Minimum dissipation rates in magnetohydrodynamics

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Minimum dissipation rate states are explored for a current-carrying channel of magnetofluid, supported by a dc magnetic field and driven by an applied electric field. The minimization is carried out subject to the constraints of constant axial (toroidal) magnetic flux and constant time-averaged rate of supply of magnetic helicity. The solutions of the resulting Euler-Lagrange equations are sensitive to boundary conditions on the current density **j**. One set of boundary conditions on **j** leads to the same consequences as Taylor's "minimum-energy" theory. A different set leads to significantly different consequences, including a departure from the "force-free" magnetic profile and a toroidal component of current density that does not reverse at the wall when the toroidal magnetic field reverses.

I. INTRODUCTION

Nonequilibrium thermodynamics as it has evolved over the last several decades has returned repeatedly to a principle of the "minimum rate of entropy production." Key contributions include those of Onsager,[1,2] Prigogine,[3] DeGroot and Mazur,[4] Glansdorff and Prigogine,[5] and Keizer,[6] whose monograph contains a rather complete bibliography.

For systems other than weakly perturbed thermal equilibrium ones, the minimum–entropy-production–rate principle appears to occupy a status intermediate between that of a conjecture and that of a deduction. In some circumstances, such as problems involving chemical reaction rates, it seems to lead naturally to convincing answers. In others, such as static Fourier heat-conduction problems, it seems to work only with some difficulty and artificiality.

The purpose of this article is to explore the application of a closely related principle to plasma physics, and in particular to the calculation of time-averaged current profiles for the case of confined magnetofluids. As far as we are aware, the first suggestion of the applicability of minimum–entropy-production principles to the inference of magnetohydrodynamic (MHD) current profiles is due to Hameiri and Bhattacharjee.[7] The principle we explore is related to but slightly different from theirs, and will be seen to lead to somewhat different conclusions.

The principle of minimum entropy production seems to be a generalization of a more pedestrian and easily accessible principle due to Helmholtz, Korteweg, and Lord Rayleigh; a detailed development is given by Lamb.[8] This earlier principle, which can be proved under rather restrictive assumptions, is that of the *minimum rate of energy dissipation*. The two principles are equivalent for the case of constant, uniform temperature if the primitive definition of changes in specific entropy is adopted. The classic formulation given by Lamb was concerned mainly with the case of incompressible hydrodynamics with constant, uniform viscosity. The principle has been proved only for more restrictive boundary conditions than one would like, but the limits of its applicability are not known.

The search for simple variational principles derives from the need to predict at least the time-averaged properties of fields for fluids or magnetofluids whose formation involves nonlinear time-dependent processes, the details of which are difficult to follow analytically or numerically. The first important attempt in this direction for toroidal Z-pinch confinement is the "minimum-energy" principle of Taylor,[9–12] a (third and distinct) variational principle used to predict "relaxed" states of toroidal Z pinches. Key features of laboratory Z-pinch operation were predicted by this calculation, which inspired several modifications, applications, elaborations, and computational tests. A relatively complete bibliography, up to date as of early 1986, is given by Taylor.[12] One of the present authors has been involved in several attempts to provide a dynamical basis for this minimum-energy principle.[13–16]

However, recent high-resolution numerical computations[17–19] have indicated certain limitations to the minimum-energy formulation. In both decaying initial-value computations and driven, steady-state ones, a dropping ratio of energy to magnetic helicity has indeed been observed, but the ratio has also been observed to stop short of its minimum. There has been observed an accompanying achievement of a core that is approximately magnetically "force free," in the sense that the magnetic field **B** and its associated current density $\mathbf{j}=\nabla\times\mathbf{B}$ are locally aligned in the interior. However, the ratio j/B varies considerably with spatial location. This spatial variation includes, but is not limited to, an irregular boundary layer near the interface between the magnetofluid and its (assumed) bounding conducting wall. In particular, the phenomenon of "field reversal" of the toroidal component of mean magnetic field near the wall shows no tendency to be accompanied by a reversal of the corresponding component of current density.

