Mirror transmission and laser phase diffusion in the quantum regime

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We consider how the finite transmission of the output mirror of a laser operating well above threshold affects laser phase diffusion in the quantum regime. In our traveling-wave analysis, the cavity loses field energy discretely only once every round trip. For a low-Q cavity, the linewidth that arises from phase diffusion can differ significantly from the Schawlow-Townes result derived using traditional theories based on a distributed-loss picture.

I. INTRODUCTION

Traditional quantum theories of the laser[1–3] have described well the statistical properties of laser light like photon statistics and the intrinsic linewidth. They are, however, based on a few simplifying approximations that are not applicable in situations where a careful quantum-mechanical treatment is not only desirable but also imperative. Examples of contemporary relevance include the analysis of maximum theoretically achievable sensitivity of an interferometric measurement employing lasers either in the active or in the passive configuration[4] and of micromasers[5] driven by a few atoms and containing only tens of photons at any time.

In this article we analyze the problem of phase diffusion in a stable laser due to spontaneous emission. What distinguishes our approach from the previous approaches[1–3] is contained primarily in the way we treat cavity losses. A consideration, such as ours, of the coupling of the partially transparent output mirror is essential in determining how much of the phase diffusion results from vacuum fluctuations leaking in from the outside world ("universe") and how much of it would be present even in a perfect, lossless cavity. We shall see that even a perfect laser cavity would produce light with a stable amplitude but with phase that degrades in time.

After the laser field builds up and settles down to a steady value inside the cavity, the energy in the field is lost primarily through scattering from the impurities in the active medium of the walls and through the partially reflecting output mirror (mirrors). One can eliminate much of the scattering losses, but the partial transmittance of the mirror (mirrors) is necessary in order to extract light from the cavity. Scattering losses may be treated as a distributed-loss mechanism, but mirror losses should not since they occur only at the cavity ends and not inside the active medium.

In view of the complexity of including mirror losses as a discrete, beam-splitter kind of loss mechanism, traditional theories have instead treated all losses as being distributed throughout the cavity. In theories based on a Langevin equation formalism,[3] losses are thus modeled by a simple term $-(\nu/2Q)\alpha$ appended to the expression for the time derivative $d\alpha/dt$ of the fluctuating field amplitude α. One then talks about the net small-signal gain coefficient

$$\mathcal{A}_1=(\mathcal{A}-\nu/2Q) \tag{1}$$

that differs from the small-signal gain $\mathcal{A}$ of the active medium alone by $\nu/2Q$. This is a standard feature of other approaches as well, namely, those based on the density-operator[1] and related Fokker-Planck equations.[1,2]

The picture we use here is a traveling-wave picture in which cavity losses occur discretely, once every round trip, a situation that is obtained with a perfectly reflecting mirror at one end of the cavity and a partially transmitting mirror at the other. Clearly a mode-locked laser with its light pulse traveling back and forth in the cavity fits this picture physically. What becomes immediately clear is that true steady state for the amplitude distribution or for the photon statistics is not attainable. In fact, light must amplify sufficiently during each trip through the cavity to compensate for the "single-shot" energy losses at the partially transmitting mirror. In other words, only a quasisteady state can be obtained in such a system in which the light field reproduces itself periodically, the period being the round-trip passage time of light.

Alternatively, if one has a single-mode field then one may say that a true steady state does indeed result, but that the radiation field amplitude has a spatial dependence on the propagation distance z. The two pictures are completely equivalent[6] for a single-mode laser, both classically and quantum mechanically, transforming into each other under the substitution $t \rightarrow z/c_{\rm ph}$, $c_{\rm ph}$ being the phase velocity of light in the cavity.

The organization of this paper is as follows. In Sec. II we discuss in greater detail the laser model that we employ and how one can implement a simple quantum description of the beam splitter[7] to address mirror losses as well as phase changes. In Sec. III the model is first used to derive the quasi-steady-state distribution for the field amplitude. The shape of that distribution, we shall show, is invariant although its width and height change due to the process of amplification. Section III B contains the most important results of this paper, namely, the phase-diffusion rate and the resulting linewidth and their dependence on the mirror reflectivity for the laser operating in the quasisteady state.

