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Phase and amplitude dynamics in the laser Lorenz model

H. Zeghlache and Paul Mandel
Optique Nonlinéaire Théorique, Université Libre de Bruxelles, Campus Plaine CP 231, 1050 Bruxelles, Belgium

N. B. Abraham and C. O. Weiss
Laboratorium 4.42, Physikalisch-Technische Bundesanstalt, D-3300 Braunschweig, Federal Republic of Germany
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Phase as well as amplitude dynamics in the complex laser Lorenz model have been studied, in contrast to previous studies limited to intensity (i.e., amplitude only). The topological nature of the chaotic intensity is not modified by a small detuning, although the electric field phase is dramatically altered, showing a steady drift with respect to the frequency of the corresponding steady-state solution in addition to small-scale modulation. For larger detunings at the same pump parameter, where one finds periodic intensity pulsations, there is a similar average drift in the phase leading to spuriously quasiperiodic representations of the attractor. After removal of this average drift (which amounts to a frequency shift) all dynamical variables, including the phase, have the same periodicity.

Recent experimental studies of Lorenz-like pulsations in a FIR single-mode laser have included measurements of the complex amplitude of the laser field, permitting analysis of the dynamics of the phase in addition to that of the intensity.[1] This has stimulated our renewed studies of the laser Lorenz model (single-mode laser with detuning) with attention to the previously neglected phase dynamics.

The equations for the detuned laser are[2]

$$E' = -\kappa(1+i\delta)E - \kappa AP\,, \tag{1a}$$

$$P' = -\gamma_\perp(1+i\Delta)P - \gamma_\perp ED\,, \tag{1b}$$

$$D' = -\gamma_\parallel(D-1) + (\gamma_\parallel/2)(P^*E + E^*P)\,, \tag{1c}$$

where E, P, and D are the slowly varying reduced complex amplitudes of the electric field, atomic polarization, and inversion, respectively, with corresponding decay rates κ, $\gamma_\perp$, and $\gamma_\parallel$. The frequency of the carrier wave ω_L of the electromagnetic field is detuned from the nearest cavity resonance ω_c and from the atomic resonance ω_a by $\kappa\delta = (\omega_c - \omega_L)$ and $\gamma_\perp\Delta = (\omega_L - \omega_a)$, respectively.

As shown by Haken,[3] for the case of perfect tuning ($\delta = \Delta = 0$) these equations are isomorphic to the Lorenz equations[4] developed to describe convective hydrodynamics. In the laser case, the control parameter is A which is identified with the Rayleigh number (typically represented by r) in the hydrodynamic case. An extension of the original Lorenz equations to complex variables was proposed by Gibbon and McGuinness.[5] This extension was motivated by an analysis of the baroclinic instability.[6] In our notation they considered complexifying the decay rates for E and P and the pump parameter A. As the complex form of A has no physical meaning for the laser, their results do not apply to Eqs. (1).

Equations (1) have a trivial steady-state solution $E = 0$ which is stable for $A < 1 + [(\omega_c - \omega_a)/(\kappa + \gamma_\perp)]^2$. The nontrivial steady-state solution has an intensity $I = |E|^2 = A - 1 - \Delta^2$ and a dispersion relation for ω_L given by $\delta = \Delta$. This solution is stable in the domain $0 < I < I_c$. The threshold intensity, $I_c(\Delta)$, is the solution of a quadratic equation. Its existence requires that $\kappa > \gamma_\perp + \gamma_\parallel$. In the limit $\kappa \gg \gamma_\perp + \gamma_\parallel$, it has been shown in Ref. 7 that this quartic reduces to

$$3I_c^2 + 2\kappa^2(1 - 3\Delta^2)I_c - 2\kappa^3(1+\Delta^2)^2 = 0\,. \tag{2}$$