What seems to have emerged from the numerical investigations just described is first of all the zeroth-order correctness of the Taylor minimum-energy conjecture. But secondly, features seem to have been consistently displayed that are not entirely contained in or implied by the conjecture. This is notably the case for the driven steady state, in which such global quantities as energy and magnetic helicity are necessarily supplied, on the average, *at the same rates* as those at which they are dissipated, and nothing decays. The purpose of this paper is to explore the use of the minimum–dissipation-rate principle, generalized to MHD, to see what its implications are for current-carrying bounded channels of magnetofluid which are magnetically supported.

One of the principal conclusions to emerge will concern the sensitivity of the results to boundary conditions on the current and magnetic field. For one set of boundary conditions, the minimum–dissipation-rate principle will be shown to exhibit *the same* consequences as Taylor's minimum-energy principle. For other boundary conditions, the consequences are very different, and in some cases, closer to what was computationally observed. Since none of the readily implementable MHD boundary conditions probably represents very well the complex processes which go on at the liner of a toroidal Z pinch, it seems likely to us that most remaining discrepancies with experimental observations may be attributable to the sensitivity of the theory to boundary conditions, perhaps for both theories.

The outline of the paper is as follows. In Sec. II the principle to be used is stated for dissipative, incompressible MHD. Its hydrodynamic antecedent is illustrated in Appendix A by calculating the correct velocity profile for plane Poiseuille flow as an introductory example. In Appendix B the extent to which the principle can be deductively extended to MHD is described. A reader who is mainly interested in the MHD predictions can safely skip the appendixes. The variational consequences of the principle are derived in Sec. III. In Sec. IV the resulting Euler-Lagrange equations are derived and solved, using some of the properties of Chandrasekhar-Kendall functions.[20] The associated magnetic profiles are calculated in Sec. V. In Sec. VI the minimization problem is solved. The results are summarized in Sec. VII.

Such subtleties as there are in the treatment revolve around the not unrelated considerations of boundary conditions and constraints. Most of the attention is directed to the cases of conducting walls, at which boundary conditions are applied to the magnetic field, and are or are not applied to the current density. It should be borne in mind that none of these is a wholly satisfactory representation of current practice in the laboratory.[21,22]

II. DISSIPATION RATES AND CONSTRAINTS

We consider an incompressible, resistive, viscous magnetofluid. In a well-known set of dimensionless units, its equations of motion are

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=\mathbf{j}\times\mathbf{B}-\nabla p+\nu\nabla^2\mathbf{v}\ , \tag{2.1}$$

$$\nabla\cdot\mathbf{v}=0\ , \tag{2.2}$$

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B})+\eta\nabla^2\mathbf{B}\ , \tag{2.3}$$

$$\nabla\cdot\mathbf{B}=0\ . \tag{2.4}$$

The symbols in the equations of motion mean the following: $\mathbf{v}$ is the fluid velocity field, $\mathbf{B}$ is the magnetic field, $\mathbf{j}=\nabla\times\mathbf{B}$ is the electric current density, p is the mechanical pressure field, ν is the kinematic viscosity, and η is the magnetic diffusivity. p is determined by taking the divergence of (2.1), using (2.2) to eliminate the time derivative, and solving the resulting Poisson equation for p, subject to whatever boundary conditions apply. The mass density has been assumed uniform throughout. For purposes of this present paper, ν and η are assumed to be independent of time and space. Some remarks on the case of variable dissipation coefficients will be made in Sec. VII.

