Theory of a quantum anharmonic oscillator

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The time evolution of a quantum single-quartic anharmonic oscillator is considered. The study is carried on in operational form by use of the raising and lowering operators of the oscillator. The equation of motion is solved by application of a new integration method based on iteration techniques, and the rigorous solutions that describe the time development of the displacement and momentum operators of the oscillator are obtained. These operators are presented as a Laplace transform and a subsequent inverse Laplace transform of suitable functionals. Finally, the results are employed to describe the time evolution of a quasiclassical anharmonic oscillator.

I. INTRODUCTION

For many years the analytical as well as numerical study of the one-dimensional anharmonic oscillator has been of considerable interest. This system was conceived as a simple and yet nontrivial model on which many of the basic field-theory axioms can be tested.

More recently, new problems in many branches of physics, ranging from molecular dynamics to solid-state physics and to optics, have intensified the interest in anharmonic models. In fact, it is generally realized that very interesting features of numerous systems are a consequence of the anharmonic nonlinear character of their oscillations. The analysis of nonlinear problems generally requires the study of systems with more degrees of freedom; nevertheless, the study of a single quantum anharmonic oscillator is a prerequisite for an understanding of more realistic and complex systems.

In this paper we discuss the time evolution of a single-quartic anharmonic oscillator characterized by the Hamiltonian

$$H=\frac{1}{2m}p^2+\frac{1}{2}\omega^2mq^2+\frac{\varepsilon}{4}q^4\ . \tag{1}$$

This Hamiltonian has been extensively studied in classical and quantum theories.

The exact solution of Newton's equation of motion for H has been written in terms of Jacobi elliptic functions and the period and action of this classical problem have been expressed in terms of generalized hypergeometric functions. Approximate solutions of Newton's equation have also been obtained. The solution provided with the classical Birkhoff-Gustavson normal-form approach appears to be in a form suitable for studying the relative quantum problem.[1–6]

A great many papers on approximate solutions of the Schrödinger equation for this anharmonic oscillator are available in literature. Energy levels and eigenfunctions have been written by using the Rayleigh-Schrödinger perturbation theory or, more recently, by applying the Krylov-Bogoliubov method of averaging and the thermodynamic perturbation theory. At present the mean values of powers of the displacement as well as the one of the energy in both quantum and JWKB theories have been obtained from a uniform algorithm by using the Rayleigh-Schrödinger method.[7–18]

The purpose of this paper is to study the time evolution of the quantum single-quartic anharmonic oscillator and to give a detailed treatment of the new mathematical techniques we have developed to analyze this nonlinear problem, since the method employed is general enough to be applied to many other nonlinear problems.

We will investigate the time development of the displacement and momentum operators of the oscillator. This study is made in operational form by introducing the raising and lowering operators of the oscillator. No particular representation is used and the final-operator functionals are presented in normal order by means of associate functions of the raising and lowering operators.

First, we look for a solution of the equation of motion of the displacement operator by applying iteration methods. Since the equation of motion is an operational nonlinear second-order differential equation, a solution expressed as a power series of time may be obtained only if we are to overcome the following difficulties.

(i) To find a recursive operational relation among the terms of the power series and, at the same time, to take into account of the expansion factor $[(n!)^{-1}]$ for the generic nth term of the series.

(ii) To order the displacement and momentum operators in each term.

We are able to find some integral operators that permit us to deal with these difficulties in different phases of the calculations, so that we can write a formal solution of the motion equation.

Then we introduce a nonlinear first-order differential equation suitably correlated with the equation of motion. A comparison between the solution of this equation and that of the motion equation permits us to condense the resultant power series into an integral of an analytical function. After further calculations we obtain a final expression in which the displacement operator appears in the shape of a Laplace transform and of a subsequent inverse Laplace transform of an operator functional. By using this result the time development of the momentum operator is easily obtained. These expressions of the displacement and momentum operators allow us to analyze the properties of the quantum anharmonic oscillator without applying usual perturbation techniques.

The feasibility of the present approach is illustrated by studying the time evolution of a quasiclassical anharmonic oscillator for which the initial value of the momentum operator is null. Conditions on the convergence are not discussed.

Section II is devoted to introducing some useful definitions of operational calculus. The exact solutions of the equations of motion for the displacement and momentum operators of the anharmonic oscillator is presented in Sec. III. Finally, in Sec. IV on the basis of these solutions the time evolution of a quasiclassical anharmonic oscillator is analyzed. The paper concludes with two Appendixes which contain problems of operational calculus necessary for our study of the anharmonic oscillator.

II. INTEGRAL OPERATORS

We will premise some mathematical considerations in order to simplify the following study of the anharmonic oscillator. We begin by noting that some theorems of operational calculus, necessary to our purposes, are enunciated in Appendix A.

