PHYSICAL REVIEW A VOLUME 38, NUMBER 7 OCTOBER 1, 1988

# Tricritical disorder transition of polymers in a cloudy solvent: Annealed randomness

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(Received 19 February 1988)

The effect of annealed randomness caused by impurities diffusing in a polymer solution (i.e., a *cloudy* solvent) is analyzed exactly within the framework of the continuum Edwards model, in dimension $d$. It is shown that an infinite series of local $P$-body interactions of all orders $P \geq 2$ is generated in the cloudy solvent, whose coefficients are calculated as functions of the impurity density. The impurities can be repulsive or attractive for the chains. The present state of polymer theory (perturbation expansion, regularization, and renormalization) allows a detailed study of the effect of these interactions to all orders, as a function of dimension $d$, $2 < d \leq 4$. For an impurity density below a special threshold $\rho_c$, the polymer solution is not cloudy and is as it would be in a pure solvent. At $\rho_c$ the polymer chains undergo a transition toward a standard tricritical $\Theta$ state, and beyond that density to a dense state with phase separation. For $3 < d \leq 4$ the transition is Gaussian. The universal tricritical behavior at the transition in $2 < d \leq 3$ is given. The limiting dimension $d = 2$ is singular and subtle and is the cornerstone of a new type of behavior for $0 \leq d < 2$.

## I. INTRODUCTION

The effect of randomness or "disorder" in statistical systems is a widely studied subject. The most difficult case is of course the *quenched* one, where the logarithm of the partition function has to be averaged over the disorder. Usually the *annealed* system, where the partition function itself is averaged over the external disorder variables, is relatively simple. In bond-diluted magnetism,[1] the effect is a renormalization of the critical exponents of the pure magnetic system near a second-order phase transition into annealed exponents according to the rules[1]

$$\begin{aligned} &\bar{\nu} = \nu/(1-\alpha) , \\ &\bar{\gamma} = \gamma/(1-\alpha) , \\ &\bar{\eta} = \eta , \\ &\bar{\alpha} = -\alpha/(1-\alpha) , \\ &|\bar{\beta}| = \beta/(1-\alpha), \quad |\bar{\delta}| = \delta , \end{aligned} \tag{1.1}$$

where $\nu$, $\gamma$, $\eta$, $\alpha$, $\beta$, and $\delta$ have their usual meaning in critical phenomena for the pure system. This is valid if the pure magnet specific-heat exponent $\alpha$ is positive. A particular consequence is that the specific heat of the dilute system does not diverge at the transition. If $\alpha$ is negative, no renormalization of exponents takes place.

All this concerns *bond*-diluted magnetism. The annealed *site*-diluted magnetic models cannot be reduced in the same way. For instance, the annealed site-diluted spin-$\frac{1}{2}$ Ising models are related to the Blume-Emery-Griffiths model,[1] which embodies a tricritical point. This also occurs in two-dimensional models.[2] Generally, there is an *effective aggregation* of diluted sites induced by the annealed average on the spin variables.

The effect of disorder on polymers has led to much debate,[3–6] probably because of an early confusion between annealed and quenched cases. In particular, the analysis of randomness in bond dilution on a lattice is easy to perform[4] for the annealed average of correlation functions or partition functions and leads to a trivial renormalization of the effective connectivity constant of the self-avoiding walk (SAW), hence to no new critical behavior. The quenched average of logarithms is much more difficult to perform and is a subject under investigation,[6] not generally solved. One expects there to be a transition towards dense states at strong enough disorder.

However, the annealed disorder for polymers can lead to nontrivial behavior of physical interest, when one considers site dilution. For instance, a model of SAW's on the two-dimensional hexagonal lattice has been recently proposed[7] (see also Ref. 8) where a proportion $p$ of hexagons are forbidden to the self-avoiding walks. It has been shown that a new transition takes places at the exact site percolation threshold $p = p_c = \frac{1}{2}$ of the centers of the forbidden hexagons. An exact mapping onto the hexagonal $O(n)$ model for $n = 1$ in the critical low-temperature phase, analyzed by Coulomb-gas techniques, leads to the two-dimensional exact values of the polymer critical exponents

$$\nu = \tfrac{4}{7}, \quad \gamma = \tfrac{8}{7}, \quad \eta = 0 , \tag{1.2}$$

$$\phi = \tfrac{3}{7} \text{ (crossover exponent)} . \tag{1.3}$$

We note at this stage that the change of exponents (1.2) induced by the annealed site disorder is not of the Fisher renormalization form (1.1). Indeed, for a pure two-dimensional SAW, one has the Nienhuis values

$$\nu = \tfrac{3}{4}, \quad \alpha = 2 - \nu d = \tfrac{1}{2}, \quad \gamma = \tfrac{43}{32}, \quad \eta = \tfrac{5}{24} ,$$

and their "renormalized" counterparts (1.1) do not coincide with (1.2) and can even be meaningless ($\bar{\nu} = 2\nu = \frac{3}{2} > 1$).

A similar transition was numerically observed for a model of "dressed" chains.[9] We proposed[7] to identify the collapse transition of the SAW at the site percolation threshold of forbidden hexagons to the standard *tricritical* $\Theta$ point of SAW's,[10-13] here in two dimensions. One can show indeed that the random forbidden-hexagon model is equivalent to attractive nearest-neighbor interactions on the polymer chain and a *subclass* of attractive next-nearest-neighbor interactions.[7,8] The model proposed in Ref. 7 was solved, in the sense that the above exponents calculated in two dimensions (2D) for the polymer chains in the presence of forbidden hexagons with annealed randomness were exactly calculated from Coulomb-gas methods and conformal invariance. However, the question arises[14-16] of the universality class of this model, and of the order of the transition (tricritical, tetracritical, etc.). A simple (and rather conventional) way to describe the tricritical $\Theta$ point on a lattice consists of allowing only attractive nearest-neighbor interactions. It is then possible (although unexpected) that adding a subclass of nearest-neighbor (NN) attractive interactions leads to a transition point of different order. (One expects that adding all NN interactions leaves the transition tricritical.) In particular, one can imagine that there are some *curvature* effects[14] which are relevant near the $\Theta$ point. (Note that in the hexagonal lattice model[7] the polymer chain turns by construction by an angle of $\pi/3$ at each step.) All these problems are genuinely open and under investigation.[14-16] At the very least the study of Ref. 7 establishes the existence of a nontrivial transition for a polymer in presence of annealed hard hexagons.

It is then worthwile to attack this problem from a different point of view and to generalize it, by studying its continuum limit in $d$ dimensions. We shall consider here a similar polymer model with annealed random forbidden points[17] in the framework of the *Edwards continuum model*,[18] and in $d$ dimensions. As we shall show, this continuous model is amenable to an exact treatment. Recently, Thirumalai[17] has studied this model for a *single* chain with excluded volume in three dimensions in the presence of annealed random obstacles. His results, obtained within the realm of an *approximate self-consistent* method, suggest a negative shift of the second virial coefficient, and that a $\Theta$-like transition appears at a certain impurity concentration, towards a "disordered, coiled state," which, in the first approximation, is Gaussian in three dimensions. But the study of Ref. 17 is not sufficient. In particular, as we shall show, the annealed disorder induces an infinite series of higher multiple-point interactions, which are not treated in Ref. 17, nor their regularization.

Rather than use a self-consistent method, we perform the annealed average over the positions of the impurities and get a new effective Edwards Hamiltonian with an *infinite* set of $P$-body interactions, $P \geq 2$, the coefficients of which are explicit functions of the impurity density. These higher-order interactions generated by the annealed disorder have alternating signs when the impurities are repulsive (pointlike obstacles). In Ref. 17 only the first effective two-body interaction (which is attractive) has been obtained, which leads to the collapse, the three-body interaction being thought to be marginal in three dimensions (3D), and the higher ones irrelevant in the (infrared) long-range limit. However, one has to study the (ultraviolet) short-range effect of all $P$-point interactions, before concluding on the existence of the disorder-induced transition. Indeed, the proper mathematical analysis of the regularization of such a model is subtle, as shown in the case of two- and three-body interactions in Ref. 12. Any high-order $P$-body interaction contributes to the interactions of order $n < P$, hence to the excluded volume ($n = 2$), and here the interactions are infinitely many. So in particular the question arises of the sign of the total shift of the second virial coefficient, of the net effect of the annealed site randomness (attractive or repulsive), and of its stability. An illustration of the need of a well defined regularization procedure, is furnished by Ref. 17. There the self-consistent Flory-like equation (4.2) for the collapse transition includes a three-body term whose coefficient $A$ [Eq. 4.3)] is just infinite.

We study the complete disorder model in detail here, by resumming for each $n$-body interaction, the contribution of higher-order ones, $P \geq n$. We confirm the existence of an universal *tricritical* transition towards a $\Theta$ state at a certain finite impurity density and for any dimension $2 < d \leq 4$, and any number of chains. For $d > 2$ this disorder-induced transition is in the same universality class as the usual $\Theta$ point. For $3 < d \leq 4$, the tricritical transition has mean-field Gaussian exponents. For $2 < d \leq 3$ it develops a nontrivial universal tricritical behavior, which is given from known results.[11,12] In reference to the preceding discussion on the two-dimensional case, the limit $d \rightarrow 2$ of the continuum model is also considered, and exhibits some subtleties, as expected.

At $d = 2$, a new instability occurs, where all the disorder-induced interactions become equally important, and with alternating signs. Hence we can no longer draw conclusions about the order of the transition, which is reminiscent of the difficulties mentioned above for the two-dimensional theory.

This article is organized as follows. In Sec. II we present the multichain continuum Edwards model with annealed random impurities, in space dimension $d$ and its transformation into a multichain model with an infinite series of multiple-point interactions. Their coefficients are calculated as functions of the impurity density $\rho$.

In Sec. III we consider a general model of chains with any multiple-point interaction (i.e., the most generalized Edwards model). We study its ultraviolet regularization, and the cascading interaction structure induced in the cutoff regularization. The transcription of cutoff-regularized partition functions into dimensionally regularized ones is performed in detail, the latter being required for locating later any multicritical disorder transition. This formalism allows us a disgression on *k-tolerant* walks, which are proven in the framework of the continuum model, with the help of its cascading structure, to be in the same universality class as usual SAW's.

In Sec. IV we apply the formalism of Sec. III to the effective disorder model of Sec. II. The *real effective* multiple-point disorder interactions are calculated as functions of the impurity density $\rho$, of the interaction strength $\beta$ with the polymer, and of the ultraviolet cutoff. Then the existence, for $2<d<4$, of a *tricritical* disorder collapse transition at a certain impurity density $\rho_c(\beta,s_0,d)$ is established.

In Sec. V the various logarithmic laws expected at a tricritical polymer $\Theta$ point in 3D are applied to the $\Theta_D$-disorder transition. In Sec. VI the nontrivial tricritical exponents, and universal scaling laws governing the $\Theta_D$ transition in $2<d<3$, $d=3-\epsilon$ are given. The limit $d\rightarrow 2$, and the instability occuring in two dimensions are then discussed. Section VII ends with a summary of results.

## II. ANNEALED-DISORDER MODEL

### A. Effective annealed-disorder action

For simplicity, we first consider a single continuous polymer chain embedded in $d$ space, with a probability weight given in a *pure* solvent by Edwards model[18]

$$\mathcal{P}\{\mathbf{r}\}=\exp(-\mathcal{A}\{r\})\ , \tag{2.1}$$

$$\mathcal{A}\{\mathbf{r}\}=\tfrac{1}{2}\int_0^S\left[\frac{d\mathbf{r}}{ds}\right]^2 ds + \frac{b}{2}\int_0^S ds\int_0^S ds'\delta^d(\mathbf{r}(s)-\mathbf{r}(s'))\ , \tag{2.2}$$

where $\mathbf{r}(s)$ is the configuration in space $\mathbb{R}^d$ and $s$ is the abscissa along the chain, $0\leq s\leq S$, $S$ being the size (essentially the "continuum" number of monomers, or the "Brownian area"[19]). $b$ is the excluded volume term, $b\geq 0$, for the model to be defined. For $b=0$, the pure Brownian path has an averaged end-to-end distance

$$R^2=\langle[\mathbf{r}(S)-\mathbf{r}(0)]^2\rangle=dS \tag{2.3}$$

and $S$ is homogeneous to an area, $S=[L]^2$.