For many purposes, it is convenient to write $\mathbf{B}$ in terms of a vector potential $\mathbf{A}$, $\mathbf{B}=\nabla\times\mathbf{A}$, pull a curl off Eq. (2.3), and write a time-evolution equation for $\mathbf{A}$,

$$\frac{\partial \mathbf{A}}{\partial t}=\mathbf{v}\times\mathbf{B}-\eta\nabla\times(\nabla\times\mathbf{A})+\nabla\phi\ . \tag{2.5}$$

ϕ is a scalar potential, and can be determined as soon as a gauge is chosen for $\mathbf{A}$. It is convenient to use the Coulomb gauge, $\nabla\cdot\mathbf{A}=0$, so that ϕ is determined, like p, as the solution to a Poisson equation,

$$\nabla^2\phi=-\nabla\cdot(\mathbf{v}\times\mathbf{B})\ , \tag{2.6}$$

which determines ϕ as a functional of $\mathbf{v}$ and $\mathbf{B}$.

A. Boundary conditions

The greatest single difficulty in applying MHD equations, such as (2.1)–(2.6), to laboratory magnetically confined plasmas probably has to do with boundary conditions. None of the readily implementable sets of boundary conditions for Eqs. (2.1)–(2.6) adequately describes the boundary of a real confined magnetized plasma.[21,22] This is still true even if kinetic effects such as particle gain and loss, chemical reactions, radiation, and steep boundary temperature drops are omitted. Two cases of interest will be singled out for consideration here. They are the rigid, perfectly conducting wall, which has been characterized by $\mathbf{v}=0$, $\mathbf{B}\cdot\hat{\mathbf{n}}=0$, and $\mathbf{j}\times\hat{\mathbf{n}}=0$, where $\hat{\mathbf{n}}$ is the unit normal to the wall, and the perfectly conducting wall coated on the inside with a very thin layer of insulating dielectric. The latter has been characterized both by $\mathbf{v}=0$, $\mathbf{B}\cdot\hat{\mathbf{n}}=0$, and $\mathbf{j}$ unrestricted, and also by $\mathbf{v}=0$, $\mathbf{B}\cdot\hat{\mathbf{n}}=0$, and $\mathbf{j}\cdot\hat{\mathbf{n}}=0$. The content of the theory to be presented depends sensitively on the choice that is made, as will be seen later.

The difficulty with all three choices is the unsatisfactory representation of the applied electric fields which sustain the current. These must be admitted, particularly in toroidal geometries, through slits and slots cut in the bounding conductors. The distortions of the slits and slots will still be present even if the conductor is coated on the inside with a thin layer of dielectric or resistive material. Geometrical complications, together with the subtleties of the interaction of the plasma with the external circuit, make such realism for all practical purposes impossible to include. Nevertheless, we will represent $\nabla\phi$ as the sum of two terms,

$$\nabla\phi=\nabla\tilde{\phi}+\mathbf{E}_{\rm ext} , \tag{2.7}$$

where $\mathbf{E}_{\rm ext}=E_{\rm ext}\hat{\mathbf{e}}_z$, and $E_{\rm ext}$ is independent of space; the spatial average of $\tilde{\phi}$ is zero. We shall ignore the fact that the presence of the externally applied electric field $\mathbf{E}_{\rm ext}$ is incompatible, physically, with our unbroken conducting boundary. As in the computation,[19] $\mathbf{E}_{\rm ext}$ is understood to drop steeply to zero at the conducting wall.

The boundary will be assumed to be a circular cylinder with radius a and periodically identified ends; other toroidal effects will be ignored. The plasma then fills, with uniform mass density, the region $0\le r=(x^2+y^2)^{1/2}<a$ and $0\le z<L_z$, where L_z is the periodicity length in the axial direction. The externally applied electric field is in the axial direction. All the variables in Eqs. (2.1)–(2.7) except ϕ will be assumed periodic in z; since only gradients of ϕ appear, this average linear increase of ϕ with z is no problem. $\tilde{\phi}$ will be periodic in z.