II. FOKKER-PLANCK DESCRIPTION OF THE LASER

The active medium of our laser consists of two-level atoms in exact resonance (frequency ν) with a single mode of the laser cavity. We do not worry about the subtleties concerning modes in a leaky cavity,[8] assuming that the single mode of interest is identical to the corresponding mode of a loss-free cavity, except that the former has a finite frequency width.

Atoms injected in the excited state at a constant rate into the cavity are stimulated by the circulating radiation field to emit in phase. In the traveling-wave picture that we have adopted, we assume that there is no loss of the field energy throughout a single round trip of the field from the partially transmitting mirror back to it. Only at the very end of each round trip does the field get partially lost through that mirror.

Under a coarse-graining assumption, the time derivative of the density operator ρ_F of the field may be easily calculated.[1] Since phase diffusion is best analyzed in the Fokker-Planck picture, we look for the equivalent equation that the P distribution[9] of the field expressed in terms of coherent states $|\alpha\rangle$,

$$\rho_F=\int P(\alpha,\alpha^*,t)\,|\alpha\rangle\langle\alpha|\,d^2\alpha\ , \tag{2}$$

satisfies. This equation may be written in terms of scaled variables as[2]

$$\frac{\partial}{\partial\hat{t}}P(\hat{\alpha},\hat{\alpha}^*,t)=-\left[\frac{\partial}{\partial\hat{\alpha}}[(a_1-|\hat{\alpha}|^2)\hat{\alpha}P(\hat{\alpha},\hat{\alpha}^*,t)]+\text{c.c.}\right]+4\frac{\partial^2P(\hat{\alpha},\hat{\alpha}^*,t)}{\partial\hat{\alpha}\partial\hat{\alpha}^*}\ , \tag{3}$$

where the variables denoted by carets are related to the ordinary, physical variables via scale factors:

$$\hat{t}=\left[\frac{\mathcal{A}\mathcal{B}}{8}\right]^{1/2}t,\quad \hat{\alpha}=\left[\frac{2\mathcal{B}}{\mathcal{A}}\right]^{1/4}\alpha\ ,$$

and (4)

$$a_1=\left[\frac{2\mathcal{A}}{\mathcal{B}}\right]^{1/2}.$$

The quantities $\mathcal{A}$ and $\mathcal{B}$ are the coefficients of linear gain (minus any distributed losses) and of gain saturation. Writing $\hat{\alpha}=\hat{r}e^{i\theta}$ and $\hat{\alpha}^*=\hat{r}e^{-i\theta}$, in which $\hat{r}^2$ and θ represent the scaled intensity and phase variables of the laser field, one finds that Eq. (3) transforms to the following Fokker-Planck equation in variables $\hat{r}$, θ, and $\hat{t}$:

$$\frac{\partial}{\partial\hat{t}}P(\hat{r},\theta,\hat{t})+\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}[(a_1-\hat{r}^2)\hat{r}^2P(\hat{r},\theta,\hat{t})]=\frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\left[\hat{r}\frac{\partial}{\partial\hat{r}}P(\hat{r},\theta,\hat{t})\right]+\frac{1}{\hat{r}^2}\frac{\partial^2}{\partial\theta^2}P(\hat{r},\theta,\hat{t})\ . \tag{5}$$

The usual phase-diffusion analysis[1-3] assumes that the amplitude $\hat{r}$ of the field is sharply defined at the classical steady-state value of $\sqrt{a_1}$, with $\mathcal{A}$ replaced by $(\mathcal{A}-\nu/2Q)$, so that the intrinsic laser linewidth is just the factor $1/a_1$ that multiplies the $(\partial^2/\partial\theta^2)P$ term. As we shall see, this is a good approximation only when the cavity is nearly perfect.