This model and its direct renormalization have been studied in detail, and we refer the reader to Refs. 19–23 for a comprehensive study (another but more complicated formalism can be found in Ref. 24). Here we shall recall only that the perturbation expansion of the weight (2.1) is naturally expressed in terms of the dimensionless parameter $z$ (Refs. 19 and 25)

$$z=(2\pi)^{-d/2}bS^{2-d/2}\ , \tag{2.4}$$

where one sees clearly that for $d>d_c=4$ the interactions are irrelevant in the asymptotic limit $S\rightarrow\infty$. A last remark: as such, the model (2.1) and (2.2) is ill defined since there are short-range ultraviolet divergences appearing when two abscissas $s$ and $s'$ coincide in the interaction term (2.2). One has to regularize the model. This can be done in two equivalent ways;[12,19,21] either by introducing a cutoff $s_0$ for the minimal distance along the chain for two interacting abscissas,[19,24]

$$|s-s'|\geq s_0\ , \tag{2.5}$$

or by *dimensional regularization*.[26] Dimensional regularization is the most powerful method, but its correct interpretation requires some care.[12] Cutoff and dimensional regularizations can be rigorously related one to another[12,19,21] and we shall return to this later, when considering multiple point interactions of all orders.

Let us now cloud the solvent by introducing impurities into it, with annealed randomness.[17] We add to the action (2.2) the *local-impurity* interaction term,

$$\mathcal{A}_{\text{dis}}\{\mathbf{r},\rho\}=\beta\int_0^S ds\int\delta^d(\mathbf{r}(s)-\mathbf{R})\rho(\mathbf{R})d^dR\ , \tag{2.6}$$

where $\rho(\mathbf{R})$ is the impurity density

$$\rho(\mathbf{R})=\sum_{i=1}^{N}\delta^d(\mathbf{R}-\mathbf{R}_i)\ , \tag{2.7}$$

where the $R_i$'s are the positions of the random scatterers (Fig. 1). $\beta$ is a coefficient which embodies all the physical parameters giving the strength of the *local* interaction of the chain with the impurities. $\beta$ is a real number, $\beta>0$ corresponds to excluded impurity sites, and $\beta<0$ to attractive ones. The dimensional analysis of (2.6) and (2.7) leads to a dimension of $\beta$,

$$\beta=[S]^{d/2-1}\ . \tag{2.8}$$

Let us consider now the partition function of the chain defined by the functional integral

$$Z\{S,\rho\}=\int d\{\mathbf{r}\}\exp(-\mathcal{A}\{\mathbf{r}\}-\mathcal{A}_{\text{dis}}\{\mathbf{r},\rho\}) \tag{2.9}$$

and depending explicitly on the impurity density $\{\rho(\mathbf{R})\}$. Since a troubled solvent is annealed, we have simply to evaluate the disorder average,

$$\langle Z\{S,\rho\}\rangle\equiv\frac{1}{V^N}\int\prod_{i=1}^{N}d^dR_iZ\{S,\rho\}\ , \tag{2.10}$$

where $V$ is the $d$ volume, with the thermodynamic limit to be taken in the end,

$$\begin{aligned}&V\rightarrow\infty\ ,\\&N\rightarrow\infty\ ,\\&\frac{N}{V}=\rho,\quad 0\leq\rho<\infty\end{aligned} \tag{2.11}$$

$\rho$ being the average density of scatterers. Hence the only disorder average to be computed is the effective probability weight

$$\begin{aligned}\mathcal{P}'\{\mathbf{r}\}&\equiv\langle\exp(-\mathcal{A}\{\mathbf{r}\}-\mathcal{A}_{\text{dis}}\{\mathbf{r},\rho\})\rangle\\&=\exp(-\mathcal{A}\{\mathbf{r}\})\langle\exp(-\mathcal{A}_{\text{dis}}\{\mathbf{r},\rho\})\rangle\ ,\end{aligned} \tag{2.12}$$

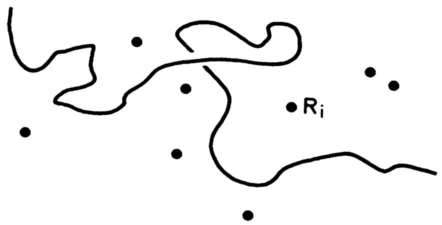

FIG. 1. Polymer chain floating in a cloudy solvent, with $N$ impurities at random positions $R_i$, $i=1,\ldots,N$.

where we define

$$\mathcal{P}_{\rm dis}\{\mathbf{r}\}=\langle \exp(-\mathcal{A}_{\rm dis}\{\mathbf{r},\rho\})\rangle$$

$$=\left\langle \exp\left[-\beta\sum_{i=1}^{N}\int_0^S \delta^d(\mathbf{r}(s)-\mathbf{R}_i)ds\right]\right\rangle . \tag{2.13}$$

It is easily factorized into

$$\mathcal{P}_{\rm dis}\{\mathbf{r}\}=\left[\int \frac{d^dR}{V}\exp\left[-\beta\int_0^S\delta^d(\mathbf{r}(s)-\mathbf{R})ds\right]\right]^N \tag{2.14}$$

and we use the usual trick of the Ursell-Mayer expansion for real gases, writing symbolically[17]

$$\int\frac{d^dR}{V}\exp\left[-\beta\int_0^S\delta^d(\mathbf{r}(s)-\mathbf{R})ds\right]$$

$$=1+\int\frac{d^dR}{V}\left[\exp\left[-\beta\int_0^S\delta^d(\mathbf{r}(s)-\mathbf{R})ds\right]-1\right] .$$

Now, taking the limit (2.11) $N\rightarrow\infty$ in (2.14) we find[17]

$$\mathcal{P}_{\rm dis}\{\mathbf{r}\}=\exp\left\{\rho\int d^dR\left[\exp\left[-\beta\int_0^S ds\,\delta^d(\mathbf{r}(s)-\mathbf{R})\right]-1\right]\right\} . \tag{2.15}$$

In Ref. 17 this effective weight was studied via the self-consistent Edwards-Singh approximate method, and also by some first-order perturbation expansions.

We depart from it here and calculate the complete effective action. For this, we expand the exponential inside the square brackets of (2.15). We write symbolically

$$[\delta]=\int_0^S ds\,\delta^d(\mathbf{r}(s)-\mathbf{R})$$

and

$$e^{-\beta[\delta]}-1=\sum_{n\geq 1}(-\beta)^n\frac{1}{n!}[\delta]^n ,$$

where

$$[\delta]^n=\int_0^S ds_1\cdots\int_0^S ds_n\delta^d(\mathbf{r}(s_1)-\mathbf{R})\cdots\times\delta^d(\mathbf{r}(s_n)-\mathbf{R}) . \tag{2.15a}$$

Hence by integrating over $\mathbf{R}$

$$\rho\int d^dR(e^{-\beta[\delta]}-1)$$

$$=\rho\sum_{n\geq 1}(-\beta)^n\frac{1}{n!}\prod_{i=1}^{n}\int_0^S ds_i\prod_{i=1}^{n-1}\delta^d(\mathbf{r}(s_i)-\mathbf{r}(s_n)) . \tag{2.16}$$

The notation for $n=1$ is to be understood in the sense that there is no $\delta^d$ distribution, and the $n=1$ term reads simply $-\rho\beta S$. It thus corresponds to the *mean-field* energy of the chain of size $S$ in the bath of impurities with density $\rho$.

Inserting (2.16) into the effective disorder weight (2.15) we find exactly

$$\begin{aligned}&\mathcal{P}_{\rm dis}\{\mathbf{r}\}=\exp(-\mathcal{A}_{\rm dis}\{\mathbf{r}\}) ,\\ &\mathcal{A}_{\rm dis}\{\mathbf{r}\}=-\rho\sum_{n\geq 1}(-1)^n\beta^n\frac{1}{n!}\\ &\qquad\times\prod_{i=1}^{n}\int_0^S ds_i\prod_{i=1}^{n-1}\delta^d(\mathbf{r}(s_i)-\mathbf{r}(s_n)) .\end{aligned} \tag{2.17}$$

Again the $n=1$ term in $\mathcal{A}_{\rm dis}$ is just $\rho\beta S$. So we see that the annealed cloudiness of the solvent due to the impurities is *exactly* equivalent to an infinite set of local interactions between multiple points of the chain, *without disorder.* The only peculiarity is that the coefficients of these interactions are correlated and depend on the variables $\rho$ and $\beta$ only. Furthermore, they alternate in sign (for $\beta>0$). So we shall have to study their resulting effect.

Finally, the total weight (2.12) for a single annealed disordered chain is of the generalized Edwards type,

$$\begin{aligned}&\mathcal{P}'\{\mathbf{r}\}\equiv\mathcal{P}\{\mathbf{r}\}\mathcal{P}_{\rm dis}\{\mathbf{r}\}=\exp(-\mathcal{A}'\{\mathbf{r}\}) ,\\ &\mathcal{A}'\{\mathbf{r}\}=\mathcal{A}\{\mathbf{r}\}+\mathcal{A}_{\rm dis}\{\mathbf{r}\}=\tfrac{1}{2}\int_0^S\left[\frac{d\mathbf{r}}{ds}\right]^2 ds+\sum_{n\geq 1}b_n\frac{1}{n!}\prod_{i=1}^{n}\int_0^S ds_i\prod_{i=1}^{n-1}\delta^d(\mathbf{r}(s_i)-\mathbf{r}(s_n)) ,\end{aligned} \tag{2.18}$$

where $b_n$ is the coefficient of the $n$-body interaction,

$$b_n=\delta_{n,2}b+B_n , \tag{2.19}$$

$$B_n=(-1)^{n+1}\rho\beta^n . \tag{2.20}$$

The shift in the excluded volume coefficient (2.19) is identical to that obtained by Thirumalai[17] using the self-consistent Edwards-Singh approximation. $b_1=\rho\beta$ and the higher-order terms are new. Of course, had we put into the pure model (2.1) the higher-order interaction terms, they would add to $b_n$ for $n>2$. For simplicity, we shall only discuss a pure model with the original excluded volume term $b$. Our conclusions can be straightforwardly applied to the more general case.

From the negative shift of (2.19) for $b_2=b-\rho\beta^2$, one can guess[17] the existence of a $\Theta$ point transition[10–12] towards a collapsed state at some critical value of the density $\rho$.

However, the higher-order terms $b_n$ all contribute to the location of the tricritical $\Theta$ point[12] and it is not obvious if this point really exists for this model. Indeed, for $\beta>0$ (forbidden impurities) the coefficients $b_n$ alternate in sign and induce similar effective two-body interactions. So the value $\rho_c=b/\beta^2$ (in our notation), where $b_2=0$, and proposed in Ref. 17 for the critical impurity density, is only suggestive.

Note also that for $\beta<0$ (*attractive* impurities) all the disorder-induced interactions in (2.19) and (2.20) are attractive. Then the existence of a collapse transition seems to be assured.