B. Variational principles

We are interested in the long-time configuration of the plasma under the conditions of a finite, constant $\mathbf{E}_{\rm ext}$. Though a wide range of possibilities exists for $\mathbf{j}$ and $\mathbf{B}$ profiles in ideal MHD, the existence of finite η and ν greatly restricts the possible steady states that may be expected. Computational experience indicates strongly that, at least above certain thresholds in $\mathbf{E}_{\rm ext}$ and below certain thresholds in η and ν, the time-averaged states are turbulent and time dependent. What we shall explore here is the rather limited question of what profiles are to be predicted by the assumption that the time-averaged profiles are profiles of minimum–energy-dissipation rate, given analytically by

$$R\equiv\int d^3x\,(\eta\mathbf{j}^2+\nu\omega^2) , \tag{2.8}$$

where the integrals will always be, unless otherwise indicated, over the three-dimensional region $0\le r<a$, $0\le z<L_z$. ($\boldsymbol{\omega}=\nabla\times\mathbf{v}$ is the vorticity.)

The only thing which constrains R away from zero is that certain time-averaged constants of the motion can be proved from Eqs. (2.1)–(2.7) when $\mathbf{E}_{\rm ext}\neq0$. These clearly do not include the classical constants of *ideal* motion such as energy or magnetic helicity, since both are being supplied and dissipated simultaneously. The two rates (supply and dissipation) must on the average be equal, but the mean energy or helicity that is maintained is not uniquely predicted by this simple requirement. A similar statement applies to other prime candidates such as the total toroidal current. We have been able to discover only one dynamical constraint which is implied by (2.1)–(2.7) under the sets of boundary conditions considered. This is the *rate of supply and dissipation of magnetic helicity*. There is, in addition, also the kinematical constraint of flux conservation.

C. Constraints

Dot Eq. (2.5) with $\mathbf{B}$, (2.3) with $\mathbf{A}$, and add the results; since $\nabla\times(\nabla\times\mathbf{A})=\mathbf{j}$, we get

$$\frac{\partial}{\partial t}(\mathbf{A}\cdot\mathbf{B})=-\eta\mathbf{j}\cdot\mathbf{B}+\mathbf{E}_{\rm ext}\cdot\mathbf{B}+\mathbf{B}\cdot\nabla\tilde{\phi}+[\nabla\times(\mathbf{v}\times\mathbf{B})]\cdot\mathbf{A}-\eta(\nabla\times\mathbf{j})\cdot\mathbf{A} . \tag{2.9}$$

It is preferable to work with a magnetic helicity H_m defined not as the volume integral of $\mathbf{A}\cdot\mathbf{B}$ alone, but in the gauge-invariant way,[12]

$$H_m\equiv\int\mathbf{A}\cdot\mathbf{B}\,d^3x-\oint\mathbf{A}\cdot dl\oint\mathbf{A}\cdot ds ,$$

where the two line integrals run, respectively, from $z=0$ to $z=L_z$ at $r=a$ and azimuthally around the cylinder at $r=a$. Integrating (2.9) over the volume of the cylinder, and using some vector identities,

$$\frac{d}{dt}H_m=-\eta\int\mathbf{j}\cdot\mathbf{B}\,d^3x+2\mathbf{E}_{\rm ext}\cdot\int\mathbf{B}\,d^3x+\int d^3x\,\nabla\cdot[(\mathbf{v}\times\mathbf{B})\times\mathbf{A}]+\int(\nabla\times\mathbf{A})\cdot(\mathbf{v}\times\mathbf{B})d^3x$$
$$-\int d^3x\,\nabla\cdot(\eta\mathbf{j}\times\mathbf{A})-\int(\nabla\times\mathbf{A})\cdot\eta\mathbf{j}\,d^3x+\int d^3x\,\nabla\cdot(\mathbf{B}\tilde{\phi}) . \tag{2.10}$$