Equation (5) governs the evolution of the quasiprobability distribution P during a pass through the amplifying medium. The field suffers an irreversible loss of energy at the partially transmitting mirror as well as phase shifts at both mirrors. We now address how one may use a previously obtained unitary description of the beam splitter[7] to treat both these effects.

In the traveling-wave picture, the laser field may be looked upon as consisting of two traveling modes with the same frequency ν—one traveling from right to left (mirror 2 to mirror 1) and the other from left to right (mirror 1 to mirror 2), as shown in Fig. 1. The two mirrors merely serve to mix the two modes, which we label 1 and 2, respectively. Let $e^{i\phi_1}$ and $\rho e^{i\phi_2}$ be the complex reflection coefficients of the two mirrors for reflection from their inner surfaces. From our analysis of the beam splitter[7] of which the two mirrors may be considered as special cases, the following transformations of two-mode coherent states become evident:

$$|\alpha,0\rangle\langle\alpha,0|\ \xrightarrow{\text{mirror 1}}\ |0,\alpha e^{i\phi_1}\rangle\langle 0,\alpha e^{i\phi_1}|\ , \tag{6}$$

and

$$|0,\beta\rangle\langle 0,\beta|\ \xrightarrow{\text{mirror 2}}\ |\beta\rho e^{i\phi_2},\beta\tau e^{i\psi_2}\rangle\langle\beta\rho e^{i\phi_2},\beta\tau e^{i\psi_2}|\ , \tag{7}$$

in which ψ_2 is the phase shift of the field on transmission though mirror 2. The combined discrete transformation brought about by the mirrors acting in sequence is then

$$|\alpha,0\rangle\langle\alpha,0|\rightarrow|\alpha\rho e^{i(\phi_1+\phi_2)},\alpha\tau e^{i(\phi_1+\psi_2)}\rangle\times\langle\alpha\rho e^{i(\phi_1+\phi_2)},\alpha\tau e^{i(\phi_1+\psi_2)}|\ . \tag{8}$$

Having thus treated the effect of mirrors, we now go back to the single-traveling-mode picture by dropping the second mode from Eq. (8), since from the standpoint of

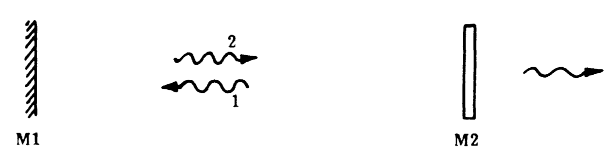

FIG. 1. A schematic diagram of a laser cavity with the two traveling-wave modes denoted by 1 and 2. Mirror $M1$ is assumed to be perfectly reflecting, while the output mirror $M2$ is partially transmitting. The complex reflection coefficients of their inner surfaces are $e^{i\phi_1}$ and $\rho e^{i\phi_2}$, respectively.

the cavity the second mode gets lost irreversibly at mirror 2:

$$|\alpha\rangle\langle\alpha| \to |\alpha\rho e^{i(\phi_1+\phi_2)}\rangle\langle\alpha\rho e^{i(\phi_1+\phi_2)}| \ . \tag{9}$$

Transformation (9) of the coherent state $|\alpha\rangle\langle\alpha|$ induces a transformation of the P distribution in Eq. (2), which follows from the scaling and shifting of the integration variables $\hat{r}$ and θ by ρ and $\phi_1+\phi_2$ in that equation:

$$P(\hat{r},\theta,\hat{t}) \to \frac{1}{\rho^2} P(\hat{r}/\rho, \theta-\phi_1-\phi_2, \hat{t}) \ . \tag{10}$$

The evolution of the field during a round trip thus consists of two stages: the amplification of the field in accordance with (5) as the time $\hat{t}$ changes by a round trip time $\hat{T}$ and the loss of the field in accordance with (10). Under these conditions, the distribution of the field amplitude attains a quasisteady state in the long-time limit.