Let us remark also that the preceding effective action (2.18) also applies to set of polymer chains. Indeed, the disorder average for $N$ chains of configurations $\mathbf{r}_a(s_a)$ and lengths $S_a$, $a=1,\ldots,\mathcal{N}$, reads as in (2.13),

$$\mathcal{P}_{\text{dis}}\{\mathbf{r}_a\}=\left\langle \exp\left[-\beta\sum_{a=1}^{\mathcal{N}}\sum_{i=1}^{N}\int_0^{S_a} ds_a\,\delta^d(\mathbf{r}_a(s_a)-\mathbf{R}_i)\right]\right\rangle .$$

It can be transformed as in Eqs. (2.14)–(2.16) to yield

$$\mathcal{P}_{\text{dis}}\{\mathbf{r}_a\}=\exp(-\mathcal{A}_{\text{dis}}\{\mathbf{r}_a\})\ , \tag{2.21}$$

$$\mathcal{A}_{\text{dis}}\{\mathbf{r}_a\}=-\rho\sum_{n\geq 1}(-\beta)^n\frac{1}{n!}\int d^dR\left[\sum_{a=1}^{\mathcal{N}}\int_0^{S_a} ds_a\,\delta^d(\mathbf{r}_a(s_a)-\mathbf{R})\right]^n . \tag{2.22}$$

This action can be rewritten explicitly as

$$\mathcal{A}_{\text{dis}}\{\mathbf{r}_a\}=-\rho\sum_{n\geq 1}(-\beta)^n\sum_{\{\nu_a\}\left[\sum_{a=1}^{\mathcal{N}}\nu_a=n\right]}\frac{1}{\prod_{a=1}^{\mathcal{N}}\nu_a!}\prod_{a=1}^{\mathcal{N}}\int_0^{S_a}\prod_{j=1}^{\nu_a} ds_{a,j}\int d^dR\prod_{a=1}^{\mathcal{N}}\prod_{j=1}^{\nu_a}\delta^d(\mathbf{r}_a(s_{a,j})-\mathbf{R})\ , \tag{2.23}$$

where $\nu_a$, $0\leq\nu_a\leq n$, is the number of times the chain $a$ ($a=1,\ldots,\mathcal{N}$) appears in the $n$-body interaction, with the sum over all configurations $\{\nu_a\}$ restricted by the condition $\sum_{a=1}^{\mathcal{N}}\nu_a=n$. For values of $a$ such that $\nu_a=0$ we set by convention that their contribution simply disappear in (2.23), i.e., be taken equal to 1.

The multiple interaction weight (2.23) is now just the symmetric form of a series of $n$-body interactions ($n\geq 1$) between $N$ chains with their proper symmetry factors.[12,23] The coefficient of the $n$-body term is $(-1)^{n+1}\rho\beta^n$, i.e., just the one-chain value (2.19) and (2.20), as expected. The integral over the dummy variable $\mathbf{R}$ just ensures the $n$-body interaction between the $n$ points $s_{a,j}$, $a\in\{1,\mathcal{N}\}$, $j\in\{1,\nu_a\}$, to take place at a same arbitrary point of the space. The integration over $\mathbf{R}$ could of course be performed to arrive at a (less symmetric) form as in Eq. (2.18), but this is not necessary here.

### B. Effective quenched-disorder action

It is interesting to remark that the preceding formalism allows also the (formal) treatment of *quenched* disorder. Indeed, let us consider for simplicity a *single* chain with disordered partition function (2.9), and average its free energy by the usual replica trick,

$$\langle \ln\mathcal{Z}\rangle=\lim_{\mathcal{N}\to 0}\frac{\langle\mathcal{Z}^{\mathcal{N}}\rangle-1}{\mathcal{N}}\ . \tag{2.24}$$

For $\mathcal{N}$ integer the average $\langle\mathcal{Z}^{\mathcal{N}}\rangle$ is then exactly the $\mathcal{N}$ chain average performed in (2.21)–(2.23). The only change is that the original excluded volume term $b$ now applies only to each replica $a=1,\ldots,\mathcal{N}$, but not to the mutual interaction of these. On the other hand, the disorder induces mutual multiple interactions between the replicas. For simplicity, we discard here the disorder-induced interactions higher than $n=2$. Then the effective action for the $\mathcal{N}$ replicas in quenched disorder reads

$$\begin{aligned}\mathcal{A}_{\mathcal{N}}=&\sum_{a=1}^{\mathcal{N}}\int_0^S\left[\frac{d\mathbf{r}_a}{ds}\right]^2 ds+\tfrac{1}{2}(b-\rho\beta^2)\sum_{a=1}^{\mathcal{N}}\int_0^S\int_0^S ds\,ds'\,\delta^d(\mathbf{r}_a(s)-\mathbf{r}_a(s'))\\&-\tfrac{1}{2}\rho\beta^2\sum_{a,a'=1_{a\neq a'}}^{\mathcal{N}}\int_0^S\int_0^S ds\,ds'\,\delta^d(\mathbf{r}_a(s)-\mathbf{r}_{a'}(s'))+\cdots\ .\end{aligned} \tag{2.25}$$

The average $\langle\mathcal{Z}^{\mathcal{N}}\rangle$ is then simply the functional integral

$$\langle\mathcal{Z}^{\mathcal{N}}\rangle=\int\prod_{a=1}^{\mathcal{N}} d\{\mathbf{r}_a\}\exp(-\mathcal{A}_{\mathcal{N}})\ . \tag{2.26}$$

For discussing the effective sign of the disorder-induced interactions, one really has to take into account all multiple interactions, as will be done below for the annealed case. The results will equally apply to the quenched-disorder action (2.25). Anticipating a little, we shall see that the structure (2.25) is formally stable when resumming the effects of high-order interactions, leading to some renormalization of $\beta$, into $\beta^{\text{reg}}$. Now we see that the polymer replicas in quenched disorder are a very peculiar polymer system. For small enough disorder, the self-interactions of the replicas remain repulsive $[b-\rho(\beta^{\text{reg}})^2$

$>0$], while the mutual interactions between replicas are always attractive [$-\rho(\beta^{\mathrm{reg}})^2$]. Hence the replicas tend to *aggregate*. This system could present some interesting instabilities. It should be possible to study it by direct renormalization methods. Clearly more work is needed to treat correctly the quenched case and take the $\mathcal{N}\to 0$ limit.

Let us now return to the annealed-disorder action (2.18) and study the effect of cascading interactions induced by $P$-body interactions at the lower levels $n<P$. This cascading effect must first be studied in the framework of the *cutoff* regularization of the model [Eq. (2.5)], which has to be linked to *dimensional* regularization in order to find the $\Theta$ point.[12]

## III. CASCADING INTERACTIONS AND REGULARIZATION

### A. General remarks

It is sufficient to discuss the ultraviolet regularization of the one-chain model, which is straightforwardly extended to $\mathcal{N}$ interacting chains.[12,19] So we consider a general action of the form

$$\mathcal{A}'\{\mathbf{r}\}=\tfrac{1}{2}\int_0^S \left|\frac{d\mathbf{r}}{ds}\right|^2 ds + \sum_{P\geq 1} b_P \frac{1}{P!}\int_0^S \prod_{i=1}^{P} ds_i \prod_{i=1}^{P-1} \delta^d(\mathbf{r}(s_i)-\mathbf{r}(s_P)) \ . \tag{3.1}$$

By dimensional analysis we find

$$[b_P]=[S]^{(P-1)d/2-P} \ , \tag{3.1a}$$

and a convenient dimensionless interaction parameter is[27]

$$z_P \equiv b_P(2\pi)^{-(P-1)d/2} S^{P-(P-1)d/2} \ . \tag{3.1b}$$

One sees here formally that the upper multicritical dimension $d_c(P)$ below which the $P$-body interaction becomes formally (infrared) relevant is

$$d \leq d_c(P)=\frac{2P}{P-1} \ , \tag{3.1c}$$

such that $z_P\to\infty$ for $S\to\infty$ (long-chain limit). A geometrical derivation of (3.1c) is possible and is given in Appendix A.

Consider now the single-chain partition function

$$Z(S)=\frac{\int d\{\mathbf{r}\}\exp(-\mathcal{A}'\{\mathbf{r}\})}{\int d\{\mathbf{r}\}\exp(-\mathcal{A}_0\{\mathbf{r}\})} \ , \tag{3.2}$$

which is normalized by the pure Brownian partition function,[19] where $\mathcal{A}_0$ is the weight (2.2) for $b=0$.

For $d<2$, the perturbation expansion of a partition function like (3.2) involves Feynman-like integrals which are (ultraviolet) convergent.[12,19,28,29] For $d\geq 2$, one has to regularize the theory and a physically appealing way[12,19,30] is to introduce a lower cutoff area $s_0$ (2.5), such that, along the *same* chain, two interacting points are separated by $s_0$. [Here this cutoff applies also to the multiple interactions induced by the impurities: its meaning in the original model is that two points $s$ and $s'$ of a same chain must be distant by an abscissa $|s-s'|\geq s_0$ for interacting with the same impurity scatterer in (2.15a)]. A new dimensionless parameter appears in such a cutoff $P$-point theory,

$$z_{0,P}\equiv b_P(2\pi)^{-(P-1)d/2} s_0^{P-(P-1)d/2} \ , \tag{3.1b'}$$

which will be useful later.

Another way to regularize the theory is by dimensional regularization.[12,19,28,29] One extends analytically the Feynman integrals of the diagrammatic expansion, from $d<2$ to $d>2$. There is then a rather subtle relationship between the cutoff and dimensional regularization.[12,19] A pure $P$-body interaction in the cutoff $s_0$ scheme can be decomposed into $P'$ body ones with $1\leq P'\leq P$, in the dimensional regularization scheme. For instance, for the standard excluded volume ($P=2$), this leads to ($P'=1$) point insertions as discussed in Ref. 19. For the tricritical case ($P=3$) this leads to cascading two-point interactions and one-point ones, as discussed in detail in Ref. 12. Note that this has led to much confusion in the field (see the discussion in Ref. 12) concerning the shift of the $\Theta$ point and the meaning of dimensional regularization for it.

### B. First-order expansion

We study now this cascading effect for an arbitrary $P$-body interaction, on a specific diagrammatic example, hopefully illuminating. We shall later derive the general structure of cascading interactions.

Let us consider the expansion of $Z$ (3.2) to *first* order in *all* $b_P$ terms. For each $P$-body interaction, it is given by a diagram like Fig. 2.[12,19] Standard diagrammatic and Gaussian integration rules[12,19,27] give (see Ref. 27 for the $P$-body case)

$$Z(S,\{b_P\},s_0,d)=1+\sum_{P\geq 1} D_P \ , \tag{3.3}$$

where the $s_0$ cutoff regularized contribution of the $P$-point diagram (Fig. 2) reads explicitly

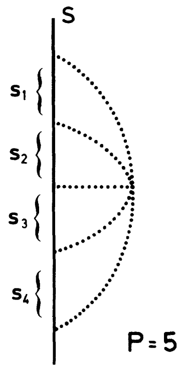

FIG. 2. First-order diagram $D_P$ induced by a $P$-point interaction and contribution to $Z(S)$ (here $P=5$). The $\{s_i\}$, $i=1,\ldots,P-1$, are the internal areas appearing in the integral representation (3.4). Their minimal value is $s_0$.

$$D_P=(-b_P)\int_{s_0}^{S}ds_1\cdots\int_{s_0}^{S}ds_{P-1}\Theta\left(S-\sum_{i=1}^{P-1}s_i\right)\left(S-\sum_{i=1}^{P-1}s_i\right)(2\pi)^{-(P-1)d/2}\prod_{i=1}^{P-1}s_i^{-d/2}\ . \tag{3.4}$$

The $S_i$, $i=1,\ldots,P-1$, are the successive segments between interaction points and have a minimal size $s_0$ (Fig. 2).