Using the divergence theorem three times, the resulting surface integrals vanish for the boundary conditions considered. [For the surface integral resulting from the $\nabla\cdot(\eta\mathbf{j}\times\mathbf{A})$ integral, a limiting argument with variable η is necessary for the case of the dielectric-coated conductor, but presents no serious difficulty.] The result is

$$\frac{d}{dt}H_m=-2\eta\int\mathbf{j}\cdot\mathbf{B}\,d^3x+2\mathbf{E}_{\rm ext}\cdot\int\mathbf{B}\,d^3x . \tag{2.11}$$

We now time average Eq. (2.11) over a very long time interval, the operation of which we indicate by a bar over the quantities. If the magnetic helicity H_m remains bounded (not a significant restriction), the time average of the left-hand side vanishes. The last term on the right-hand side of (2.11) is time independent and equal to $2E_{\rm ext}B_0\pi a^2L_z$, where B_0 is the (constant) average toroidal magnetic field; $\pi a^2B_0\equiv\Phi_B$ is the constant toroidal magnetic flux. The content of Eq. (2.11) is then

$$2\eta\int\overline{\mathbf{j}\cdot\mathbf{B}}\,d^3x=2E_{\rm ext}\pi a^2L_zB_0 . \tag{2.12}$$

In the most general case, $\mathbf{j}$ and $\mathbf{B}$ will have time-averaged values and temporally fluctuating parts, so that $\mathbf{j}=\overline{\mathbf{j}}+\delta\mathbf{j}$, $\mathbf{B}=\overline{\mathbf{B}}+\delta\mathbf{B}$, and $\overline{\mathbf{j}\cdot\mathbf{B}}=\overline{\mathbf{j}}\cdot\overline{\mathbf{B}}+\overline{\delta\mathbf{j}\cdot\delta\mathbf{B}}$, since the cross terms vanish upon time averaging.

If the turbulence level is sufficiently low (a typical rms value for $\delta\mathbf{B}/B$ might be estimated at $\lesssim 10\%$, for reversed field pinches), the $\overline{\delta\mathbf{j}\cdot\delta\mathbf{B}}$ may be neglected compared to $\overline{\mathbf{j}}\cdot\overline{\mathbf{B}}$ and we shall restrict ourselves to this limit hereafter. The strongly turbulent case, in which $\overline{\delta\mathbf{j}\cdot\delta\mathbf{B}}$ is comparable to $\overline{\mathbf{j}}\cdot\overline{\mathbf{B}}$, is considerably more formidable and will be deferred to future consideration. Hereafter, we shall assume that we are dealing with time averages, and drop the bars over all the various variables. The content of (2.12) is now (hereafter, $\langle\ \rangle$ indicates a *spatial* average)

$$\langle \mathbf{j}\cdot\mathbf{B}\rangle = (B_0 E_{\text{ext}})/(\eta) = \text{const} , \tag{2.13}$$

which is our main constraint. The reader will be able to convince himself that a similar argument involving, say, energy will fail, because the integrals corresponding to the rightmost term of (2.11) do not reduce to constants of the motion.

Whether other undiscovered dynamical constants beyond (2.13) exist is, of course, an open question, as is usually the case for nonlinear systems with many degrees of freedom. In the past, numerical evidence has proved to be the most useful method of determining the answer; it is anticipated that the same will be true here.

Since the system is not mechanically driven, it seems that nothing will constrain the time-averaged $\mathbf{v}$ away from zero. Again assuming that time averages of fluctuation products are negligible compared to products of time averages, the problem has been reduced, from (2.8), to minimizing $\eta\langle j^2\rangle$ subject to the constraint (2.13) and a kinematical constraint yet to be discussed. (Notice that the analogous statement of the Taylor problem is minimizing $\langle \mathbf{B}^2\rangle$ subject to the constancy of H_m; the differences in the implications have mainly to do with the boundary conditions.)