Now we introduce the integral operator

$$\hat{I}(t;t_1)=\int_0^t dt_1 \ , \tag{2}$$

for which we let

$$\hat{I}^{\,n}(t;t_n)=\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \ . \tag{3}$$

Obviously, we have

$$\hat{I}^{\,n}(t;t_n)(t_n)^h=\frac{h!}{(h+n)!}t^{h+n} \tag{4}$$

and, in particular,

$$\hat{I}^{\,n}(t)f\equiv\hat{I}^{\,n}(t;t_n)f=\frac{1}{n!}t^n f \ , \tag{5}$$

provided the function f is chosen so that

$$\frac{df}{dt_i}=0 \ .$$

Then we consider the operators $\hat{\mathcal{J}}^{(+)}$ and $\hat{\mathcal{J}}^{(-)}$ defined by the following relations:

$$\hat{\mathcal{J}}^{(+)}(\eta;z)z^n=n!\eta^n \tag{6a}$$

and

$$\hat{\mathcal{J}}^{(-)}(z;\eta)\eta^n=\frac{1}{n!}z^n \ . \tag{6b}$$

We will give an explicit form to these operators, which from definitions (6) we see to be integral operators. They may be expressed, for instance, by using integral transforms. If

$$\hat{\mathcal{L}}(\eta;z)f(z)\equiv\int_0^\infty dz\,\exp(-\eta z)f(z)=\varphi(\eta)$$

is the Laplace transform of the function $f(z)$ and

$$\hat{\mathcal{L}}^{-1}(z;\eta)\varphi(\eta)\equiv\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}d\eta\,\exp(z\eta)\varphi(\eta)=f(z)$$

is the inverse Laplace transform of the function $\varphi(\eta)$, we have

$$\hat{\mathcal{L}}(\eta;z)z^n=n!\eta^{-n-1} \tag{7a}$$

and

$$\hat{\mathcal{L}}^{-1}(z;\eta)\eta^{-n-1}=\frac{1}{n!}z^n \ . \tag{7b}$$

Consequently, we can write

$$\hat{\mathcal{J}}^{(+)}(\eta;z)=\hat{\mathcal{L}}(\eta^{-1};z)\eta^{-1} \tag{8a}$$

and

$$\hat{\mathcal{J}}^{(-)}(z;\eta)=\hat{\mathcal{L}}^{-1}(z;\eta^{-1})\eta \ . \tag{8b}$$

It is trivial to note that the operators $\hat{\mathcal{J}}^{(+)}$ and $\hat{\mathcal{J}}^{(-)}$ can be also expressed in other integral forms.

III. METHOD OF SOLUTION

Let us briefly review the equation underlying the operator time evolution for a system described by Hamiltonian $\hat{\mathcal{H}}$. From elementary quantum mechanics the evolution of an operator $\hat{O}$ is found by writing the solution of the equation

$$\frac{d\hat{O}}{dt}=\frac{1}{i\hbar}[\hat{O},\hat{\mathcal{H}}] \tag{9}$$

as

$$\hat{O}(t)=\hat{O}+\frac{t}{i\hbar}[\hat{O},\hat{\mathcal{H}}]+\frac{1}{2!}\left[\frac{t}{i\hbar}\right]^2[[\hat{O},\hat{\mathcal{H}}]\hat{\mathcal{H}}]+\cdots \ , \tag{10}$$

provided operators $\hat{\mathcal{H}}$ and $\hat{O}$ have no explicit time dependence.

If the characteristic parameters of the anharmonic oscillator are expressed in suitable form, Hamiltonian (1) can be written as

$$\hat{H}=\beta\hat{p}^{\,2}+\gamma\hat{q}^{\,2}+\chi\hat{q}^{\,4} \ . \tag{11}$$

A similar Hamiltonian is also obtained using the Symanzik suggestion of rescaling the anharmonic oscillator variable q in Eq. (1).[19] This scaling process must be introduced since a power expansion of the energy eigenvalues is divergent for any $\varepsilon>0$, no matter how small.

According to Eq. (9) the displacement and momentum operators $\hat{q}(t)$ and $\hat{p}(t)$ of the anharmonic oscillator obey the equations

$$\frac{d\hat{q}}{dt}=2\beta\hat{p} \tag{12a}$$

and

$$\frac{d\hat{p}}{dt}=-2\hat{q}(2\chi\hat{q}^{\,2}+\gamma) \ . \tag{12b}$$

Consequently, the displacement operator $\hat{q}(t)$ satisfies the following equation of motion:

$$\frac{d^2\hat{q}}{dt^2}=g(\hat{q}) , \tag{13}$$

where the functional $g(\hat{q})$ is defined through the function

$$g(x)=-4\beta x(2\chi x^2+\gamma) . \tag{14}$$

In order to study the time evolution of the operator $\hat{q}$, we must solve Eq. (13). For this integration we will use a new mathematical technique which appears to be very convenient to study nonlinear equations.

We begin by writing a formal solution of the system of Eqs. (12). When Eq. (10) is applied, this solution is given by

$$\hat{q}(t)=\hat{Y}(t)+\hat{I}^2(t)g\{\hat{Y}(t)+\hat{I}^2(t)g\{\hat{Y}(t)+\hat{I}^2(t)g\{\cdots\}\}\} , \tag{15}$$

where the operator $\hat{I}(t)$ has been introduced by Eq. (2) and the operator $\hat{Y}(t)$ is defined as

$$\hat{Y}(t)=\hat{q}_0+2\beta\hat{I}(t)\hat{p}_0 , \tag{16}$$

with

$$\hat{q}_0=\hat{q}(t=0), \quad \hat{p}_0=\hat{p}(t=0) .$$

It is trivial to verify that Eq. (15) is solution of the equation of motion (13), since the second derivative of Eq. (15) gives

$$\frac{d^2\hat{q}}{dt^2}=g\{\hat{Y}(t)+\hat{I}^2(t)g\{\hat{Y}(t)+\hat{I}^2(t)g\{\cdots\}\}\} .$$