III. QUASISTEADY STATE AND PHASE DIFFUSION

As we mentioned earlier, in the traveling-wave picture true steady state for the laser amplitude distribution is not possible, and one must be content with a quasisteady state. This is, of course, a formal artifact, and indeed if one were to allow for a spatial variation of the field envelope inside the laser, a true steady state would be obtained. What we wish to consider here first is the actual quasi-steady-state amplitude distribution and then how a sharp phase distribution imposed upon it diffuses in time due to random spontaneous-emission events.

If we write the θ-periodic $P(\hat{r},\theta,\hat{t})$ in terms of its Fourier decomposition

$$P(\hat{r},\theta,\hat{t}) = \sum_{m=-\infty}^{\infty} P_m(\hat{r},\hat{t}) e^{im\theta} \ , \tag{11}$$

then

$$P_0(\hat{r},\hat{t}) = \frac{1}{2\pi}\int_0^{2\pi} P(\hat{r},\theta,\hat{t})\, d\theta \tag{12}$$

is proportional to the amplitude distribution, while

$$P_{-1}(\hat{r},\hat{t}) = \frac{1}{2\pi}\int_0^{2\pi} P(\hat{r},\theta,\hat{t}) e^{i\theta} d\theta \tag{13}$$

describes the expectation value of the field at time $\hat{t}$. In other words, P_{-1} represents phase diffusion. (There is no ambiguity here concerning the definition of the phase operator, since it is $\hat{r}e^{i\theta}$, and not θ itself, for which the expectation value is being considered.) More precisely, at time $\hat{t}$

$$\langle \hat{r}e^{i\theta}\rangle = 2\pi \int_0^{\infty} P_{-1}(\hat{r},\hat{t})\hat{r}^2 d\hat{r} \ , \tag{14}$$

in which the usual normalization of the P distribution is assumed:

$$\int P(\hat{r},\theta,\hat{t})\hat{r}\, d\hat{r}\, d\theta = 2\pi \int P_0(\hat{r},\hat{t})\hat{r}\, d\hat{r} = 1 \ . \tag{15}$$

A. Quasi-steady-state amplitude distribution

In quasisteady state the amplitude distribution $P_0(\hat{r},\hat{t})$ must retrace itself precisely after each complete pass of the active medium and reflection from the mirrors. Thus, if $P_{\rm QSS}(\hat{r},0)$ represents the quasi-steady-state amplitude distribution at the beginning of a pass starting at mirror 2, then in accordance with (10), we must have the relation

$$\frac{1}{\rho^2} P_{\rm QSS}(\hat{r}/\rho,\hat{T}) = P_{\rm QSS}(\hat{r},0) \ . \tag{16}$$

If one were to assume a true steady state, then Eq. (5) could be solved exactly for such a distribution, say, $P_{\rm SS}(\hat{r})$ (Refs. 1–3):

$$P_{\rm SS}(\hat{r}) = N e^{-(\hat{r}^2-a_1)^2/4} \ , \tag{17}$$

which for large a_1 is a rather sharply peaked function of $\hat{r}^2$. For the quasi-steady-state problem at hand, one may look for a similar solution that approximately satisfies (5). Since the transformation (10) preserves the shape of the distribution function, it is clear that one must try a shape-invariant form for $P_{\rm QSS}(\hat{r},\hat{t})$. We use the ansatz

$$P_{\rm QSS}(\hat{r},\hat{t}) = N(\hat{t}) \exp\left\{-\frac{1}{4\bar{\sigma}(\hat{t})}[\hat{r}^2-\bar{a}_1(\hat{t})]^2\right\} \tag{18}$$

in which the time dependences of the center and width of the distribution are expressed in terms of those of the quantities η and Δ defined by the relations

$$\bar{a}_1(\hat{t}) \equiv a_1 + \eta(\hat{t}) \tag{19}$$

and

$$\bar{\sigma}(\hat{t}) \equiv 1 + \Delta(\hat{t}) \ . \tag{20}$$

In the analysis that follows, we only consider operation well above the threshold, $a_1 \gg 1$.