The toroidal flux is a *kinematic* constant of the motion for a magnetofluid surrounded by a perfect conductor. This functions as a second constraint which can be introduced into the formalism in more than one way. It is useful to introduce it into the formalism here by setting

$$\int d^3x\, \mathbf{B}\cdot\nabla\mathcal{S} = \text{const} , \tag{2.14}$$

where $\mathcal{S}$ is a general function not of time but of spatial position, and $\nabla\mathcal{S}$ is periodic in z. The temporal constancy of the expression (2.14) can be readily demonstrated by applying (2.3), integrating by parts, and invoking the **B**-field boundary conditions. The apparently more general content of the expression (2.14) is illusory; it will be seen to restrict only the toroidal flux. The advantage of writing flux conservation in this way is that it leads to Euler-Lagrange equations that are of a form general enough to permit the satisfaction of both the boundary conditions on $\mathbf{j}$ and on $\mathbf{B}$. [See also the remarks between Eqs. (3.2) and (3.3).]

In terms of two Lagrange multipliers, say α and 2β, the variational problem can then be given the compact and economical statement

$$\delta\int d^3x (\mathbf{j}^2 + \alpha\mathbf{j}\cdot\mathbf{B} + 2\beta\mathbf{B}\cdot\nabla\mathcal{S}) = 0 , \tag{2.15}$$

where all field variables are to be hereafter considered as time averages, and α and β are to be chosen to be compatible with specified values of $\langle \mathbf{j}\cdot\mathbf{B}\rangle$ and $\langle \hat{\mathbf{e}}_z\cdot\mathbf{B}\rangle$. The determination of $\mathcal{S}$ follows, in a way illustrated in Sec. III, from the need to satisfy boundary conditions.

III. EULER-LAGRANGE EQUATION

The variations implied in (2.15) are of a standard type. We let the fields acquire small variations,

$$\mathbf{B}\to\mathbf{B}+\delta\mathbf{B}, \quad \mathbf{j}\to\mathbf{j}+\delta\mathbf{j} , \tag{3.1}$$

where $\mathbf{j}=\nabla\times\mathbf{B}$, $\delta\mathbf{j}=\nabla\times\delta\mathbf{B}$, and $\delta\mathbf{B}=0$ at $r=a$. $\delta\mathbf{B}$ is also assumed periodic in z with period L_z. Integrating by parts twice and collecting the first-order part of the integral in (2.15) gives (independently of which boundary conditions on $\mathbf{j}$ are imposed)

$$\int d^3x\, \delta\mathbf{B}\cdot(\nabla\times\mathbf{j}+\alpha\mathbf{j}+\beta\nabla\mathcal{S}) = 0 . \tag{3.2}$$

The vanishing of (3.2) for any such $\delta\mathbf{B}$ is *necessary* that the integral (2.15) be stationary.

The vanishing of a volume integral $\int \mathbf{V}\cdot\delta\mathbf{B}\, d^3x$, for a solenoidal vector field $\mathbf{V}$ and for an arbitrary solenoidal $\delta\mathbf{B}$ which vanishes at $r=a$ (or even only satisfies $\hat{\mathbf{n}}\cdot\delta\mathbf{B}=0$ there) and is periodic in z, does not imply that $\mathbf{V}=0$, but only that $\mathbf{V}$ is the gradient of a periodic scalar U, say. U and $\beta\mathcal{S}$, since they are still to be determined, may be combined into a single scalar s. The states of minimum $\int d^3x\, \mathbf{j}^2$ are then contained among the solutions of the Euler-Lagrange equations

$$\nabla\times\mathbf{j}+\alpha\mathbf{j}+\nabla s = 0 . \tag{3.3}$$

The possibilities for the solutions to (3.3) appear very wide, but are greatly restricted by the observation that both $\mathbf{j}$ and $\nabla\times\mathbf{j}$ are solenoidal fields. Taking the divergence of Eq. (3.3) gives Laplace's equation for s,

$$\nabla^2 s = 0 . \tag{3.4}$$

In the geometry considered, this will mean that s is almost completely determined by imposing the boundary conditions on $\mathbf{j}$ in Eq. (3.3).