For evaluating $D_P$ we use the identity

$$X\Theta(X)=\int_{-i\infty+\sigma}^{+i\infty+\sigma}\frac{da}{2\pi i a^2}e^{aX}\ , \tag{3.5}$$

where $\sigma$ is a positive real number. Let us denote the imaginary axis $[-i\infty+\sigma,\ +i\infty+\sigma]$ by $\mathcal{C}$. Then

$$D_P=-b_P\mathcal{J}_P\ ,$$

$$\mathcal{J}_P=\int_{\mathcal{C}}\frac{da}{2\pi i a^2}e^{aS}\left[(2\pi)^{-d/2}\int_{s_0}^{\infty}ds\, s^{-d/2}e^{-as}\right]^{P-1}\ . \tag{3.6}$$

Note that on this expression (3.6) one sees immediately that the $s$ integral converges for $d<2$ up to $s_0=0$, whereas it diverges for $d\geq 2$, requiring a finite $s_0$. It is sufficient to study the loop integral

$$\mathcal{J}(s_0,d)\equiv\int_{s_0}^{\infty}ds\, s^{-d/2}e^{-as}\quad(\mathrm{Re}\, a>0)\ . \tag{3.7}$$

For $d<2$ the integral converges at $s_0=0$ and we have

$$\mathcal{J}(s_0,d)=\Gamma\left(1-\frac{d}{2}\right)a^{d/2-1}+O\left(as_0^{2-d/2}\right)\quad(d<2)\ . \tag{3.8}$$

For $2<d<4$, it is convenient to rewrite (3.7) identically as

$$\mathcal{J}(s_0,d)=\int_{0}^{\infty}(e^{-as}-1)s^{-d/2}ds+\int_{s_0}^{\infty}s^{-d/2}ds$$
$$-\int_{0}^{s_0}(e^{-as}-1)s^{-d/2}ds\quad(2<d<4)\ , \tag{3.9}$$

where for $2<d<4$ the first integral converges, whereas the third one leads to a contribution of order $as_0^{2-d/2}\rightarrow 0$, when $s_0\rightarrow 0$ and $d<4$. [Note that if we had $d>4$, we would subtract more terms $\tau e^{-as}$ of the Taylor expansion[28] $\tau$ of $e^{-as}$ near the origin in order to define a convergent integral $\int_0^{\infty}(1-\tau)e^{-as}s^{-d/2}ds$].

In all cases this convergent Taylor-subtracted integral is *just identical* to the analytic extension of the $d<2$ value (3.8) to $d>2$ (dimensional regularization),

$$\int_0^{\infty}e^{-as}s^{-d/2}ds\,|_{\text{dim. reg.}}=\Gamma\left(1-\frac{d}{2}\right)a^{d/2-1}$$
$$(\text{any } d \text{ not an even integer})\ . \tag{3.10}$$

Hence for $2<d<4$ we find

$$\mathcal{J}(s_0,d)=\Gamma\left(1-\frac{d}{2}\right)a^{d/2-1}-\frac{s_0^{1-d/2}}{1-d/2}+O(as_0^{2-d/2})$$
$$(2<d<4) \tag{3.11}$$

and we see that the jump from $d<2$ to $d>2$ across the pole $d=2$ has generated a singular cutoff-dependent part [compare (3.8) and (3.11)]. We now evaluate $\mathcal{J}_P$ (3.6) for $2<d<4$,

$$\mathcal{J}_P=\int_{\mathcal{C}}\frac{da}{2\pi i a^2}e^{aS}[A(a)+X]^{P-1}\ , \tag{3.12}$$

where $A$ is the dimensionally regularized amplitude,

$$A(a)=\Gamma\left(1-\frac{d}{2}\right)a^{d/2-1}(2\pi)^{-d/2}\ ,$$

and $X$ the cutoff part,

$$X=-\frac{1}{1-d/2}s_0^{1-d/2}(2\pi)^{-d/2}\ , \tag{3.12a}$$

and where we have suppressed the term $O(as_0^{2-d/2})$ in the limit $s_0\rightarrow 0$ for $2<d<4$. Expanding (3.12) we get

$$\mathcal{J}_P=\int_{\mathcal{C}}\frac{da}{2\pi i a^2}e^{aS}\sum_{p=0}^{P-1}\binom{P-1}{p}[A(a)]^{P-1-p}X^p\ . \tag{3.13}$$

On the other hand, the dimensionally regularized value is simply obtained by retaining only the $p=0$ term [see (3.10)],

$$\mathcal{J}_P\,|_{\text{dim. reg.}}=\int_{\mathcal{C}}\frac{da}{2\pi i a^2}e^{aS}[A(a)]^{P-1}\ . \tag{3.14}$$

Hence we find for the cutoff regularized integral (3.6) and (3.13)

$$\mathcal{J}_P=\mathcal{J}_P\,|_{\text{dim. reg.}}+\sum_{p=1}^{P-1}\binom{P-1}{p}X^p\mathcal{J}_{P-p}\,|_{\text{dim. reg.}}\ . \tag{3.15}$$

So we see the aforementioned generation of (dimensionally regularized) $n$-point interactions with $1\leq n=P-p\leq P$ from the $P$-point interaction of the cutoff theory. Since the contribution of the diagram of order $P$ (Fig. 2) is [Eq. (3.6)] $D_P=-b_P\mathcal{J}_P$, we see that the coefficient of the dimensionally-regularized. $(P-p=n)$-point interaction induced by the cutoff $P$-point interaction is just (to this order)

$$B_n(P)\equiv b_PX^p\binom{P-1}{p}$$
$$n=P-p,\quad 0\leq p\leq P-1,\quad 1\leq n\leq P\ . \tag{3.16}$$

The case $n=P$, $p=0$ gives just $B_P(P)=b_P$, which is of course still the coefficient of the $P$-body term in dimensional regularization induced by itself. The identity (3.16) is exact to first order in $b_P$. Higher-order diagrams would give the terms $O(b_P^2)$, etc.

Let us now ascend this cascade and calculate the total effective interaction induced at fixed order $n$ in dimensional regularization by higher-order ones $P\geq n$,

$$b_n^{\text{reg}}=\sum_{P,\,P=n}^{\infty}B_n(P)\ . \tag{3.17}$$

Equation (3.16) gives to *first order* in all $B_P$'s

$$b_n^{\text{reg}}=\sum_{P=n}^{\infty} b_P X^{P-n}\begin{bmatrix}P-1\\P-n\end{bmatrix} \qquad (3.18\text{a})$$

$$=\sum_{p\geq 0} b_{n+p}X^{p}\begin{bmatrix}n+p-1\\p\end{bmatrix}, \qquad (3.18\text{b})$$

where we recall that for $2<d<4$

$$X=-\frac{1}{1-d/2}s_0^{1-d/2}(2\pi)^{-d/2}. \qquad (3.19)$$

Note that for $d<2$, (3.18) is still valid by simply setting $X\equiv 0$.

### C. General structure

The pedagogical example given here clearly exhibits the general downward cascading structure (3.17) obtained when going from a cutoff polymer partition function like (3.2) to its dimensionally regularized twin[12,19,21] (in the limit $s_0\to 0$)

$$\mathcal{Z}_{s_0\to 0}(S,\{b_P,P\geq 1\},s_0,d)$$

$$\to \mathcal{Z}(S,\{b_P^{\text{reg}},P\geq 1\},d)\,|_{\text{dim. reg.}}\,. \qquad (3.20)$$

We can be more specific for the one-point interactions (point insertions).[19] In an Edwards action like (3.1) it simply gives a dimensionless contribution $b_1S$, which is exponentiated in the partition function (3.2). By dimensional analysis we can write the induced one-point insertion $b_1^{\text{reg}}$ (3.17) as

$$b_1^{\text{reg}}=s_0^{-1}F_1\{z_{0,P},P\geq 1,d\}, \qquad (3.21)$$

where the $z_{0,P}$ are the dimensionless coefficients (3.1d) associated with $b_P$ and $s_0$,

$$z_{0,P}=b_P(2\pi)^{-(P-1)d/2}s_0^{1+(P-1)(1-d/2)}. \qquad (3.22)$$

We can calculate function $F$ to first order in the $z_{0,P}$ from Eq. (3.18a)

$$b_1^{\text{reg}}=\sum_{P=1}^{\infty} b_P X^{P-1}.$$

Using $X$ (3.19) and the definition of $z_{0,P}$ gives the simple result for the point-insertion function

$$F_1\{z_{0,P},d\}=\sum_{P=1}^{\infty}\left[-\frac{1}{1-d/2}\right]^{P-1}z_{0,P}+O(\{z_{0,P}^2\}). \qquad (3.23)$$

Hence Eq. (3.20) is finally written as

$$\mathcal{Z}_{s_0\to 0}(S,\{b_P,P\geq 1\},s_0,d)$$

$$\to \exp[-(S/s_0)F_1\{z_{0,P},P\geq 1,d\}]$$

$$\times\mathcal{Z}(S,\{b_P^{\text{reg}},P\geq 2\},d)\,|_{\text{dim. reg.}}\,. \qquad (3.24)$$

This is the full generalization of the "first renormalization" of point insertions induced by two-body interactions of Ref. 19, and of the dimensional regularization scheme given in Ref. 12 for two- and three-body terms. It is important to realize that the $b_n^{\text{reg}}$, given by Eq. (3.17), are functions of the $\{b_{P\geq n}\}$ and of $s_0$ and $d$ only, and *not* of $S$. One can then write them as in (3.21) [see (3.1b)],

$$b_n^{\text{reg}}=b_n+s_0^{(d/2-1)(n-1)-1}F_n\{z_{0,P},P>n,d\}. \qquad (3.25)$$

Finally, let us note that this *local* study of ultraviolet divergences applies equally well to any *multichain* partition function (possibly) with the following constraints:

$$\mathcal{Z}_{\mathcal{N}\atop s_0\to 0}(\ldots,\{S_a\},\{b_P,P\geq 1\},s_0,d)$$

$$\to \exp\left[-\sum_{a=1}^{N}\frac{S_a}{s_0}F_1\right]$$

$$\times\mathcal{Z}_{\mathcal{N}}(\ldots,\{S_a\},\{b_P^{\text{reg}},P\geq 2\},d)\,|_{\text{dim. reg.}}\,. \qquad (3.26)$$

Hence, in dimensional regularization, all ultraviolet divergences ($s_0\to 0$) have been suppressed. This works actually for all divergences stronger than logarithmic. The latter appear for a discrete set of dimensions of measure zero on $d\in[2,4]$.

One can show that this general set of "anomalous" dimensions is for a $P$-body interaction (generalizing the argument of Ref. 31)

$$d(P,q)=d_c(P)-\frac{2}{q(P-1)}$$

$$=\frac{2P}{P-1}-\frac{2}{q(P-1)},\quad P=2,3,\ldots;\quad q\in N^*.$$

Note that for (Ref. 31)

$$P=2,\quad d(2,q)=4-2/q,$$

and for (Ref. 12)

$$P=3,\quad d(3,q)=3-1/q,$$

where $q$ is a positive integer. In dimensional regularization, these dimensions occur as poles for the dimensionally regularized amplitudes and can be reached only after renormalization. Note that this is the case of dimension $d=3$ for the relevant interactions[11,12] $P=2,3$ of the tricritical $\Theta$ point since $3=d(P=2,q=2)=d(P=3,q=\infty)$. Note also that the set $d(P,q)$ always contains for any $P$, $d(P,1)=2$, which thus plays a singular role for any $P$-body interaction.