On substituting Eq. (18) into Eq. (5) and neglecting a term proportional to $(\hat{r}^2-\bar{a}_1)^3(1/\bar{\sigma}-1/\bar{\sigma}^2)$, we obtain the following ordinary differential equations for Δ and η:

$$\dot{\Delta} + 4(a_1+\eta)\Delta + 4\eta(1+\Delta) = 0 \tag{21}$$

and

$$\dot{\eta} + 2(a_1+\eta)\eta - 8\Delta = 0 \ . \tag{22}$$

[Neglecting such a term is a satisfactory procedure sufficiently near the peak of the distribution (18), but one that may not determine the width of (18) correctly. However, as we shall see later, for laser operation well above threshold, this width is of no consequence in the phase-diffusion problem.]

We assume, and we shall see so *a posteriori*, that the term -8Δ contributes negligibly in Eq. (22) and may be neglected. The resulting nonlinear equations are straightforwardly integrated in terms of the initial conditions, $\eta(0)=\eta_0$, $\Delta(0)=\Delta_0$:

$$\eta(\hat{t}) = a_1/[(a_1/\eta_0+1)e^{2a_1\hat{t}}-1] \tag{23}$$

and

$$\Delta(\hat{t}) = \frac{\Delta_0 - 2\beta_0^4 I(t)}{(e^{a_1\hat{t}} + 2\beta_0 \sinh a_1\hat{t})^4}, \tag{24}$$

where $\beta_0 \equiv \eta_0 / a_1$ and

$$I(t) \equiv (1+\beta_0^{-1})^3 (e^{2a_1\hat{t}} - 1) - 6(1+\beta_0^{-1})^2 a_1\hat{t} + 3(1+\beta_0^{-1})(1-e^{-2a_1\hat{t}}) - \tfrac{1}{2}(1-e^{-4a_1\hat{t}}) .$$

One may now enforce the quasi-steady-state condition (16) and determine η_0 and Δ by insisting that

$$\rho^4[1+\Delta(\hat{T})] = 1+\Delta_0$$

and (25)

$$\rho^2[a_1+\eta(\hat{T})] = a_1+\eta_0 .$$

In view of Eqs. (23) and (24), these conditions imply that

$$\eta_0 = -\frac{a_1(1-\rho^2)e^{2a_1\hat{T}}}{(e^{2a_1\hat{T}} - 1)} \tag{26}$$

and a similar result for Δ_0.

We have evaluated η_0, Δ_0, η, and Δ for a variety of values for the pump parameter a_1, the reflectivity ρ^2, and the linear-gain parameter $2a_1\hat{T}$. For $0.85 < \rho^2 < 1.0$ and $0.05 < 2a_1\hat{T} < 0.2$, the value of $8|\Delta|$ is typically much smaller than 1% and never exceeds 3% of the value of $2\eta(a_1+\eta)$ for $a_1 \geq 10$.

There is a third consistency condition, in addition to (25), involving the normalization $N(\hat{t})$ of relation (18) but that is automatically satisfied since the total probability $2\pi \int P_0(\hat{r},\hat{t})\hat{r}\, d\hat{r}$ is conserved under the action of Eqs. (5) and (10). This completes the evaluation of the quasi-steady-state distribution of amplitude fluctuations in a single-mode laser.

B. Phase fluctuations and the spectral linewidth

We now investigate how the phase information of a laser operating well above threshold in the quasisteady state degrades in time due to spontaneous emission. In this paper we assume that the phase and amplitude fluctuations are uncorrelated. This assumption is justified only for laser operation well above threshold (large a_1), but fails sufficiently near threshold.

Under this assumption, one may write the expectation value of the positive-frequency part of the field as

$$\langle a(\hat{t})\rangle = \langle \hat{r} e^{i\theta}\rangle \simeq \langle \hat{r}\rangle \langle e^{i\theta}\rangle .$$

However, since the amplitude distribution $P_{QSS}(\hat{r},\hat{t})$ retraces itself once every round trip, one may suppress $\langle \hat{r}\rangle$ from the time dependence of $\langle \alpha(\hat{t})\rangle$ on time scales much greater than the round-trip time $\hat{T}$. Since the laser phase diffuses significantly only over many round trips, this is not an unreasonable restriction. Thus $\langle a(\hat{t})\rangle \sim \langle e^{i\theta}\rangle$.