The nontrivial solution is a plane wave of constant intensity and constant frequency with the "steady-state" electric field given by

$$\sqrt{A - 1 - \Delta^2}\exp(-i\omega_L t)\,. \tag{3}$$

As it is defined in the reference frame rotating at ω_L, E is a constant; in any other reference frame, the electric field becomes a simple harmonic function with a frequency given by the frequency shift from ω_L. Therefore, the stability analysis of the nontrivial "steady-state solutions" of Ref. 7 is identical to the stability analysis of the periodic solution of Ref. 8, as the two solutions only differ in the choice of reference frame. We shall consider decay rates such that for small detunings (and in resonance) the instability at I_c is a subcritical Hopf bifurcation leading to chaos; for larger detunings, the instability is a supercritical Hopf bifurcation to stable periodic solutions.[7]

The phase dynamics in the resonant case is limited to changes of sign of the real variable E and, hence, the phase corresponding to the well-known spirals in the E-D plane is restricted to alternating jumps of π radians as shown in Fig. 1(b) (see also, Ref. 9 for a related observation). We present the corresponding amplitude dynamics in Fig. 1(a) by a plot of $|E|^2$ vs t as this representation is physically more relevant in the laser case.

For small detunings the intensity dynamics remain essentially unchanged as seen by comparing Figs. 1(c) and 1(a). In contrast, the phase as displayed in Fig. 1(d) has three new characteristics. (1) The "transitions" between

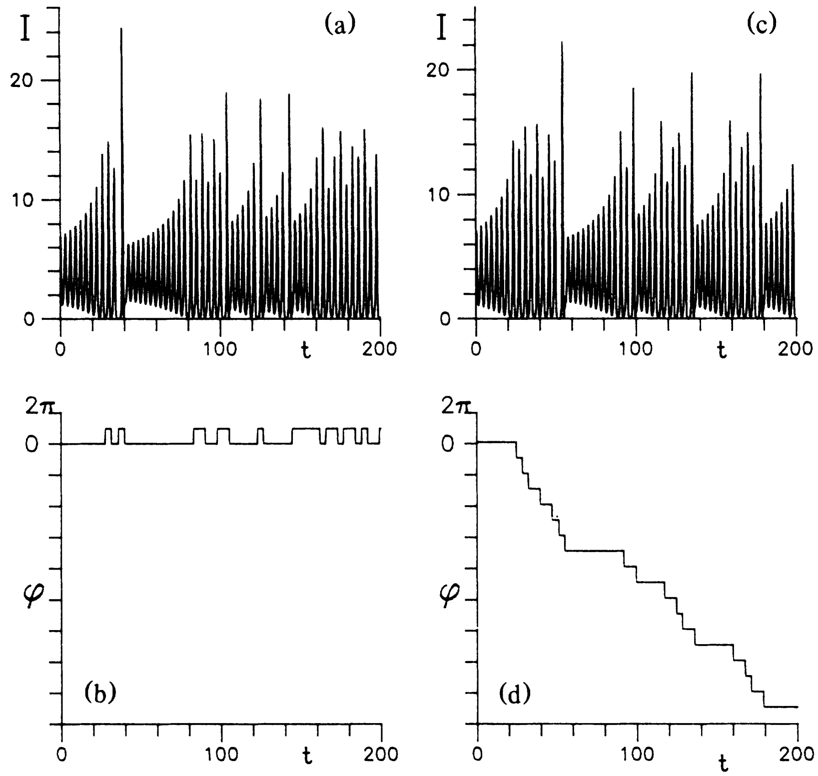

FIG. 1. Solutions of Eqs. (1) for $\gamma_{\parallel}/\gamma_{\perp}=0.25$, $A=15$, and $\kappa/\gamma_{\perp}=2$ with the time t measured in units of $\gamma_{\perp}^{-1}$; (a),(b) intensity and phase of the field at $\delta=0$; (c),(d) intensity and phase of the field at $\delta=0.001$.

plateaus are no longer sharp jumps but are reduced to changes by less than π and the evolution during a transition is smoothed. (2) The transitions no longer alternate in sign but always have the same sign determined by the sign of Δ. (3) The plateaus are no longer constant but show both a fine structure correlated with the intensity pulsations and an average slope. These results are consistent with the experimental results[1] shown in Fig. 2. The only difference is an occasional change of sign in the transition which we attribute to jitter in the cavity frequency of the laser.