### D. Digression on $k$-tolerant walks

Incidentally, the preceding study proves a now generally admitted result concerning $k$-tolerant walks.[32] A $k$-tolerant walk on a lattice is a walk which can visit up to $k$ times the same site, but not more. A long-lasting question[32] was whether it is in the same universality class as the standard SAW (one-tolerant walk). The answer is definitely *yes*. The $k$-tolerant walk on a lattice is entirely analogous in the continuum model to a single $(k+1)$-point local interaction (with cutoff) with a coefficient $b_{k+1}>0$. The present study shows that it induces interactions $b_n^{\text{reg}}=B_n(k+1)$ with $1\leq n\leq k+1$ in dimensional regularization, including a standard excluded volume coefficient $b_2^{\text{reg}}=B_2(k+1)>0$. Equation (3.24) reads here

$$Z(S,b_{k+1},s_0,d)\,|_{k\text{-tolerant walk}}\to\exp\left[-\frac{S}{s_0}F_1\right]Z_{\text{dim. reg.}}(S,b_2^{\text{reg}},\ldots,b_{k+1}^{\text{reg}},d)\ .$$

Now, in *dimensional regularization* and for $2<d<4$, since $b_2^{\text{reg}}>0$ interactions higher than $n=2$ are really irrelevant in the (infrared) long-chain limit and the universal behavior is dominated by the excluded volume term $b_2^{\text{reg}}>0$ and is that of the good solvent or SAW-QED.

## IV. EFFECTIVE INTERACTIONS IN ANNEALED DISORDER

### A. Calculation

We are now in a position to apply the above results to the annealed model (2.18) of polymer chains in a troubled solvent. The bare interactions are [Eqs. (2.19) and (2.20)]

$$b_n=\delta_{n,2}b+(-1)^{n+1}\rho\beta^n\ .$$

Equation (3.18b) leads in dimensional regularization to first order to

$$\begin{aligned}b_n^{\text{reg}}&=\sum_{p\geq 0}b_{n+p}X^p\left[\begin{matrix}n+p-1\\p\end{matrix}\right]\\&=(-1)^{n+1}\rho\beta^n\sum_{p=0}^{\infty}(-\beta X)^p\left[\begin{matrix}n+p-1\\p\end{matrix}\right]\\&\quad+\delta_{n,1}bX+\delta_{n,2}b\ .\end{aligned}\tag{4.1}$$

We set

$$y_0=-\beta X=\beta(2\pi)^{-d/2}\frac{s_0^{1-d/2}}{1-d/2}\ .\tag{4.2}$$

Note that $y_0$ is *negative* for $d>2$. We have to evaluate

$$\begin{aligned}f_n(y_0)&=\sum_{p=0}^{\infty}y_0^p\left[\begin{matrix}n-1+p\\p\end{matrix}\right]\\&=\frac{1}{(n-1)!}\frac{d^{n-1}}{dy_0^{n-1}}\left[\sum_{p=0}^{\infty}y_0^{p+n-1}\right]\\&=\frac{1}{(n-1)!}\frac{d^{n-1}}{dy_0^{n-1}}\left[y_0^{n-1}\frac{1}{1-y_0}\right]\ .\end{aligned}$$

Subtracting and adding 1 to $y_0^{n-1}$, the $(n-1)$th derivative is easily evaluated to yield

$$f_n(y_0)=(1-y_0)^{-n}\ .\tag{4.3}$$

Hence we find the very simple result

$$b_n^{\text{reg}}=\delta_{n,1}bX+\delta_{n,2}b+(-1)^{n+1}\rho\left[\frac{\beta}{1-y_0}\right]^n\ .\tag{4.4}$$

The first two terms are trivial and due to $b$.

Comparing the disorder induced $n$-point interactions (2.20) to their counterpart in (4.4), we see that (to first order in all disorder $P$-point interactions) the net result is a simple renormalization of the impurity interaction energy

$$\beta\to\beta^{\text{dim. reg.}}\equiv\beta/(1-y_0)\ .\tag{4.5}$$

Let us now generalize to include all disorder-induced interactions to all orders. The bare disorder-induced interaction is $B_n=(-1)^{n+1}\rho\beta^n$. Forgetting the pure solvent parameter $b$, which induces only trivial point insertions for $n=1$ which we discard here, $B_n$ is renormalized in dimensional regularization *necessarily* as

$$\begin{aligned}&b_n^{\text{reg}}=\delta_{n,2}b+B_n^{\text{reg}}\ ,\\&B_n^{\text{reg}}=(-1)^{n+1}\rho\beta^nA_n^{\text{reg}}(\rho s_0^{d/2},y_0)\ ,\end{aligned}\tag{4.6}$$

where $A_n^{\text{reg}}$ is a function of $x\equiv\rho s_0^{d/2}$ and of $y_0$ (4.2), i.e., of the only two basic independent *dimensionless* parameters which one can form with $\rho$, $\beta$, and $s_0$.

Note that for $\beta\to 0$ there is no renormalization, hence $A_n^{\text{reg}}(x,0)=1$. Moreover, we have just performed in (3.17) the resummation retaining all $B_P\sim\rho\beta^P$ (2.20) to first order, which *apparently* amounts to taking the low-density limit. Actually, the cutoff also plays a role and a more precise study (Appendix) shows that the first-order expansion (4.4) and (4.5) in all $B_P$'s amounts to the limit $x=\rho s_0^{d/2}\to 0$, and $y_0$ *finite*. Note that since $x\sim\rho s_0^{d/2}$, $y_0\sim\rho s_0^{1-d/2}$ for $d>2$, the zero cutoff limit $s_0\to 0$, which we are primarily interested in [see (3.26)], seems to lead in a natural way (i.e., for $\rho$ and $\beta$ fixed) to $x\to 0, y_0\to\infty$. So for $y_0$ fixed, we have to take also $\beta\to 0$ simultaneously (Appendix). Therefore what we have computed at first order in (4.4) and (4.5) is really the limit $x=\rho s_0^{d/2}\to 0$, i.e.,

$$A_n^{\text{reg}}(0,y_0)\equiv f_n(y_0)=\frac{1}{(1-y_0)^n}\ .\tag{4.7}$$

Hence we conclude at this order $x=0$ that $A_n$ factorizes into

$$A_n^{\text{reg}}(x=\rho s_0^{d/2},y_0)_{x\to 0}=[A(0,y_0)]^n$$

in such a way that

$$\begin{aligned}&b_n^{\text{reg}}=\delta_{n,2}b+B_n^{\text{reg}}\ ,\\&B_n^{\text{reg}}=(-1)^{n+1}\rho(\beta^{\text{reg}})^n\ ,\end{aligned}\tag{4.8}$$

$$\beta^{\text{reg}}\equiv\beta A(0,y_0),\quad A(0,y_0)=(1-y_0)^{-1}\ .\tag{4.9}$$

This factorization statement is valid here only in the low-density limit $x=\rho s_0^{d/2}\to 0$, and to all orders in $y_0$. Its validity for all orders in $x$ is not obvious. If it were true, it would mean that disorder-induced parameters in dimensional regularization all retain the *same* form

$$(-1)^{n+1}\rho(\beta^{\text{reg}})^n\ .$$

Note that in general we cannot predict the sign of $A_2^{\text{reg}}(x,y_0)$ for any value of $x$. However, we know that in the limit $x\to 0$, $A_2^{\text{reg}}=[A(0,y_0)]^2>0$, and this proves that there exists a certain range of parameters where the disorder-induced shift (4.8) in the two-body interaction is *negative*,

$$B_2^{\text{reg}} < 0\,,$$

$$x = \rho s_0^{d/2} \to 0\,,$$

$$y_0 = \frac{1}{1-d/2}(2\pi)^{-d/2}\beta s_0^{1-d/2} \quad \text{finite}\,.$$

The existence of this domain is discussed in Appendix B. This establishes at least in this range the assertion of Ref. 17. By continuity, $A_2^{\text{reg}} > 0$ will continue to hold true for at least a certain interval of values of $x > 0$. For simplicity we shall mainly place ourselves in the limit $x \to 0$ and use the factorized form (4.8). Otherwise one should use everywhere the general form (4.6).

### B. Existence of the tricritical disorder transition

We finally end up with a full disorder-induced model in dimensional regularization,

$$\mathcal{A}'\{\mathbf{r}\} = \frac{1}{2}\int_0^s \left[\frac{d\mathbf{r}}{ds}\right]^2 ds + \frac{1}{2}(b - \rho\beta^2 A_2^{\text{reg}})\int_0^S ds \int_0^S ds'\,\delta^d(\mathbf{r}(s) - \mathbf{r}(s'))$$
$$+ \sum_{n\geq 3}(-1)^{n+1}\rho\beta^n A_n^{\text{reg}}\frac{1}{n!}\prod_{i=1}^{n}\int_0^S ds_i \prod_{i=1}^{n-1}\delta^d(\mathbf{r}(s_i) - \mathbf{r}(s_n)) \tag{4.10}$$

(where one-point insertions are suppressed). With the short-range ultraviolet divergences ($s_0 \to 0$) being eliminated, we focus on the long-chain infrared limit $S \to \infty$. The effective parameters (3.1b) are now in dimensional regularization for the $n$-body interaction

$$(z_n)_{n\geq 2} = b_n^{\text{reg}}(2\pi)^{-d(n-1)/2} S^{n(1-d/2)+d/2}\,. \tag{4.11}$$

In dimensional regularization their strength is really the *effective* one. Hence we see that for increasing $n$ they have decreasing effect, provided $d > 2$,

$$(z_n/z_{n+p})_{p\in\mathbb{N}^*} \sim S^{p(1-d/2)} \to 0, \quad 2 < d\,. \tag{4.12}$$

Furthermore, at a given dimension $d > 2$, only the interactions such that their upper critical dimension (3.1e) $d_c(n) \geq d$ are relevant, i.e., for

$$n < \frac{d}{d-2}, \quad z_n \to \infty$$

and for (if integer),

$$n = \frac{d}{d-2}\,,$$
$$z_n = \text{const} \quad (\text{marginally relevant})\,. \tag{4.13}$$

So at fixed dimension $2 < d < 4$, only a finite set of interactions are thus infrared relevant, and because of the order relation (4.12), one keeps only the *lowest* $n$, $n \leq d/(d-2)$, such that $b_n^{\text{reg}} \neq 0$, which will dominate the (multi)critical behavior.

As analyzed in detail in Ref. 12, the tricritical $\Theta$ point transition occurs in the Edwards model at the exact point in dimensional regularization

$$b_2^{\text{reg}} = 0, \quad b_{n=3}^{\text{reg}} > 0, \quad b_{n\geq 4}^{\text{reg}} \ \text{arbitrary}\,. \tag{4.14}$$

Hence in our model (4.10) with cloudy solvent the true $\Theta$ point is obtained for the density,

$$b_2^{\text{reg}} = b - \rho\beta^2 A_2^{\text{reg}} = 0\,, \tag{4.15}$$

with

$$A_2^{\text{reg}}(x, y_0) = (1 - y_0)^{-2} \quad \text{for } x = \rho s_0^{d/2} \to 0 \tag{4.16}$$

and

$$y_0 = \frac{1}{1-d/2}(2\pi)^{-d/2}\beta s_0^{1-d/2} \quad \text{finite}\,.$$

Hence we see that this disorder-induced $\Theta$ point occurs for a tricritical density

$$\rho_c = b\beta^{-2}[A_2^{\text{reg}}(x, y_0)]^{-1} \quad [= b\beta^{-2}A^{-2}(0, y_0)],$$
$$x \to 0$$

which always exists (at least for $x$ small), even if for $\beta > 0$ the successive interactions in the bare effective action $\mathcal{A}'$ (2.18) have alternate signs.

In Appendix B we study in more detail the existence of a domain of parameters $\rho$, $\beta$, and $s_0$, where the tricritical condition (4.15) holds, and where the resummation (4.7) and (4.16) is still valid, i.e., in the simultaneous limits $s_0 \to 0$, $x = \rho s_0^{d/2} \to 0$, $y_0 \sim \beta s_0^{1-d/2} < \infty$, and $b = \rho\beta^2(1 - y_0)^{-2} < \infty$. The solution for $2 < d < 4$ is $s_0 \to 0$, $\rho \sim b s_0^{2-d} \to \infty$, and $\beta \sim s_0^{d/2-1} \to 0$, at $\rho\beta^2$ fixed and $x = \rho s_0^{d/2} \sim s_0^{2-d/2} \to 0$. This establishes for $d > 2$ the existence of a standard tricritical collapse transition in the annealed-disorder model (2.6), in the same universality class as the usual $\Theta$ point, QED.