Now we must find the analytical function to which the series (15) converges. For this purpose we consider the simpler series

$$G_1(\tau;\xi)=\xi+\hat{I}(\tau)g\{\xi+\hat{I}(\tau)g\{\xi+\hat{I}(\tau)g\{\cdots\}\}\} , \tag{17}$$

where ξ and τ are independent c-number variables and the function g is defined in Eq. (14). Our first task is to find the function to which the series (17) converges. In Appendix B we see this function to be written in the following form:

$$G_1(\tau;\xi)=\exp[\tau g(\xi)\hat{D}(\xi)]\xi ,$$

where

$$\hat{D}(x)=\frac{d}{dx} .$$

In the same Appendix, on making use of this result we obtain that the auxiliary function G_1 may be expressed as

$$G_1(\tau;\xi)=\gamma^{1/2}\xi\exp(-4\beta\gamma\tau)\times\{2\chi\xi^2[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{-1/2} . \tag{18}$$

Then we modify the function G_1 suitably. If we let

$$G_2(t;\xi)=\xi+\hat{I}^2(t)g\{\xi+\hat{I}^2(t)g\{\xi+\hat{I}^2(t)g\{\cdots\}\}\} , \tag{19}$$

we see from Eqs. (6) and (17) that

$$G_2(t;\xi)=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta^2;\tau)G_1(\tau;\xi) . \tag{20}$$

In Eq. (19) we have used the function g and the operator $\hat{I}$, as in Eqs. (15) and (17), where we have defined $\hat{q}(t)$ and $G_1(\tau;\xi)$. If we compare Eq. (19) with Eq. (15), we have from theorem (A1) that

$$\hat{q}(t)=\exp[\hat{Y}(t)\hat{D}(\xi)]G_2(t;\xi)|_0 , \tag{21}$$

since the operator $\hat{Y}(t)$, introduced by Eq. (16), commutes with all terms of the series (19). We mean by subscript 0 that the function must be evaluated for $\xi=0$. When Eq. (20) is introduced into Eq. (21), we obtain

$$\hat{q}(t)=\exp[\hat{Y}(t)\hat{D}(\xi)]\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta^2;\tau)G_1(\tau;\xi)|_0 . \tag{22}$$

Therefore we have written the operator $\hat{q}(t)$ in a compact form.

Now we will free Eq. (22) from the operator $\hat{I}(t)$, present in this equation through the operator $\hat{Y}(t)$. With the help of the definition (6b) we see that

$$[\hat{Y}(t)]^n\hat{\mathcal{J}}^{(-)}(t;\eta)=\hat{\mathcal{J}}^{(-)}(t;\eta)[\hat{y}(\eta)]^n ,$$

where from Eq. (16) we have let

$$\hat{y}(\eta)=\hat{q}_0+2\beta\eta\hat{p}_0 . \tag{23}$$

Therefore we can write Eq. (22) as

$$\hat{q}(t)=\hat{\mathcal{J}}^{(-)}(t;\eta)\exp[\hat{y}(\eta)\hat{D}(\xi)]\hat{\mathcal{J}}^{(+)}(\eta^2;\tau)G_1(\tau;\xi)|_0 . \tag{24}$$

We point out that the operational part in $\hat{q}_0$ and $\hat{p}_0$ of Eq. (24) is completely held in the operator

$$\hat{E}=\exp[\hat{y}(\eta)\hat{D}(\xi)] . \tag{25}$$

It is useful to introduce the raising and lowering operators of the oscillator, $\hat{a}^\dagger$ and $\hat{a}$, which obey the commutation relation

$$[\hat{a},\hat{a}^\dagger]=1 .$$

On making use of the transformations for the displacement and momentum operators of the oscillator,[20]

$$\hat{q}_0=\sigma(\hat{a}^\dagger+\hat{a}) \tag{26a}$$

and

$$\hat{p}_0=i\theta(\hat{a}^\dagger-\hat{a}) , \tag{26b}$$

with

$$\sigma=\left[\frac{\hbar^2}{4}\frac{\beta}{\gamma}\right]^{1/4}, \quad \theta=\left[\frac{\hbar^2}{4}\frac{\gamma}{\beta}\right]^{1/4}$$

for the operator (23) we obtain the following expression:

$$\hat{y}(\eta)=(\sigma+i2\beta\theta\eta)\hat{a}^{\dagger}(\sigma-i2\beta\theta\eta)\hat{a} .$$

Thus for the exponential operator (25) we have

$$\hat{E}=\exp\{[V(\eta)\hat{a}^{\dagger}+V^{*}(\eta)\hat{a}]\hat{D}(\xi)\} , \tag{27}$$

where

$$V(\eta)=\sigma+i2\beta\theta\eta . \tag{28}$$

If we apply theorem (A3) to Eq. (27), for the exponential operator $\hat{E}$ we can write

$$\hat{E}=\exp[\hat{a}^{\dagger}V(\eta)\hat{D}(\xi)]\exp[\hat{a}V^{*}(\eta)\hat{D}(\xi)] \times\exp[\tfrac{1}{2}V^{*}(\eta)V(\eta)\hat{D}^{2}(\xi)] . \tag{29}$$

It is well known that an operator $\hat{f}(\hat{a},\hat{a}^{\dagger})$, a function of operators $\hat{a}$ and $\hat{a}^{\dagger}$, may be easily studied when it is written in normal order, i.e., when in each term of the series defining the functional $\hat{f}$ all annihilation operators appear to the right of all creation operators.[21] In order to express the functional $\hat{f}(\hat{a},\hat{a}^{\dagger})$ into normal order we can use the associate function $f(\alpha,\alpha^{*})$, which is obtained by the diagonal elements of $\hat{f}(\hat{a},\hat{a}^{\dagger})$ in the coherent state representation,

$$f(\alpha,\alpha^{*})=\langle\alpha|\hat{f}(\hat{a},\hat{a}^{\dagger})|\alpha\rangle , \tag{30}$$

with $\hat{a}|\alpha\rangle=\alpha|\alpha\rangle$. We find the normal form of $\hat{f}(\hat{a},\hat{a}^{\dagger})$ if we replace α by $\hat{a}$, α^{*} by $\hat{a}^{\dagger}$, and write each term in normal order in the function $f(\alpha,\alpha^{*})$.