We interpret the new amplitude and phase dynamics as follows: From the lack of sharp jumps by π we deduce that the amplitude of the complex field never passes exactly through zero. As Eqs. (1) are written in the rotating reference frame of the steady-state solution, any slope of the phase means that there is an additional frequency shift of dynamical origin. A careful analysis of the numerical data indicates that there are coarse and fine contributions to phase evolution. On a coarse scale, there is an average negative drift of the phase. This suggests that the rotating reference frame we are using is inadequate and that it must be corrected by an additional frequency shift. On a fine scale, each plateau displays both frequency modulation and an average positive slope. This is better analyzed by considering the instantaneous frequency $\phi'(t)$ as will be presented in Ref. 10.

As the detuning is further increased, the spiral chaotic

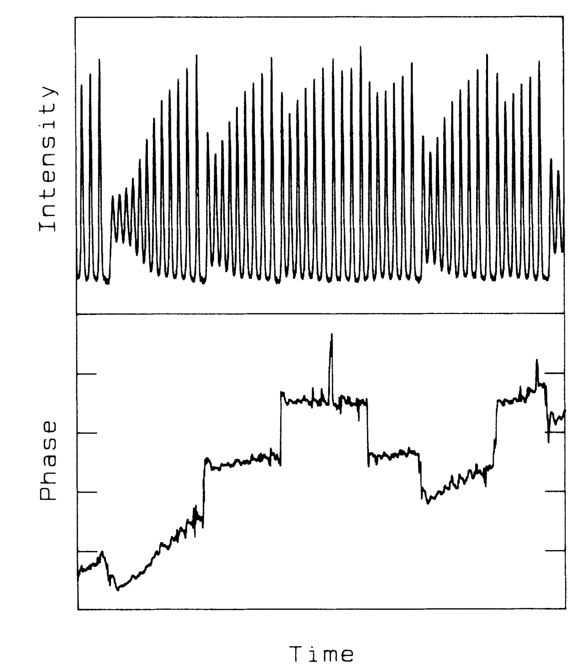

FIG. 2. Intensity and phase for the FIR ammonia laser studied in Ref. 1. The operating parameters correspond approximately to those used in Figs. 1(c) and 1(d). The vertical axis for the phase has tick marks every π.

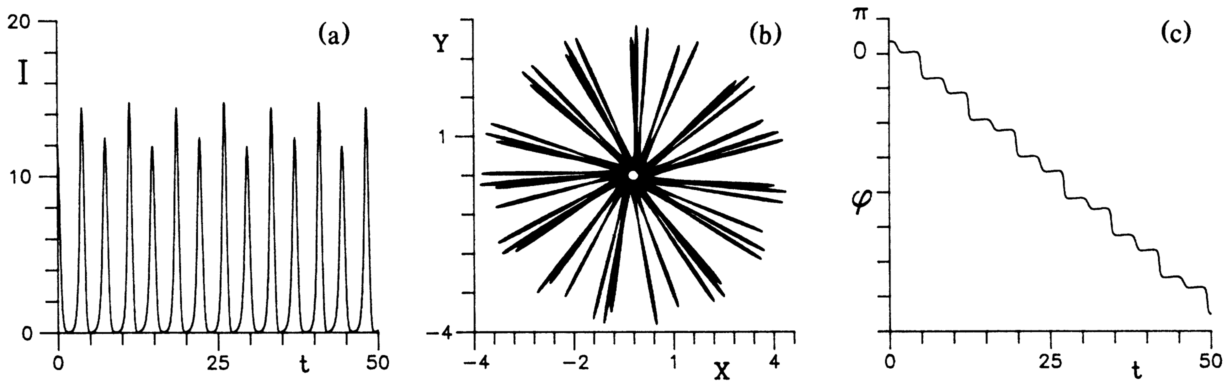

FIG. 3. A period-four solution of Eqs. (1) for $\gamma_{\parallel}/\gamma_{\perp}=0.25$, $A=15$, $\kappa/\gamma_{\perp}=2$, and $\delta=0.22$ with time t measured in units of $\gamma_{\perp}^{-1}$; (a) intensity vs time; (b) imaginary vs real part of the electric field; (c) phase of the field.