By multiplying Eq. (5) by $\hat{r}e^{i\theta}$ and integrating over $d\hat{r}$, one obtains during a round trip through the amplifying medium the following exact evolution equation for the laser field:

$$\frac{d}{dt}\langle e^{i\theta}\rangle = -\left\langle \frac{1}{\hat{r}^2} e^{i\theta}\right\rangle . \tag{27}$$

Once again, under the decorrelation approximation this reduces to a simple equation for $\langle e^{i\theta}\rangle$ that may be easily solved in an exponential form:

$$\langle e^{i\theta}\rangle_{\hat{t}} = \langle e^{i\theta}\rangle_0 \exp\left[-\int_0^{\hat{t}} \left\langle \frac{1}{\hat{r}^2}\right\rangle_{\hat{t}'} d\hat{t}'\right], \tag{28}$$

in which the subscript of the angular brackets represents the time at which the average of the enclosed quantity is evaluated. Equation (28) describes the decay of the average value of the field phasor as a function of time during each round trip. Thus, if one were to begin with an ensemble of lasers in the quasisteady state with a precisely determined phase θ_0 at the start of a pass so that $|\langle e^{i\theta}\rangle_0| = 1$, then at the end of that pass the phase of one laser relative to any other would have randomly changed, and the average $\langle e^{i\theta}\rangle$ would be less than 1 by the factor $\exp[-\int_0^{\hat{T}} \langle 1/\hat{r}^2\rangle_{t'} dt')$.

Reflection from the mirrors, on the other hand, given by transformation (10), implies the following transformation for $\langle e^{i\theta}\rangle$:

$$\langle e^{i\theta}\rangle \to \langle e^{i\theta}\rangle e^{i(\phi_1+\phi_2)} . \tag{29}$$

On combining Eqs. (28) and (29), it becomes clear that after N complete round trips, the expectation value of the field phasor will have changed from its initially sharp value of $e^{i\theta_0}$ to

$$\langle e^{i\theta}\rangle_{N\hat{T}} = e^{i\theta_0} \exp\left[-N\int_0^{\hat{T}} \left\langle \frac{1}{\hat{r}^2}\right\rangle_{t'} dt'\right] e^{iN(\phi_1+\phi_2)} . \tag{30}$$

In terms of the elapsed time $\hat{t}$, Eq. (30) may be written suggestively as

$$\langle e^{i\theta}\rangle_{\hat{t}} = e^{i\theta_0} e^{-(\hat{D}-i\Delta\hat{\nu})\hat{t}} \tag{31}$$

with the following definitions of the phase-diffusion constant $\hat{D}$ and the frequency shift $\Delta\hat{\nu}$:

$$\hat{D} \equiv \frac{1}{\hat{T}} \int_0^{\hat{T}} \left\langle \frac{1}{\hat{r}^2}\right\rangle_{t'} dt' \quad \text{and} \quad \Delta\hat{\nu} \equiv \frac{(\phi_1+\phi_2)}{\hat{T}} . \tag{32}$$

The quantity $\hat{D}$ measures the rate at which the phase information of a stable laser is lost in time. Since the validity of the quantum regression hypothesis[3] for our present problem implies that Eq. (31) also describes the time dependence of the field autocorrelation as a function of the delay $\hat{t}$, $\hat{D}$ also represents the linewidth of the laser spectrum. In particular it is the half width of the Lorentzian line shape of this spectrum. The quantity $\Delta\hat{\nu}$ is the frequency shift of the laser center frequency $\hat{\nu}$ due to mirror reflections.