When Eq. (29) is introduced into Eq. (24), the associate function of the displacement operator,

$$\langle\hat{q}(t)\rangle=\langle\alpha|\hat{q}(t)|\alpha\rangle ,$$

is readily obtained and is given by

$$\langle\hat{q}(t)\rangle=\hat{\mathcal{J}}^{(-)}(t;\eta)\exp\{[\alpha^{*}V(\eta)+\alpha V^{*}(\eta)]\hat{D}(\xi)\} \times\exp[\tfrac{1}{2}V^{*}(\eta)V(\eta)\hat{D}^{2}(\xi)]\hat{\mathcal{J}}^{(+)}(\eta^{2};\tau) \times G_{1}(\tau;\xi)|_{0} . \tag{31}$$

In Appendix A we study the operator $\exp[k\hat{D}^{2}(\xi)]$ and from Eq. (A5) we see for a function $f(\xi)$ that

$$\exp[k\hat{D}^{2}(\xi)]f(\xi) =(4\pi k)^{-1/2}\int_{-\infty}^{+\infty}d\bar{\xi}\exp\left[-\frac{(\xi-\bar{\xi})^{2}}{4k}\right]f(\bar{\xi}) .$$

If this property is applied to Eq. (31) the associate function of the displacement operator becomes

$$\langle\hat{q}(t)\rangle=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta^{2};\tau)[2\pi|V(\eta)|^{2}]^{-1/2} \times\int_{-\infty}^{+\infty}d\bar{\xi}\exp\left[-\frac{\bar{\xi}^{2}}{2|V(\eta)|^{2}}\right] \times G_{1}[\tau;\xi=\bar{\xi}+\alpha V^{*}(\eta)+\alpha^{*}V(\eta)] .$$

Then a straightforward calculation yields

$$\langle\hat{q}(t)\rangle=\pi^{-1/2}\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta^{2};\tau)\int_{-\infty}^{+\infty}d\bar{\xi}\exp(-\bar{\xi}^{2})G_{1}[\tau;\xi=\sqrt{2}\bar{\xi}|V(\eta)|+\alpha V^{*}(\eta)+\alpha^{*}V(\eta)] , \tag{32}$$

where the functions G_1 and V have been defined in Eqs. (18) and (28), respectively. This result gives the desired expression that describes the time evolution of the displacement operator of the anharmonic oscillator.

Now we will write the associate function of the operator $\hat{p}(t)$,

$$\langle\hat{p}(t)\rangle=\langle\alpha|\hat{p}(t)|\alpha\rangle .$$

From Eq. (12a) we see that

$$\langle\hat{p}(t)\rangle=(2\beta)^{-1}\frac{d}{dt}\langle\hat{q}(t)\rangle . \tag{33}$$

Since from the definition (6b) we have

$$\frac{d}{dt}[\hat{\mathcal{J}}^{(-)}(t;\eta)f(\eta)]=\hat{\mathcal{J}}^{(-)}(t;\eta)\{\eta^{-1}[f(\eta)-f(0)]\} ,$$

by using Eq. (32) we can write the following expression for the associate function of the momentum operator:

$$\langle\hat{p}(t)\rangle=\pi^{-1/2}(2\beta)^{-1}\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta^{2};\tau) \times\int_{-\infty}^{+\infty}d\bar{\xi}\exp(-\bar{\xi}^{2})\eta^{-1}\{G_{1}[\tau;\xi=\sqrt{2}\bar{\xi}|V(\eta)|+\alpha V^{*}(\eta)+\alpha^{*}(\eta)]-G_{1}[\tau;\xi=\sqrt{2}\bar{\xi}\sigma+(\alpha+\alpha^{*})\sigma]\} . \tag{34}$$

For the sake of completeness we will rewrite Eqs. (32) and (34) in explicit form. By making use of Eqs. (8) we find that the associate functions $\langle\hat{q}(t)\rangle$ and $\langle\hat{p}(t)\rangle$ are given by

$$\langle\hat{q}(t)\rangle=\pi^{-1/2}\hat{\mathcal{L}}^{-1}(t;\eta^{-1})\hat{\mathcal{L}}(\eta^{-2};\tau)\gamma^{1/2}\eta^{-1} \times\int_{-\infty}^{+\infty}d\bar{\xi}\exp(-\bar{\xi}^{2})\exp(-4\beta\gamma\tau)\Xi(\bar{\xi};\eta)\{2\chi[\Xi(\bar{\xi};\eta)]^{2}[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{-1/2} \tag{35}$$

and

$$\langle \hat{p}(t)\rangle = \pi^{-1/2}(2\beta)^{-1}\hat{\mathcal{L}}^{-1}(t;\eta^{-1})\hat{\mathcal{L}}(\eta^{-2};\tau)\gamma^{1/2}\eta^{-2} \times \int_{-\infty}^{+\infty} d\xi \exp(-\xi^2)\exp(-4\beta\gamma\tau)(\Xi(\xi;\eta)\{2\chi[\Xi(\xi;\eta)]^2[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{1/2} - \Xi_0(\xi)\{2\chi[\Xi_0(\xi)]^2[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{-1/2}) , \quad (36)$$

with

$$\Xi(\xi;\eta) = \sqrt{2}\xi\,|V(\eta)| + \alpha V^*(\eta) + \alpha^* V(\eta)$$

and

$$\Xi_0(\xi) \equiv \Xi(\xi;\eta=0) = (\sqrt{2}\xi + \alpha + \alpha^*)\sigma .$$

The displacement and momentum operators of the oscillator can be easily obtained by replacing the variables α and α^* with the operators $\hat{a}$ and $\hat{a}^\dagger$ in the associate functions, as has been previously indicated.