attractor is transformed into a period-doubling chaotic attractor and then through an inverse period-doubling sequence into a periodic attractor.[7b] Thus far, this transition has only been recorded and studied for the resulting intensity pulsations. Similar inverse period doublings have been observed in FIR laser experiments.[11]

Figure 3 shows a stable four-cycle intensity pulsation pattern and the corresponding representations of the solution in the [Re(E),Im(E)] plane and for the phase versus time. In Fig. 3(b) the attractor is quasiperiodic with two incommensurate frequencies. One frequency is the intensity pulsation frequency (and its subharmonics) while the other is given by the average slope of the phase. As a consequence, when we subtract this average slope from the frequency of the rotating frame, we obtain the new results displayed in Fig. 4. These graphs clearly show four-cycle behavior in this new rotating frame. A similar transformation of the frequency of the rotating frame can be performed for each periodic detuned solution. This transformation provides a kind of irreducible representation of the periodic solution for all variables and not only for the intensity.

The range of modulation of the phase, as displayed in Fig. 4(b), is 0.68π. The steep transitions [which correspond to the jumps between plateaus in Fig. 3(c)] occur between pulses and depend on the height of the preceding intensity peak (larger jumps occur after larger peaks). The largest jump is 0.64π and the smallest is 0.12π. The frequency, shown in Fig. 4(c), is nearly constant during each pulse though there is a slight difference for pulses of different heights.

With reference to Figs. 3 and 4 taken as an example of the various detuned periodic solutions, we can compare the various frequencies and frequency shifts. The detuning of the laser cavity eigenfrequency from the atomic resonance $(\omega_c-\omega_a)/\gamma_{\perp}$ is given by $(1+\kappa/\gamma_{\perp})\Delta=0.66$ while the corresponding detuning of the steady-state laser frequency from the atomic resonance $(\omega_L-\omega_a)/\gamma_{\perp}$ is 0.22. The detuning from the atomic resonance of the reference frame frequency used in Fig. 4 is 0.70. The intensity pulsation frequency is 1.7 and the spectrum includes the first two subharmonics of this frequency at 0.85 and 0.425.

Therefore, the phase dynamics of the detuned laser Lorenz model has distinctive signatures which can be compared with experimental measurements using heterodyne detection of the electric field.

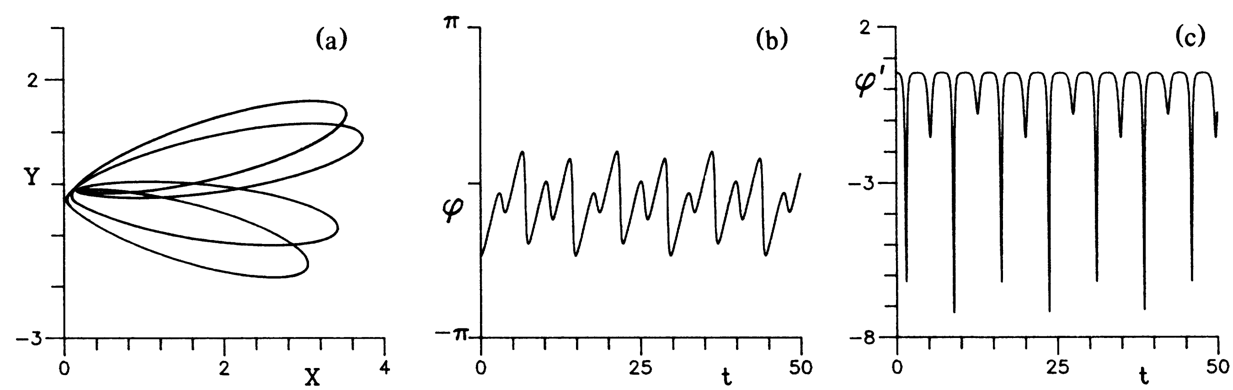

FIG. 4. For the same parameters as in Fig. 3, solutions of Eqs. (1) after subtracting from the electric field phase the contribution $0.48\gamma_{\perp}t$ which defines a new rotating frame; (a) imaginary vs real part of the electric field; (b) phase of the field; (c) instantaneous field frequency.

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