## V. TRICRITICAL DISORDER TRANSITION

### A. Minimal model

Since the existence of a $\Theta$-like tricritical transition is now established in the annealed-disorder model (2.18) and (4.10), we shall for $2 < d < 4$ work with analytic continuation in $d$ and use the dimensionally regularized minimal theory (forgetting the trivial one-point insertion term)

$$\mathcal{A}'\{\mathbf{r}\}=\tfrac{1}{2}\int_0^S\left[\frac{d\mathbf{r}}{ds}\right]^2 ds+\tfrac{1}{2}[b-\rho(\beta^{\mathrm{reg}})^2]\int_0^S ds\int_0^S ds'\delta^d(\mathbf{r}(s)-\mathbf{r}(s'))$$
$$+\frac{1}{3!}\rho(\beta^{\mathrm{reg}})^3\int_0^S ds\int_0^S ds'\int_0^S ds''\delta^d(\mathbf{r}(s)-\mathbf{r}(s'))\delta^d(\mathbf{r}(s')-\mathbf{r}(s'')) \ , \tag{5.1}$$

since higher interactions are less relevant in the (infrared) $S\rightarrow\infty$ limit.

The tricritical laws obtained from this model (5.1) are now well established[11,12,20] and we shall not repeat the renormalization calculations which are rather involved. We simply state here the physical results. In order to simplify the notations and to make contact with Ref. 11 and 12 we set (in dimensional regularization)

$$b_2^{\mathrm{reg}}=b-\rho(\beta^{\mathrm{reg}})^2\equiv b' \ , \tag{5.2}$$

$$b_3^{\mathrm{reg}}=\rho(\beta^{\mathrm{reg}})^3\equiv c' \ . \tag{5.3}$$

We assume $\beta>0$ (repulsive impurities), hence $\beta^{\mathrm{reg}}>0$, and $c'>0$. Otherwise $(c'<0)$ the model is unstable at $b'=0$. In this case of attractive impurities $(\beta<0)$, a similar $\Theta$-point transition could nevertheless exist, and would depend on the pure solvent three-body interaction $c$ (like $b$ here), which has been neglected until now. Then, if $c'=c+\rho(\beta^{\mathrm{reg}})^3>0$, one is still in the universality class of the *tricritical* $\Theta$ point. If one has the simultaneous conditions

$$b'=b-\rho(\beta^{\mathrm{reg}})^2=0 \ , \tag{5.4}$$

$$c'=c+\rho(\beta^{\mathrm{reg}})^3=0 \tag{5.5}$$

(hence $\beta^{\mathrm{reg}}<0$, attractive impurities), then the system will undergo a *tetracritical* transition, with relevant four-body interactions, and an upper critical dimension $d_c(4)=\frac{8}{3}$. This seems not entirely impossible: one can first adjust the impurity density to satisfy (5.5), and then vary the temperature $T$ of the solvent to adjust $b$ to (5.4), $b$ being very sensitive to $T$, while $c$ is not. However, experimentally the difficulties would probably be considerable.

Finally, for the model (5.1)–(5.3), in dimensional regularization, near the tricritical point, there are only two dimensionless relevant parameters,[11,12,23] namely, the two- and three-body interaction parameters

$$z'=(2\pi)^{-d/2}b'S^{2-d/2} \ , \tag{5.6}$$

$$y'=(2\pi)^{-d}c'S^{3-d} \ . \tag{5.7}$$

The tricritical $\Theta$ point corresponds to

$$z'=0, \quad S\rightarrow\infty \ . \tag{5.8}$$

One sees that the three-body parameter $y'$ is relevant in the long-chain (infrared) limit $S\rightarrow\infty$, only if $d<3$, $y'\rightarrow\infty$. Hence three situations are possible for a tricritical transition in $d$ dimensions.[33]

(a) $3<d$. In this case $y'\rightarrow 0$, for $S\rightarrow\infty$. The tricritical transition is of the mean-field type, all exponents take the Gaussian values. The polymer chains are simply Gaussian at the $\Theta$ point $z'=0$.

(b) $3=d$. One has $y'=(2\pi)^{-3}c'$, and $y'$ is a marginal parameter. At this upper tricritical dimension $d=3$, logarithmic divergences occur,[10,30] which are experimentally observable.[34,35]

(c) $d<3$. In this case $S\rightarrow\infty, y'\rightarrow\infty$, and new tricritical nontrivial exponents appear.

We refer the reader to previous studies (Refs. 11, 12, and 23 and references therein) for the tricritical direct renormalization of the model (5.1) for $d\leq 3$. The universal results are as follows.

### B. Tricritical logarithmic laws in 3D (Refs. 10-12,30)

The size of an isolated polymer chain can be characterized by the end-to-end square distance

$$R^2=\langle[\mathbf{r}(S)-\mathbf{r}(0)]^2\rangle \tag{5.9}$$

or by its radius of gyration

$$R_G^2=\frac{1}{2S^2}\int_0^S ds\int_0^S ds'\langle[\mathbf{r}(s)-\mathbf{r}(s')]^2\rangle \ . \tag{5.10}$$

Near the $\Theta$ point of the cloudy solvent they read, respectively, in $d=3$ (Refs. 11, 12, and 30)

$$R^2=3S[A_0(y')(1-\tfrac{148}{33}\pi h)+\tfrac{4}{3}z'(h/y')^{4/11}A_4(y')] \tag{5.11}$$

and (Ref. 12 correcting Ref. 11)

$$R_G^2=\frac{S}{2}\left[A_0(y')\left(1-\frac{493}{(33)4}\pi h\right]+\tfrac{134}{105}z'(h/y')^{4/11}A_4(y')\right] \ , \tag{5.12}$$

where $h$ is the effective *renormalized* three-body dimensionless virial coefficient[11,12,30]

$$h=\frac{1}{44\pi\ln S/s_0} \quad (S\rightarrow\infty) \tag{5.13}$$

and where $A_0$ and $A_4$ are regular functions of $y'$, such that $A_0(y')=1+\frac{16}{33}\pi y'+\cdots$ and $A_4(0)=1$. [The functions[11] $A_0(y')$ and $A_4(y')$ are to be distinguished from the $A_n^{\mathrm{reg}}(x,y_0)$ (4.6)]. In our disorder cloudy solvent, let us recall that in $d=3$

$$z'=(2\pi)^{-3/2}[b-\rho(\beta^{\mathrm{reg}})^2]S^{1/2} \ ,$$
$$y'=(2\pi)^{-3}\rho(\beta^{\mathrm{reg}})^3 \ . \tag{5.14}$$

It is important to define the domain of validity of such tricritical logarithmic laws.[30] They apply to long chains such that $S\rightarrow\infty, h\rightarrow 0$, and[30]

$$h/y'\ll 1, \quad y'44\pi\ln(S/s_0)\gg 1 \ . \tag{5.15}$$

Furthermore, the effect of two-body interactions must be dominated by the logarithmic (infrared) divergences due to the three-body terms. The renormalized $z'$ parameter in the tricritical theory is [see (5.11) and (5.12)]

$$g' \equiv z'(h/y')^{4/11} A_4(y') , \tag{5.16}$$

and one can show that the preceding condition reads[30]

$$g'^2 \ll h , \tag{5.17}$$

i.e.,

$$z'^2 h^{-3/11} y'^{-8/11} A_4(y') \ll 1 .$$

Note that the first-order perturbation expansion of $R^2$ and $R_G^2$ is simply[12]

$$\begin{aligned} R^2 &= 3S(1-4\pi y' + \tfrac{4}{3}z') , \\ R_G^2 &= \frac{S}{2}(1-\tfrac{13}{4}\pi y' + \tfrac{134}{105}z') . \end{aligned} \tag{5.18}$$

Hence we see in (5.11) and (5.12) that the smooth linear dependence of the polymer size on $z'$ (5.14), i.e., on the solvent temperature through $b$ (Ref. 30) or on the impurity density $\rho$, is renormalized by a large logarithmic factor

$$(h/y')^{4/11} \sim [\rho \ln(S/s_0)]^{-4/11} . \tag{5.19}$$

Such a reduction factor has been observed[34] ( $\sim 0.62$) in light-scattering data[35] of polystyrene in cyclohexane at the $\Theta$ temperature. It would be most interesting to test the $(\rho \ln S)^{-4/11}$ dependence (5.19) in an experimental realization of a troubled solvent.

The virial expansion of the osmotic pressure near the cloud $\Theta$ point reads explicitly[11]

$$\beta_B \Pi = C[1+\tfrac{1}{2}gC(2\pi R^2/3)^{3/2} + \tfrac{1}{3}hC^2(2\pi R^2/3)^3 + \cdots ] , \tag{5.20}$$

where $\beta_B = 1/k_B T$, where $C$ is the chain concentration, and where the dimensionless second virial coefficient $g$ reads

$$g = g' - 8h = z'(h/y')^{4/11} A_4(y') - 8h \tag{5.21}$$

and depends on $\rho$ in a nontrivial way as in (5.14) and (5.16),

$$\begin{aligned} g' &= (2\pi)^{-3/2}[b-\rho(\beta^{\mathrm{reg}})^2]S^{1/2} \\ &\quad \times [22(2\pi)^{-2}\rho(\beta^{\mathrm{reg}})^3 \ln(S/s_0)]^{-4/11} . \end{aligned}$$

The *partition function* $\mathcal{Z}(S)$ of a single polymer chain in the troubled solvent is also interesting. It reads[11,12,30]

$$\begin{aligned} \mathcal{Z}(S) = A_1(y')[1 &- \tfrac{49}{11}\pi h + 4z'(h/y')^{4/11} A_4(y') \\ &+ \tfrac{1}{6}z'^2 h^{-3/11} y'^{-8/11} A_4^2(y')] , \end{aligned} \tag{5.22}$$

where $A_1(y')$ is a regular function of $y'$,

$$A_1(y') = 1 + \tfrac{5}{11}\pi y' + \cdots$$

The *specific heat* of a single polymer chain exhibits a logarithmic peak at the usual $\Theta$ point.[30,36,37] Here again a similar phenomenon occurs. The two-body parameter $b$ usually depends on the temperature $T$ of the solvent. Hence here $z'$ (5.14) will be considered as depending phenomenologically on $T$.[12,30] The energy reads

$$E = -\frac{\partial}{\partial \beta_B} \ln \mathcal{Z}(S) \quad (\beta_B = 1/k_B T) , \tag{5.23}$$

yielding a specific heat

$$C/k_B = -\beta_B^2 \frac{\partial}{\partial \beta_B} E . \tag{5.24}$$

From the partition function we find an energy

$$\begin{aligned} E = -\frac{dz'}{d\beta_B}[4(h/y')^{4/11} A_4(y') \\ + \tfrac{1}{3}z' h^{-3/11} y'^{-8/11} A_4^2(y')] . \end{aligned} \tag{5.25}$$

Usually[30] the variation of $b$ and $z$ with $T$ is taken as linear near the Flory $\Theta_F$ temperature, where $z=0$. Here the disorder-induced $\Theta_D$ point (5.14) $z'=0$ is shifted towards $b>0$ at a temperature $\Theta_D > \Theta_F$. Hence the variation of $b$ with the temperature can no longer be linear around $\Theta_D$. So we find

$$\begin{aligned} C/k_B = \beta_B^2 \Bigg[ &\frac{d^2 z'}{d\beta_B^2} 4(h/y')^{4/11} A_4(y') \\ &+ \frac{1}{3}\left[\frac{dz'}{d\beta_B}\right]^2 h^{-3/11} y'^{-8/11} A_4^2(y') \Bigg] \end{aligned}$$