We point out that the expressions (35) and (36) only contain a Laplace transform, an inverse Laplace transform, and an integration in ξ, which rises in the ordering operation of the annihilation and creation operators. Therefore these expressions can be applied to study the properties of the anharmonic oscillator or to calculate handy approximate values of the quantities which describe particular anharmonic systems. We note that the expression which describe the displacement and momentum of the classical quartic anharmonic oscillator are immediately obtained from Eqs. (35) and (36) when in these equations the function $[\exp(-\xi^2)]$ is replaced by $[\sqrt{\pi}\delta(\xi)]$.

It may be of some interest to deduce from Eq. (35) the well-known expression of the associate function for the displacement operator of the harmonic oscillator $\langle \hat{q}(t)\rangle_{\text{harm}}$. In this case we must assume $\chi=0$, so that Eq. (35) becomes

$$\langle \hat{q}(t)\rangle_{\text{harm}} = \pi^{-1/2}\hat{\mathcal{L}}^{-1}(t;\eta^{-1})\hat{\mathcal{L}}(\eta^{-2};\tau)\eta^{-1} \times \int_{-\infty}^{+\infty} d\xi \exp(-\xi^2)\Xi(\xi;\eta) . \quad (37)$$

When we perform the integrations, from Eq. (37) we find

$$\langle \hat{q}(t)\rangle_{\text{harm}} = (\alpha+\alpha^*)\sigma\cos[2(\beta\gamma)^{1/2}t] + i(\alpha^*-\alpha)\left[\frac{\beta}{\gamma}\right]^{1/2}\sin[2(\beta\gamma)^{1/2}t] .$$

Consequently, on making use of Eqs. (26) we can write the following relation:

$$\hat{q}_{\text{harm}}(t) = \hat{q}_0\cos[2(\beta\gamma)^{1/2}t] + \left[\frac{\beta}{\gamma}\right]^{1/2}\hat{p}_0\sin[2(\beta\gamma)^{1/2}t] . \quad (38)$$

Thus the expression that describes the displacement operator of the harmonic oscillator is obtained.

IV. APPLICATION

We will verify the feasibility of the present method by applying the results to a simple case. Therefore we study the time evolution of an anharmonic oscillator which at time $t=0$ is in a minimum-uncertainty state described by a coherent state $|\psi\rangle$ with $\hat{a}|\psi\rangle = \psi|\psi\rangle$. Moreover, in order to simplify the following calculations the amplitude ψ is assumed to be real and such that $\psi^2 \gg 1$. Therefore the oscillator is initially in a quasiclassical state for which the mean value of the momentum operator is null and the one of the displacement operator is $(2\sigma\psi)$.

We begin by considering the mean value of the displacement operator

$$\langle \hat{q}(t)\rangle_\psi = \langle \psi|\hat{q}(t)|\psi\rangle .$$

To evaluate $\langle \hat{q}(t)\rangle_\psi$ we must perform the integrations of Eq. (32) where we let $\alpha=\alpha^*=\psi$. When we expand the function $G_1(\tau;\xi)$ as a power series of ξ, we find that

$$G_1(\tau;\xi) = \left[\frac{\gamma}{2\chi}\right]^{1/2} \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} c_{nl}\exp[-4(2n+1)\beta\gamma\tau]\left[\left[\frac{2\chi}{\gamma}\right]^{1/2}\xi\right]^{2(n+l)+1} , \quad (39)$$

where

$$c_{nl} = \frac{(-1)^l(2n+2l)!}{4^{n+l}n!\,l!\,(n+l)!} .$$

Then, for $G_1(\tau;\xi)$ we write the relation

$$\hat{\mathcal{J}}^{(+)}(\eta^2;\tau)G_1(\tau;\xi) = \hat{\mathcal{J}}^{(+)}(\eta;\mu)\bar{G}_1(\mu;\xi) ,$$

which is satisfied by the function

$$\bar{G}_1(\mu;\xi) = \left[\frac{\gamma}{2\chi}\right]^{1/2} \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} c_{nl}\cos(\omega_n\mu)\left[\left[\frac{2\chi}{\gamma}\right]^{1/2}\xi\right]^{2(n+l)+1} , \quad (40)$$

with

$$\omega_n = 2[(2n+1)\beta\gamma]^{1/2} .$$

Now we consider the integral

$$Z_h(\eta)=\pi^{-1/2}\int_{-\infty}^{+\infty}d\xi\exp(-\xi^2)[\sqrt{2}\xi\,|V(\eta)|+\alpha V^*(\eta)+\alpha^*V(\eta)]^{2h+1}\ , \tag{41}$$

where the function $V(\eta)$ has been defined in Eq. (28). For our particular oscillator the integral (41) becomes

$$Z_h(\eta)=(2\sigma\psi)^{2h+1}\sum_{s=0}^{h}\frac{(2h+1)!}{[2(h-s)+1]!s!}\left[\frac{1}{8\psi^2}\right]^s\left[1+\frac{4\beta^2\theta^2}{\sigma^2}\eta^2\right]^s\ , \tag{42}$$

since the amplitude ψ is real. If we introduce Eqs. (39) and (42) into Eq. (32) we find that

$$\langle\hat{q}(t)\rangle_\psi=\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\mu)\sum_{n,l}c_{nl}\cos(\omega_n\mu)\left[\frac{2\chi}{\gamma}\right]^{n+l}Z_{n+l}(\eta)\ . \tag{43}$$

