change the anomalous exponent of the second moment of the effective random-walk propagator. At long times the external noise only introduce a "renormalized" diffusion coefficient.

This analysis gives us insight into the study of static and dynamic disorder. We have shown that the *resolvent matrix* appears to be a useful technique to attack the problem of global dynamical disorder.

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APPENDIX: CONNECTION BETWEEN EQS. (1.1) AND (1.6)

The generalization of the Markovian chain with internal states, Eq. (1.6), to a continuous-time random walk with internal states[9] (as was used in this paper) follows using a separable model. The transition probability density matrix is

$$\Psi(s-s',\beta,\beta',t)=\psi(t)\Psi(s-s',\beta,\beta') . \tag{A1}$$

Here $\psi(t)$ is the waiting-time density and the transition matrix Ψ is given in (1.7). Our model of random walk with internal states then corresponds to a particular separable model given (in Fourier) by

$$\Psi_{\beta,\beta'}(k,t)\equiv\psi(t)\begin{bmatrix}\Psi_{\Delta,\Delta}(k) & \Psi_{\Delta,-\Delta}(k)\\ \Psi_{-\Delta,\Delta}(k) & \Psi_{-\Delta,-\Delta}(k)\end{bmatrix} , \tag{A2}$$

where $\psi(t)$ is the waiting-time density of the CTRW theory.

The model defined by the one-step *stochastic* master equation (1.1) can be written as composite Markov processes.[19] Then the evolution equation for the joint probability distribution $P(s,\alpha,t)$ is a Liouville master equation.[19,20] As is well known, there is a close connection between the multistates continuous-time random walk and the generalized master equation with internal states.[9,18] For the composite Markov processes associated with (1.1) we can write the following waiting-time density matrix Ψ (in Fourier representation):[8]

$$\Psi=\begin{bmatrix}B_{\Delta}(k)\exp[-B_{\Delta}(k=0)t]\phi_{\Delta,\Delta}(t) & \exp[-B_{-\Delta}(k=0)t]f_{\Delta,-\Delta}(t)\\ \exp[-B_{\Delta}(k=0)t]f_{-\Delta,\Delta}(t) & B_{-\Delta}(k)\exp[-B_{-\Delta}(k=0)t]\phi_{-\Delta,-\Delta}(t)\end{bmatrix} , \tag{A3a}$$

where

$$B_{\pm\Delta}(k) \equiv 2r_0\cos(k) \mp 2r_1\Delta i\sin(k)\ , \qquad \text{(A3b)}$$

which follows immediately from Eq. (1.1) for each fixed value of the external noise $\alpha(t)$. $\phi_{\pm\Delta,\pm\Delta}(t)$ is the probability that the noise remains fixed in the level $\pm\Delta$ at time t since it arrives at this level at time 0 where

$$\phi_{\pm\Delta,\pm\Delta}(t) = 1 - \int_0^t f(t')dt' \qquad \text{(A4)}$$

and $f_{\pm\Delta,\mp\Delta}(t)dt = f(t)dt$ is the probability that the noise at level $\pm\Delta$ at $t=0$ makes it transitional to $\mp\Delta$ at time between t and $t+dt$. For the two-level Markovian noise characterized by Eq. (1.3), $f(t) = (\nu/2)e^{-(\nu/2)t}$.

From Eqs. (A3) we can see that the model defined by (1.1) will correspond to a separable model if and only if the parameters take the following values:

$$2r_0 = (1-\Delta)\lambda = 2r\lambda;\quad r_1 = \lambda;\quad \nu = 2\lambda\Delta\ . \qquad \text{(A5)}$$

In this case the multistate CTRW constructed in (A2) from the Markovian chain (1.6) is equivalent to the CTRW associated with the composite Markov process equivalent to (1.1). In a more general case the waiting-time density matrix (A3) is nonseparable: a site-independent time-dependent waiting-time density cannot be factorized. As a consequence a direct connection between (1.1) and (1.6) no longer exists. In such cases a *stochastic* master equation such as (1.1) can be studied in terms of multistate CTRW theory[9] with a nonseparable Ψ matrix.[8]

The values given in (A5) imply that the mean time between steps of the walker $\langle t_W \rangle$ and the mean time between changes of the external noise $\langle t_N \rangle$ are proportional

$$\begin{aligned} \langle t_W \rangle &= \int_0^\infty t\psi_W(t)dt = (2r_0)^{-1} = [(1-\Delta)\lambda]^{-1}\ ,\\ \langle t_N \rangle &= \int_0^\infty t\psi_N(t)dt = (\nu/2)^{-1} = (\lambda\Delta)^{-1}\ , \end{aligned} \qquad \text{(A6)}$$

where

$$\begin{aligned} \psi_W(t) &= B_\Delta(k=0)\exp[-B_\Delta(k=0)t]\\ &= B_{-\Delta}(k=0)\exp[-B_{-\Delta}(k=0)t]\\ &= 2r_0\exp(-2r_0t)\ ,\\ \psi_N(t) &= f_{\pm\Delta,\mp\Delta}(t) = (\nu/2)\exp[-(\nu/2)t]\ . \end{aligned} \qquad \text{(A7)}$$

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[1] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981).

[2] H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1975).

[3] T.Odagaki and M. Lax, Phys. Rev. B **24**, 5284 (1981).

[4] J. Machta, J. Stat. Phys. **30**, 305 (1983).

[5] M. Lax and H. Scher, Phys. Rev. Lett. **39**, 781 (1977).

[6] M. F. Schlesinger and B. D. Hughes, Physica **109A**, 597 (1981).

[7] J. M. Sancho and M. San Miguel, J. Stat. Phys. **37**, 151 (1984).

[8] C. E. Budde and M. O. Cáceres, Phys. Rev. Lett. **60**, 2712 (1988).

[9] M. O. Cáceres, Phys. Rev. A **33**, 647 (1986).

[10] M. O. Cáceres and C. E. Budde, Phys. Lett. **125A**, 369 (1987).

[11] A. K. Harrison and R. Zwanzig, Phys. Rev. A **32**, 1072 (1985).

[12] R. C. Bourret, U. Frisch, and A. Pouquet, Physica **65**, 303 (1973).

[13] N. G. Van Kampen, Physica **96A**, 435 (1979).

[14] D. Bedeaux, K. L. Lindenberg, and K. E. Schuler, J. Math. Phys. **12**, 2116 (1971).

[15] M. O. Cáceres and C. E. Budde, Physica A (to be published).

[16] E. W. Montroll and G. H. Weiss, J. Math. Phys. **6**, 167 (1965).

[17] M. O. Cáceres, Phys. Scr. **37**, 214 (1988).

[18] G. H. Weiss and R. J. Rubin, Adv. Chem. Phys. **52**, 363 (1983); **52**, 389 (1983).

[19] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).

[20] R. Kubo, J. Math. Phys. **4**, 174 (1963).