(whereas usually $d^2 z'/d\beta_B^2 \approx 0$). However, it is easy to compare the dependence on $h/y' \rightarrow 0$ of the two terms, the second one being divergent in the long-chain limit (5.15). Hence we find, using (5.14), a singular specific heat

$$\begin{aligned} C/k_B = T^2 \left[\frac{db}{dT}\right]^2 \frac{1}{24\pi^3}(44\pi)^{3/11} y'^{-8/11} A_4^2(y') \\ \times S(\ln S)^{3/11} . \end{aligned} \tag{5.26}$$

One also expects tricritical laws in the *semidilute* regime.[30] Let us simply mention that the are $S$ of a single chain is replaced by the monomer concentration

$$\mathcal{C} = CS \tag{5.27}$$

in such a way that[30,11] the dimensionless third virial coefficient becomes

$$h \rightarrow h(\mathcal{C}) = \frac{1}{44\pi \ln(1/\mathcal{C}^2 s_0)} . \tag{5.28}$$

One has still $\mathcal{C}^2 s_0 \ll 1$, $h \ll 1$, and the solution is never dense. We predicted some years ago[30] the universal equation of the phase separation curve in the $(\mathcal{C}, T)$ plane, in the infinite-chain limit. The osmotic pressure in the simedilute regime[30] is given by

$$\begin{aligned} \beta_B \Pi = \tfrac{1}{3}h(\mathcal{C})\mathcal{C}^3[2\pi A_0(y')]^3 \\ + \tfrac{1}{2}b'\mathcal{C}^2[h(\mathcal{C})/y']^{4/11} A_4(y')[A_0(y')]^{3/2} . \end{aligned} \tag{5.29}$$

This osmotic pressure depends only on the monomer concentration $\mathcal{C}$ and is valid in the limit $S/s_0 \rightarrow \infty$, $\mathcal{C}^2 s_0$

fixed. In the *infinite*-chain limit, the coexistence curve separates two phases, one is *infinitely* dilute, the other semidilute. Both phases thus have $\Pi \to 0$ when $S \to \infty$.[30] Hence the semidilute branch is obtained from $\Pi = 0$ in (5.29),

$$\begin{aligned} b' &= b - \rho(\beta^{\text{reg}})^2 \\ &= -\tfrac{2}{3}\mathcal{C}[44\pi \ln(1/\mathcal{C}^2 s_0)]^{-7/11} y'^{4/11}(2\pi)^3 \\ &\qquad \times [A_0(y')]^{3/2}[A_4(y')]^{-1} . \end{aligned} \tag{5.30}$$

Note that the classical Flory theory would give simply

$$b' = -\tfrac{2}{3}\mathcal{C} y'(2\pi)^3 .$$

So we observe a new logarithmic equation $b' \sim -\mathcal{C}(\ln \mathcal{C}^{-1})^{-7/11}$ predicted by the tricritical theory. Note that $\rho = 0$ in (5.30) gives the usual coexistence curve.[30] Hence the presence of annealed impurities *shifts* the coexistence curve along the temperature axis $b$ by $\rho(\beta^{\text{reg}})^2$, *towards higher temperatures.*

## VI. TRANSITION BELOW THREE DIMENSIONS

### A. $\epsilon$-asymptotic expansion

For $2 < d < 3$, one has $3 < d/(d-2) < \infty$ in Eq. (4.13) when $d > 2$ (Refs. 38 and 39) and more and more $n$ interactions, higher than three-body, become relevant. However, for $d > 2$, they are all strictly ordered [Eq. (4.12)] and $n = 3$ is still dominant. Hence three-body effects become fully relevant and dominate the physics of the $\Theta$ point by leading to nontrivial scaling behavior. The analytical calculations can be performed for

$$d = 3 - \epsilon, \quad \epsilon > 0 \tag{6.1}$$

(see Ref. 11 and references therein), and lead to asymptotic expansions in $\epsilon$. Let us briefly recall some results[11] which apply as such to our disorder model.

The relevant parameters are as in Eq. (5.6) and (5.7),

$$\begin{aligned} z' &= (2\pi)^{-d/2} b' S^{2-d/2} , \\ y' &= (2\pi)^{-d} c' S^{3-d} . \end{aligned} \tag{6.2}$$

The tricritical regime corresponds to the limit $y' \to +\infty$. These two parameters disappear from the universal physical laws which are expressed in terms of renormalized two- and three-body parameters $g'$ and $h$. More precisely, the osmotic pressure calculated in naive first-order perturbation theory is

$$\begin{aligned} \beta_B \Pi \,|_{\text{pert}} &= C + \tfrac{1}{2}(-8y' + z')C^2(2\pi S)^{d/2} \\ &\qquad + \tfrac{1}{3} y' C^3 (2\pi S)^d + \cdots , \end{aligned} \tag{6.3}$$

while its actual universal virial expansion is[11]

$$\begin{aligned} \beta_B \Pi &= C + \tfrac{1}{2}(-8h + g')C^2(2\pi R^2/d)^{d/2} \\ &\qquad + \tfrac{1}{3} h C^3 (2\pi R^2/d)^d + \cdots , \end{aligned} \tag{6.4}$$

where $R^2$ is the physical effective size of the chain (5.9), and the renormalized third virial coefficient reaches a fixed point value (in the infinite-chain limit)

$$h^* = \frac{\epsilon}{44\pi} + O(\epsilon^2) . \tag{6.5}$$

$g'$ measures the departure from the $\Theta$ point ($b' = 0$, $z' = 0$, $g' = 0$). It reads[11]

$$g' = z'(y')^{(\phi - 2 + d/2)/\epsilon} A_4(\epsilon) \tag{6.6}$$

[$A_4(\epsilon)$ being a calculable amplitude] or, using (6.2),

$$g' = (2\pi)^{-d/2} b' S^{\phi} A'_4(c') , \tag{6.7}$$

where $\phi$ is the tricritical crossover exponent[38]

$$\phi = \frac{1}{2} + \frac{3\epsilon}{22} + O(\epsilon^2) . \tag{6.8}$$

The squared end-to-end distance $R^2$ (5.9) and the squared radius of gyration $R_G^2$ (5.10) scale like[11]

$$\begin{aligned} R^2 &= dS\, A_0(\epsilon) y'^{(2\nu - 1)/\epsilon}(1 + \tfrac{4}{3} g' + \cdots ) , \\ R_G^2 &= dS\, A_G(\epsilon) y'^{(2\nu - 1)/\epsilon}(1 + \tfrac{134}{105} g' + \cdots ) , \end{aligned} \tag{6.9}$$

where $A_0$ and $A_G$ are calculable critical amplitudes and where $\nu$ is the usual tricritical exponent[38,11]

$$\nu = \frac{1}{2} + \frac{2}{3}\frac{\epsilon^2}{(11)^2} + O(\epsilon^3) \tag{6.10}$$

obtained from field theory[38] or directly in ploymer theory.[11]

There exists a universal ratio (Ref. 11, corrected in Ref. 12)

$$\aleph = 6R_G^2/R^2 = 1 + \tfrac{3}{4}\pi h - \tfrac{2}{35} g' . \tag{6.11}$$

At the $\Theta$ point $g' = 0$, and at the fixed point (6.5), the universal amplitude ratio is

$$\aleph = 6A_G(\epsilon)/A_0(\epsilon) = 1 + \frac{3\epsilon}{4^2(11)} + O(\epsilon^2) . \tag{6.12}$$

Finally, the tricritical partition function of a single chain reads[11]

$$Z(S) = A_1(\epsilon) y'^{(\gamma - 1)/\epsilon} \left[ 1 + 4g' + \frac{2\pi}{2\phi - 1} g'^2 \right] , \tag{6.13}$$

where $A_1(\epsilon)$ is a calculable amplitude, and where $\gamma$ is the "magnetic susceptibility" exponent

$$\gamma = 1 + 5\frac{\epsilon^2}{(22)^2} + O(\epsilon^3) \tag{6.14}$$

obtained from field theory[38] or directly in polymer theory.[11] Note that in terms of bare parameters (6.2) and (6.7), this partition function becomes explicitly

$$\begin{aligned} Z(S) = A'_1(c') S^{\gamma - 1} \Bigg[ & 1 + 4(2\pi)^{-d/2} b' S^{\phi} A'_4(c') \\ & + \frac{2\pi}{2\phi - 1}(2\pi)^{-d} b'^2 S^{2\phi} [A'_4(c')]^2 \Bigg] \end{aligned} \tag{6.15}$$

(where $A'_1$ and $A'_4$ are some amplitudes). Hence the specific heat (5.24) scales like

$$C/k_B = T^2 \left[ \frac{db}{dT} \right]^2 2(2\pi)^{1-d} \frac{S^{2\phi}}{2\phi - 1} [A_4'(c')]^2 \quad (6.16)$$

at the $\Theta$ point, and diverges when $S \to \infty$.

Let us stress again that all of these universal tricritical laws apply as such to the annealed-disorder $\Theta_D$ transition. One knows,[40] however, that the $\epsilon$ expansion is highly asymptotic for a tricritical point below 3D. Hence analytical results like those given in this section are exact, but of rather academic value. They cannot in particular be extended towards $d=2$, a case which we briefly study now.

### B. Two-dimensional case

When $d \to 2$, new phenomena occur in the annealed-disorder model (2.18) and (5.1). First, we see that the parameter $y_0$ (4.2),

$$y_0 = \beta (2\pi)^{-d/2} \frac{s_0^{1-d/2}}{1-d/2} ,$$

which appeared in the resummation of short-range ultraviolet divergences in Sec. IV becomes singular when $d \to 2$ and takes the form

$$(y_0)_{d\to 2} = \frac{\beta}{2\pi} \frac{1}{1-d/2} + \beta (2\pi)^{-1} \ln s_0 . \quad (6.17)$$

Hence new logarithmic divergence occur. Moreover, as is well known, all $n$-body interactions become equally relevant in $d=2$, and this appears clearly in Eqs. (4.12) and (4.13). So we can no longer predict the behavior of the disorder model (5.1) in two dimensions, where the interactions are all effective and with alternating signs (for $\beta > 0$, repulsive impurities). It is possible then that new local details of the model become relevant in two dimensions, since we may interpret this blow up of multiple interactions in our model (5.1) as a signal of *instability* in two dimensions.

This brings us back to the discussion of the introduction. The two-dimensional annealed-disorder model, introduced in Ref. 7, and describing a chain in the presence of forbidden hexagons on the hexagonal lattice, does have a nontrivial transition [Eq. (1.2)] at the percolation threshold of the hexagons. Its universality class was shown to be that of the *geometric* properties of the low-temperature phase of the Ising model, or equivalently of the $q=1$ critical $q$-state Potts model (percolation hulls). The present study shows that the continuum limit of this annealed polymer model with site-percolation has for $2 < d < 4$ a transition which is just the usual *tricritical* $\Theta$ transition. Unfortunately, we cannot rigorously extend this to $d=2$. Hence the order of the quite similar transition induced by disorder in the hard hexagon model of Ref. 7 is still an open question: Is it tricritical, tetracritical, etc.? However, the striking fact that for $d > 2$ it is always *tricritical* may support the tantalizing suggestion of Refs. 7 and 8 of its equivalence to the usual $\Theta$ point, even in $d=2$.

### C. Below two dimensions

It is interesting to analytically continue the generalized Edwards model (3.1) and (4.10) to dimensions below $d=2$. Then all diagrammatic integrals of the perturbation expansion are short-range *ultraviolet convergent.* So no cutoff $s_0$ is needed anymore. But the infrared behavior of the theory is probably highly nontrivial. We remark indeed that the dimensionless parameters $z_P$ (3.1b) have the form

$$z_P = b_P (2\pi)^{-(P-1)d/2} S^{P(1-d/2)+d/2}$$

and all $P$ interactions are more and more relevant when $P \to \infty$ for $S$ large. So the disorder-induced model (4.10) becomes "unstable" when the interactions alternate on sign ($\beta > 0$), $d=2$ being the dimension where this "instability" occurs.