From the definitions (6) we see that Eq. (43) can be written after a straightforward calculation as follows:

$$\langle\hat{q}(t)\rangle_\psi=\left[\frac{\gamma}{2\chi}\right]^{1/2}\sum_{n,l}c_{nl}\left[2\left[\frac{2\chi}{\gamma}\right]^{1/2}\sigma\psi\right]^{2(n+l)+1}\times\sum_{s=0}^{n+h}\frac{1}{s!}\left[\frac{1}{8\psi^2}\right]^s[\Delta_{n,l,s}^{(0)}(\varepsilon=1)\cos(\omega_n t)+\Delta_{n,l,s}^{(1)}(t;\varepsilon=1)]\ , \tag{44}$$

where

$$\Delta_{n,l,s}^{(0)}(\varepsilon)=(1-\Omega_n^{-2})^s\hat{D}^{2s}(\varepsilon)\varepsilon^{2(n+l)+1}$$

and

$$\Delta_{n,l,s}^{(1)}(t;\varepsilon)=\sum_{g=0}^{s}(-1)^{g+1}\frac{s!}{(s-g)!g!}\Omega_n^{-2g}\times C_g(\omega_n t)\hat{D}^{2s}(\varepsilon)\varepsilon^{2(n+l)+1}\ ,$$

with

$$C_g(\omega_n t)=\sum_{v=0}^{g-1}\frac{[-(\omega_n t)^2]^v}{(2v)!}$$

and

$$\Omega_n=\sigma\omega_n/2\beta\theta\ .$$

Then in order to give a compact form to this result we introduce some approximations. We recall that the oscillator is initially in a quasiclassical state, so that it is $\psi^2\gg1$. This condition allows us to neglect the terms of superior order with respect to $(8\psi^2)^{-1}$ in Eq. (44). Thus after a little algebra we find that the mean value $\langle\hat{q}(t)\rangle_\psi$ can be expressed as

$$\langle\hat{q}(t)\rangle_\psi\approx Q_\psi^{(c)}(t)+Q_\psi^{(q)}(t)\ , \tag{45}$$

where

$$Q_\psi^{(c)}(t)=\left[\frac{\gamma}{2\chi}\right]^{1/2}\sum_{n=0}^{\infty}c_n'\left[\frac{\Psi^2}{1+\Psi^2}\right]^{n+1/2}\cos(\omega_n t)$$

and

$$Q_\psi^{(q)}(t)=\left[\frac{\gamma}{2\chi}\right]^{1/2}\sum_{n=0}^{\infty}c_n'\Delta_n^{(A)}(8\Omega_n^2\psi^2)^{-1}\times\left[\frac{\Psi^2}{1+\Psi^2}\right]^{n+1/2}\times[1-\cos(\omega_n t)]\ ,$$

with

$$\Delta_n^{(A)}=(2n+1)(1+\Psi^2)^{-1}[(2n+3)(1+\Psi^2)^{-1}-3]\ .$$

Here it is

$$c_n'=\frac{(2n)!}{4^n(n!)^2},\quad \Psi=2\left[\frac{2\chi}{\gamma}\right]^{1/2}\sigma\psi\ ,$$

and for the convergence $\Psi^2<1$. We point out that in Eq. (45) the term $Q_\psi^{(c)}(t)$ represents the exact mean value of the displacement for a completely defined classical anharmonic oscillator whose initial displacement is $(2\sigma\psi)$ and the momentum is null, whereas the term $Q_\psi^{(q)}(t)$ is given by the contributions of the commutation rule between $\hat{a}$ and $\hat{a}^\dagger$, which have been considered in the present approximations. The mean value of the momentum operator can be directly obtained by applying Eq. (33) to Eq. (45).

For the sake of completeness we will evaluate the mean values of other more complex quantities for the actual oscillator. As an example, we briefly analyze the mean value of $(\hat{q}^2)$,

$$\langle\hat{q}^2(t)\rangle_\psi=\langle\psi|\hat{q}^2(t)|\psi\rangle\ .$$

For our purposes it is useful to remember the following property for the associate functions defined in Eq. (30). If $f_1(\alpha,\alpha^*)$ and $f_2(\alpha,\alpha^*)$ are the associate functions of the functionals $f_1(\hat{a},\hat{a}^\dagger)$ and $f_2(\hat{a},\hat{a}^\dagger)$, respectively, the associate function $f_3(\alpha,\alpha^*)$ of the functional

$$f_3(\hat{a},\hat{a}^\dagger)=f_1(\hat{a},\hat{a}^\dagger)f_2(\hat{a},\hat{a}^\dagger)$$

is given by

$$f_3(\alpha,\alpha^*)=f_1[\alpha+\hat{D}(\alpha^*),\alpha^*]f_2(\alpha,\alpha^*)\ , \tag{46}$$

where the derivatives $\hat{D}(\alpha^*)$ act on the function $f_2(\alpha,\alpha^*)$ only.[21] Since our formulas must be specialized for $\psi^2\gg1$, we can approximate Eq. (46) as follows:

$$f_3(\alpha,\alpha^*)\approx f_1(\alpha,\alpha^*)f_2(\alpha,\alpha^*)+\frac{\partial f_1(\alpha,\alpha^*)}{\partial\alpha}\frac{\partial f_2(\alpha,\alpha^*)}{\partial\alpha^*} .$$