From Eq. (18) it becomes clear that for $a_1 \gg 1$ (operation well above threshold), during a round trip

$$\left\langle \frac{1}{\hat{r}^2} \right\rangle_{t'} \simeq \frac{1}{\langle \hat{r}^2 \rangle_{t'}} \simeq \frac{1}{\bar{a}_1(t')} = \frac{1}{a_1} \left[1 - \frac{e^{-2a_1 t'}}{(a_1/\eta_0+1)} \right] , \quad (33)$$

in which use has also been made of Eqs. (19) and (23). On substituting this expression into (32), one obtains

$$\hat{D} = \frac{1}{a_1} \left[1 - \frac{(1-\rho^2)(e^{2a_1 \hat{T}} - 1)}{2a_1 \hat{T}(1-\rho^2 e^{2a_1 \hat{T}})} \right] . \quad (34)$$

In terms of the ordinary, unscaled variables, the linewidth D is

$$D = \hat{D} \left[\frac{\mathcal{A}\mathcal{B}}{8} \right]^{1/2} = \frac{\mathcal{B}}{4} \left[1 - \frac{(1-\rho^2)(e^{\mathcal{A}T} - 1)}{\mathcal{A}T(1-\rho^2 e^{\mathcal{A}T})} \right] . \quad (35)$$

A more familiar form of expression (35) is obtained when $\bar{n}$, the average number of photons inside the laser cavity, is introduced. This number is defined by the relation

$$\bar{n} \equiv \left[\frac{1}{\hat{T}} \int_0^{\hat{T}} [a_1 + \eta(t)] dt \right] \left[\frac{\mathcal{A}}{2\mathcal{B}} \right]^{1/2}$$

and, in view of Eqs. (23) and (26), may be shown to have the value

$$\bar{n} = \frac{\mathcal{A}}{\mathcal{B}} \left[1 + \frac{\ln \rho^2}{\mathcal{A}T} \right] . \quad (36)$$

Then from (35) it follows that

$$D = \frac{\mathcal{A}}{4\bar{n}} \left[1 + \frac{\ln \rho^2}{\mathcal{A}T} \right] \left[1 - \frac{(1-\rho^2)(e^{\mathcal{A}T} - 1)}{\mathcal{A}T(1-\rho^2 e^{\mathcal{A}T})} \right] , \quad (37)$$

in which the mirror-loss factor $(1-\rho^2)$ may be replaced by a nominal quality factor Q of the cavity, $\nu/Q = (1-\rho^2)/T$.

Equation (37) differs little from the usual Schawlow-Townes linewidth,[10] $\mathcal{A}/4\bar{n}$, in the limit of a perfect cavity, $Q \rightarrow \infty$ (or, equivalently, $\rho \rightarrow 1$). In fact, even for a moderately leaky cavity, since the product of the square brackets in (35) may be shown to differ from 1 only by terms of order $O(1-\rho^2)^2$, the Schawlow-Townes result is quite valid.

However, if one relates the linewidth to the observed laser power P_{obs}, which is the power that leaks out of the cavity, then the result (37) and the Schawlow-Townes expression differ by terms of order $O(1-\rho^2)$ and thus quite significantly for a poor-quality cavity. In terms of P_{obs}, the Schawlow-Townes linewidth obtained by previous workers may be written as

$$D_{\text{ST}} = \frac{\mathcal{A}\hbar\nu(1-\rho^2)}{4P_{\text{obs}}T} , \quad (38)$$

while our expression (35) for the linewidth for the same observed power is

$$D = D_{\text{ST}} \left[1 + \frac{(1-\rho^2)}{\rho^2} \left[\frac{1}{\mathcal{A}T} - \frac{1}{e^{\mathcal{A}T} - 1} \right] \right] . \quad (39)$$

To illustrate these differences, consider the following example of a low-quality cavity oscillating well above threshold: $\rho^2 = 0.90$, $\mathcal{A}T = 0.2$. For this cavity D exceeds D_{ST} by over 5% for a given output power level. On the other hand, for a given average power inside this laser cavity, expression (37) differs from the Schawlow-Townes linewidth $\mathcal{A}/4\bar{n}$ by less than 1%.

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