## VII. CONCLUSION

Let us summarize briefly the findings of this article. The continuum Edwards model modified by an impurity statistical weight

$$\exp \left[ -\beta \int_0^S ds \, \delta^d(\mathbf{r}(s) - \mathbf{R}) \right] , \quad (7.1)$$

where $\mathbf{r}(s)$ is the configuration of the chain and $\mathbf{R}$ the location of a single impurity, has been exactly solved for the annealed case, for any number of polymer chains. An effective generalized Edwards model results from the disorder, with coefficients of multiple point interactions $(-1)^{n+1}\rho\beta^n$, $\rho$ being the density of impurities. After a necessary regularization of the model, the real coefficients are "renormalized" into

$$(-1)^{n+1} \rho \beta^n A_n^{\text{reg}}(x, y_0) , \quad (7.2)$$

where $A_n^{\text{reg}}$ is a function of

$$x = \rho s_0^{d/2}, \quad y_0 = (2\pi)^{-d/2} \beta s_0^{1-d/2} / (1-d/2) ,$$

where $s_0$ is the ultraviolet short-range cutoff. We have calculated exactly $A_n^{\text{reg}}(0, y_0) = (1-y_0)^{-n}$. This was sufficient to establish the existence of a $\Theta_D$-tricritical transition at a certain (calculable) density $\rho_c$ and for any dimension $2 < d < 4$. This transition is in the same universality class as the usual $\Theta$-point transition of polymers, and we gave its universal tricritical scaling behavior, nontrivial for $2 < d \leq 3$.

Interestingly enough, the limit $d \to 2$ of this continuum disorder model exhibits a new kind of instability: all the disorder-induced interactions become equally relevant, and also the regularized disorder-induced interactions (7.2) depend on a variable $y_0$ singular at $d=2$.

This instability at $d=2$ appearing in our analytical continuum Edwards model is reminiscent of the direct studies of the polymer $\Theta$ point in two dimensions. There also, in a quite different formalism (conformal invariance, Coulomb-gas methods) the collapse transition seems to be sensitive to local details, and perhaps even presents several different universality classes. More work is thus needed to elucidate the two-dimensional case.

But in $d > 2$, the annealed-disorder *site*-impurity model may be considered now as being solved, exhibiting a tricritical Θ-like transition; in contradistinction to the trivial annealed bond model, which has no transition. Also, a by-product of our analysis of ultraviolet divergences in a general continuum polymer model with $(k+1)$-body interactions, was the analytical proof of the generally accepted idea that a $k$-tolerant walk is, in the asymptotic limit, in the same universality class as a usual polymer with excluded volume.

It would be interesting to search for experimental realizations of the cloudy solvent and to try and observe the disorder collapse transition. One could think of troubling the solvant with little balls, but metastable aggregation effects could occur, which have to be overcome to realize an annealed system. Also, a further theoretical investigation of the quenched-disorder model (2.25) by direct renormalization methods would be quite interesting.

## ACKNOWLEDGMENTS

We thank M. E. Fisher for a question about the multicritical dimension $d_c(n)$, which led to Appendix A, and J. des Cloizeaux for a critical reading of the manuscript.

## APPENDIX A: GEOMETRICAL DERIVATION OF THE UPPER MULTICRITICAL DIMENSION $d_c(n)$

We show that the upper multicritical dimension (3.1c) and (4.13),

$$d_c(n)=\frac{2n}{n-1}\ , \tag{A1}$$

below which $n$-point interactions are relevant $[d \le d_c(n)]$, has a simple geometrical meaning. This dimension must clearly be related to the intersection properties of $n$ random walks. In the case of $n=2$ walks, the upper critical dimension of polymers (or of a $\varphi^4$ theory) is $d_c(2)=4$, and is often derived simply as $d_c(2)=2D$, where $D=2$ is the Hausdorff or fractal dimension of a Brownian path. This is based on the seemingly obvious additivity rule, according to which two objects of dimensions $D$ and $D'$ intersect in dimension $d$ if and only if $d \le D+D'$. But clearly the generalization of this argument would not lead to (A1), but to an incorrect $d_c(n)=2n$. Actually, the above rule $d \le D+D'$, even if correct, must *not* be interpreted as an additivity rule for dimensions, but is rather a degenerate case of another additivity property, namely that of codimensions, which we derive now.

Let us consider $n$ objects or subspaces $D_i$ of dimensions $\dim D_i$, $i=1,\ldots,n$, embedded in Euclidean $d$-dimensional space $\mathbb{E}^d$. The $\dim D_i$'s are usual dimensions (e.g., 1 for a straight line, 2 for a plane, etc.), but at the end they will also be generalized to *fractal* dimensions.

In $\mathbb{E}^d$, the number of equations necessary to define a subspace $D_i$ of dimension $\dim D_i$ is the codimension

$$\operatorname{codim} D_i = d - \dim D_i\ . \tag{A2}$$

The intersection $\bigcap_{i=1}^{n} D_i$ of the $n$ objects is then described by the collection of all equations defining the $D_i$'s, hence their number is simply

$$\operatorname{codim}\left[\bigcap_{i=1}^{n} D_i\right]=\sum_{i=1}^{n}\operatorname{codim} D_i\ . \tag{A3}$$

Substituting (A2) on both sides of (A3) gives immediately

$$d-\dim\left[\bigcap_{i=1}^{n} D_i\right]=\sum_{i=1}^{n}(d-\dim D_i) \tag{A4}$$

$$=nd-\sum_{i=1}^{n}\dim D_i\ . \tag{A5}$$

This can be rewritten as

$$(n-1)d=\sum_{i=1}^{n}\dim D_i-\dim\left[\bigcap_{i=1}^{n} D_i\right]\ . \tag{A6}$$

For the intersection of the $n$ objects *not* to be *empty*, we must have

$$\bigcap_{i=1}^{n} D_i \neq \varnothing \quad \text{or} \quad \dim\left[\bigcap_{i=1}^{n} D_i\right] \ge 0\ .$$

So Eq. (A6) implies

$$d \le \frac{1}{n-1}\sum_{i=1}^{n}\dim D_i\ . \tag{A7}$$

So we see that for $n$ objects, the condition of nonempty intersection in $\mathbb{E}^d$ is not the (wrong) "additivity rule" $d \le \sum_{i=1}^{n}\dim D_i$, but (A7) with a normalization factor $(n-1)^{-1}$. A special case is that of two objects $(n=2)$, where $d \le \dim D_1+\dim D_2$, which is often interpreted as an "obvious" additivity rule for dimensions. According to the general formula (A7), it is rather a degenerate misleading case. As obvious from the derivation given in (A3), additivity holds for the codimensions (A2), not for the dimensions.

Finally, the inequality (A7) is applied to a set of $n$ random walks, or Brownian paths, each having fractal dimension 2. Hence the condition of nonempty intersection (A7) becomes $d \le [1/(n-1)]2n$, which is just (A1), QED. This applies to several physical systems, all based on a random walk underlying picture. $d_c(n)$ is derived here as the upper multicritical dimension of the intersections of $n$ random walks[27], and is also that of the $n$-body interactions in the generalized Edwards model (3.1) for polymers, and of the $(\varphi^2)^n$ field theory.

## APPENDIX B: DOMAIN OF PARAMETERS $\rho, \beta$, AND $s_0$

Let us first discuss the meaning of the first-order expansion (3.17) and (4.1) in all disorder-induced interactions (2.19) and (2.20),

$$B_P=(-1)^{P+1}\rho\beta^P\ , \tag{B1}$$

which led for $n \ge 2$ to (4.4),

$$b_n^{\text{reg}}=\delta_{n,2}b+B_n^{\text{reg}}\ ,$$

$$B_n^{\text{reg}}=(-1)^{n+1}\rho[\beta A(0,y_0)]^n\ , \tag{B2}$$

$$A(0,y_0)=(1-y_0)^{-1}\ , \tag{B3}$$

$$y_0=\frac{1}{1-d/2}(2\pi)^{-d/2}\beta s_0^{1-d/2} \tag{B4}$$

(note that $y_0<0$ for $d>2$, and that the singularity of (A3) at $y_0=1$ is irrelevant). According to Eq. (3.23), the next-order terms are power series of the dimensionless cutoff contributions $z_{0,P}$'s (3.22), associated with the disorder-induced interactions (2.19) and (2.20), and reading for $P\geq 3$

$$z_{0,P}=(-1)^{P+1}\rho\beta^P s_0^{1+(P-1)(1-d/2)}(2\pi)^{-(P-1)d/2} . \tag{B5}$$

This can be rearranged from (4.2) into

$$\begin{aligned} z_{0,P}&=(-1)^{P+1}\rho s_0^{d/2}[(1-d/2)y_0]^P \\ &=(-1)^{P+1}(1-d/2)^P x y_0^P . \end{aligned} \tag{B6}$$

Now notice that

$$x=\rho s_0^{d/2},\quad y_0\sim\beta s_0^{1-d/2} . \tag{B7}$$

Hence for $d>2$, the zero cutoff limit $s_0\to 0$, which we are primarily interested in [see (3.26)], leads for $\rho$ and $\beta$ fixed to

$$x\to 0,\quad y_0\to\infty .$$

In this case, for $P$ large enough, $z_{0,P}$ (B6) will grow indefinitely with $y_0$, even if $x\to 0$. To prevent this, we can keep $y_0$ finite, i.e., take the limit $\beta\to 0$, such that

$$s_0\to 0,\quad y_0<\infty,\quad \beta\sim s_0^{d/2-1}\to 0\quad (d>2) . \tag{B8}$$

Then, for

$$x=\rho s_0^{d/2}\to 0 , \tag{B9}$$

we have shown that (B6) $z_{0,P}\sim x y_0^P\to 0$ for any $P$, which justifies retaining in (3.23) only the linear terms $O(z_{0,P})$, for any $P$. This leads then to the simple regularized effective intractions (B2).

It remains to check that the limits (B8) and (B9) are consistent with Eqs. (4.15) and (4.16) (exact in this range), giving the location of the tricritical point (in dimensional regularization),

$$0<b=\rho\beta^2/(1-y_0)^2<\infty . \tag{B10}$$

$b$ is a parameter given by the physics of the polymer in the pure solvent, and is a fixed quantity. From (B8) we get $\rho\beta^2(1-y_0)^{-2}\sim\rho s_0^{d-2}\sim b$. Hence from (B10) we find that $\rho$ must scale like

$$\rho_{s_0\to 0}\sim b s_0^{2-d}\to\infty\quad \text{for } d>2 , \tag{B11}$$

i.e., the density of impurities must be very large. This was expected since we required that their effective interaction $\beta$ with the polymer goes to zero (B8), and that they nevertheless induce a collapse transition (B10). Now, from (B11) we calculate the equivalent of

$$x_{s_0\to 0}=\beta s_0^{d/2}\sim b s_0^{2-d/2}\to 0\quad \text{for } d<4 , \tag{B12}$$

which is exactly the required condition (B9), granted for $d<4$.

So we have established for $2<d<4$ the existence of a set of parameters ($\rho$, $\beta$, and $s_0$) scaling like $\rho\sim b s_0^{2-d}$, $\beta\sim s_0^{d/2-1}$, and $s_0\to 0$, such that the simple renormalization equation (B2) and the tricritical transition (B10) are simultaneously valid, QED.

Of course, by continuity, the existence of a tricritical transition in the disorder model can be expected over a larger set of parameters, where the real effective interactions should be rather of the full form (4.6),

$$B_n^{\text{reg}}=(-1)^{n+1}\rho\beta^n A_n^{\text{reg}}(x,y_0) .$$

The argument presented here has the academic value of a solvable model, and the physical interactions in a real cloudy system would be quite complicated.

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