Consequently, for the actual oscillator we obtain that

$$\langle\hat{q}^{\,2}(t)\rangle_\psi\approx(\langle\hat{q}(t)\rangle_\psi)^2+\langle\hat{D}(\alpha)\hat{q}(t)\rangle_\psi\langle\hat{D}(\alpha^*)\hat{q}(t)\rangle_\psi , \quad (47)$$

where, as written in Eq. (43) for $\langle\hat{q}(t)\rangle_\psi$, it is

$$\langle\hat{D}(\alpha^*)\hat{q}(t)\rangle_\psi\approx\hat{\mathcal{J}}^{(-)}(t;\eta)\hat{\mathcal{J}}^{(+)}(\eta;\mu)\times\sum_{n,l}c_{nl}\cos(\omega_n t)\left[\frac{2\chi}{\gamma}\right]^{n+l}\times Z^{(R)}_{n+l}(\eta) ,$$

with

$$Z^{(R)}_h(\eta)=(2h+1)(2\psi)^{-1}(2\sigma\psi)^{2h+1}\left[1+i\frac{2\beta\theta}{\sigma}\eta\right] .$$

After a straightforward calculation we find that

$$\langle\hat{D}(\alpha^*)\hat{q}(t)\rangle_\psi=[\langle\hat{D}(\alpha)\hat{q}(t)\rangle_\psi]^*=\left[\frac{\gamma}{2\chi}\right]^{1/2}(2\psi)^{-1}\sum_{n=0}^{\infty}c_n'(2n+1)(1+\Psi^2)^{-1}\left[\frac{\Psi^2}{1+\Psi^2}\right]^{n+1/2}\times[\cos(\omega_n t)+i\Omega_n^{-1}\sin(\omega_n t)] . \quad (48)$$

When Eqs. (45) and (48) are introduced into Eq. (47), the desired value for $\langle\hat{q}^{\,2}(t)\rangle_\psi$ is found. We note that in Eq. (47) the term

$$[\langle\hat{D}(\alpha)\hat{q}(t)\rangle_\psi\langle\hat{D}(\alpha^*)\hat{q}(t)\rangle_\psi] ,$$

as the term $Q^{(q)}_\psi(t)$ of Eq. (45), is given by the contributions of the commutation rule for $\hat{a}$ and $\hat{a}^\dagger$. Therefore for a completely defined classical oscillator Eq. (47) becomes

$$\langle\hat{q}^{\,2}(t)\rangle^{(c)}_\psi=[Q^{(c)}_\psi(t)]^2 .$$

Since the preceding method of calculation can be applied to every quantity, the complete description of the time evolution of our particular anharmonic oscillator can be obtained.

V. CONCLUSIONS

We have studied the time evolution of a single-quartic anharmonic oscillator. The equation of motion for the displacement and momentum operators of the oscillator has been solved by using iteration methods. The solutions have been presented as a Laplace transform and a subsequent inverse Laplace transform of suitable functionals of raising and lowering operators of the oscillator. These solutions permit us to analyze the properties of the anharmonic oscillator and to calculate handy approximate values of quantities which describe particular anharmonic systems. In fact, the Laplace transform and the inverse Laplace transform can be evaluated by using the convolution law or one of the many approximate methods of calculation reported in literature. So the operators that describe the anharmonic oscillator can be expressed in forms which facilitate the study of particular problems and give the possibility of analyzing anharmonic oscillators without applying the usual perturbation techniques.

In order to illustrate the feasibility of the present approach we have analyzed the time evolution of a quasiclassical anharmonic oscillator and in this particular case we have been able to write the quantities of the oscillator in analytical form. The necessary integrations have been performed by using a suitable power expansion. Finally, we speculate that many other nonlinear processes can be studied by using a similar approach to that adopted in this paper.

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APPENDIX A: SOME THEOREMS OF OPERATIONAL CALCULUS

In this paper we make use of some theorems of operational calculus. For convenience we will list the statements of these theorems.

Theorem 1. If $f(\xi)$ is an arbitrary function and $dk/d\xi=0$, it is found that

$$\exp[k\hat{D}(\xi)]f(\xi)=f(\xi+k) . \quad (A1)$$

Theorem 2. If $\hat{A}$ and $\hat{B}$ are two fixed noncommuting operators, ξ a parameter, and f an arbitrary function, it then follows that

$$\exp(\xi\hat{A})f(\hat{B})\exp(-\xi\hat{A})=f[\exp(\xi\hat{A})\hat{B}\exp(-\xi\hat{A})] . \quad (A2)$$

Theorem 3. If $\hat{A}$ and $\hat{B}$ are two fixed noncommuting operators and ξ a parameter, then we have

$$\exp(\xi\hat{A})\hat{B}\exp(-\xi\hat{A})=\hat{B}+\xi[\hat{A},\hat{B}]+\frac{\xi^2}{2}[\hat{A},[\hat{A},\hat{B}]]+\cdots . \quad (A3)$$

Theorem 4. If $\hat{A}$ and $\hat{B}$ are two noncommuting operators that satisfy the conditions

$$[\hat{A},[\hat{A},\hat{B}]]=[\hat{B},[\hat{A},\hat{B}]]=0\ ,$$

we have the following relation:

$$\exp(\hat{A}+\hat{B})=\exp(\hat{A})\exp(\hat{B})\exp(-\tfrac{1}{2}[\hat{A},\hat{B}])\ , \qquad (A4)$$

which is a special case of the well-known Baker-Hausdorff theorem.

Theorem 5. If $f(\xi)$ is an arbitrary function and $dk/d\xi=0$, it is seen that

$$\exp[k\hat{D}^2(\xi)]f(\xi) = (4\pi k)^{-1/2}\int_{-\infty}^{+\infty} d\bar{\xi}\exp\left[-\frac{(\xi-\bar{\xi})^2}{4k}\right]f(\bar{\xi})\ . \qquad (A5)$$

The proofs of the first four theorems can be found in literature, whereas the proof of the last theorem is given as follows.[22] If we put

$$\varphi(\xi)=\exp[k\hat{D}^2(\xi)]f(\xi)$$

and

$$\Phi(x)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty} d\xi\exp(ix\xi)f(\xi)\ ,$$

we can write that

$$\varphi(\xi)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty} dx\ \exp(-ix\xi)\exp(-kx^2)\Phi(x)\ . \qquad (A6)$$

Since

$$\exp(-kx^2)=(4\pi k)^{-1/2}\int_{-\infty}^{+\infty} d\xi\exp(ix\xi)\times\exp\left[-\frac{\xi^2}{4k}\right]\ ,$$

from the convolution law of Fourier transforms we see that Eq. (A6) becomes

$$\varphi(\xi)=(4\pi k)^{-1/2}\int_{-\infty}^{+\infty} d\bar{\xi}\exp\left[-\frac{(\xi-\bar{\xi})^2}{4k}\right]f(\bar{\xi})\ .$$

Therefore the effect of the exponential operator $\exp[k\hat{D}^2(\xi)]$ on a function $f(\xi)$ can be expressed by Eq. (A5).

APPENDIX B: AUXILIARY FUNCTION

In this appendix we study the function $G_1(\tau;\xi)$, defined in Eq. (17) as

$$G_1(\tau;\xi)=\xi+\hat{I}(\tau)g\{\xi+\hat{I}(\tau)g\{\xi+\hat{I}(\tau)g\{\cdots\}\}\}\ , \qquad (B1)$$

where ξ and τ are independent c-number variables, the operator $\hat{I}(\tau)$ is given by Eq. (2), and the function g is expressed as

$$g(x)=-4\beta x(2\chi x^2+\gamma)\ .$$

Let us write the function (B1) in a more useful form. Since it is

$$\frac{\partial G_1}{\partial\tau}=g\{\xi+\hat{I}(\tau)g\{\xi+\hat{I}(\tau)g\{\cdots\}\}\}\ ,$$

the function G_1 obeys the following equation:

$$\frac{\partial G_1}{\partial\tau}=g(G_1)\ , \qquad (B2)$$

with the initial condition

$$G_1(\tau=0;\xi)=\xi\ .$$

Consequently, we can write

$$G_1(\tau;\xi)=\exp[\tau g(\xi)\hat{D}(\xi)]\xi\ , \qquad (B3)$$

where

$$\hat{D}(\xi)=\frac{d}{d\xi}\ .$$

It may be of interest to verify the assertion. By usual rules of differentiation we have

$$\frac{\partial G_1}{\partial\tau}=\exp[\tau g(\xi)\hat{D}(\xi)]g(\xi)\ .$$

Then from theorem (A2) we obtain

$$\frac{\partial G_1}{\partial\tau}=g\{\exp[\tau g(\xi)\hat{D}(\xi)]\xi\exp[-\tau g(\xi)\hat{D}(\xi)]\}\ .$$

Since Eq. (B3) can be written as

$$G_1(\tau;\xi)=\exp[\tau g(\xi)\hat{D}(\xi)]\xi\exp[-\tau g(\xi)\hat{D}(\xi)]\ ,$$

it follows that the function G_1 really satisfies Eq. (B2).

Now we will evaluate the function G_1. If we put

$$\hat{D}(\mu)=g(\xi)\hat{D}(\xi)\ ,$$

μ can be written as function of ξ in the form

$$\mu=-\frac{1}{8\beta\gamma}\ln[2\chi\xi^2(2\chi\xi^2+\gamma)^{-1}]\ .$$

We therefore have

$$\xi=\left[\frac{\gamma}{2\chi}\right]^{1/2}\exp(-4\beta\gamma\mu)[1-\exp(-8\beta\gamma\mu)]^{-1/2} \qquad (B4)$$

and, when we express $G_1(\tau;\xi)$ in terms of μ, we find

$$G_1(\tau;\mu)=\exp[\tau\hat{D}(\mu)]\left[\left[\frac{\gamma}{2\chi}\right]^{1/2}\exp(-4\beta\gamma\mu)\times[1-\exp(-8\beta\gamma\mu)]^{-1/2}\right]\ .$$

On making use of theorem (A1) we obtain

$$G_1(\tau;\mu)=\left[\frac{\gamma}{2\chi}\right]^{1/2}\exp[-4\beta\gamma(\mu+\tau)]\times\{1-\exp[-8\beta\gamma(\mu+\tau)]\}^{-1/2}\ . \qquad (B5)$$

Finally, on substituting Eq. (B4) into Eq. (B5) we can write

$$G_1(\tau;\xi)=\gamma^{1/2}\xi\exp(-4\beta\gamma\tau)\times\{2\chi\xi^2[1-\exp(-8\beta\gamma\tau)]+\gamma\}^{-1/2}\ .$$

This is the desired expression of the analytical function to which the series (B1) converges.

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