It is to be emphasized that the role of s in Eqs. (3.3) and (2.14) is to be able to impose boundary conditions on $\mathbf{j}$. If $\mathbf{j}$ is regarded as unrestricted at the wall, we may simply choose $s=0$, so that $\nabla\times\mathbf{j}+\alpha\mathbf{j}=0$ and the boundary conditions are only to be imposed upon $\mathbf{B}$. At this level, the content of the theory seems to be equivalent to that of Taylor's Euler equation, $\nabla\times\mathbf{B}=\lambda\mathbf{B}$. Indeed, one Euler equation may be obtained from the other by taking the curl, and the present principle appears as an alternative basis for Taylor's theory. (We may also note the disarmingly simple solution, $\mathbf{j}=-\nabla s/\alpha=\text{const}\times\hat{\mathbf{e}}_z$.)

A richer set of solutions emerges if we require $\mathbf{j}$ to obey the perfectly conducting boundary condition at the wall, $\mathbf{j}\times\hat{\mathbf{n}}=0$, and the rest of this paper will explore this case. As will be seen in Sec. IV, there are still an infinite number of solutions, each with its own nonzero s.

IV. SOLUTION OF EQ. (3.3)

In order to save space, we shall not reproduce the deductive path by which Eqs. (3.3) and (3.4) may be straightforwardly solved, but will simply state the solutions. The solutions are identified by a triple of integers n,m,q, where $n=0,\pm1,\pm2,\ldots$, $m=0,\pm1,\pm2,\ldots$, and $q=1,2,3,\ldots$. For each n,m,q there is a solution $\mathbf{j}=\mathbf{j}_{nmq}$,

$$\mathbf{j}_{nmq}=\zeta_{nmq}\mathbf{J}_{nmq}+\frac{\nabla s_{nmq}}{\lambda_{nmq}} , \tag{4.1}$$

and the symbols are now to be explained. The amplitude ζ_{nmq} is an arbitrary real number. $\mathbf{J}_{nmq}$ will be understood as the (real part of the) expression[20]

$$\mathbf{J}_{nmq}=\lambda_{nmq}\nabla\times\hat{\mathbf{e}}_z\psi_{nmq}+\nabla\times(\nabla\times\hat{\mathbf{e}}_z\psi_{nmq}) , \tag{4.2}$$

where $\psi_{nmq}=J_m(\gamma_{nmq}r)\exp(im\varphi+ik_nz)$. We are in cylindrical coordinates r, φ, z, and J_m is the Bessel function of the first kind, of integer order. k_n is $2\pi n/L_z$ and $\lambda_{nmq}^2\equiv\gamma_{nmq}^2+k_n^2$. The γ_{nmq} are determined by boundary conditions in a way yet to be described. The $\mathbf{J}_{nmq}$ are the eigenfunctions of the curl introduced by Chandrasekhar and Kendall,

$$\nabla\times\mathbf{J}_{nmq}=\lambda_{nmq}\mathbf{J}_{nmq} , \tag{4.3}$$

as can be determined by direct differentiation of Eq. (4.2) and use of the fact that $(\nabla^2+\lambda_{nmq}^2)(\psi_{nmq})=0$.

For s_{nmq} in Eq. (4.1), we have a general solution of Laplace's equation (3.4) for which ∇s is periodic in z,

$$s_{nmq}=\beta_{nmq}z+d_{nmq}\times\begin{cases}I_m(k_nr)e^{i(m\varphi+k_nz)}, & n\neq0\\ r^me^{im\varphi}, & n=0\end{cases} \tag{4.4a}$$

and

$$\nabla s_{nmq}=\beta_{nmq}\hat{\mathbf{e}}_z+\begin{cases}(d_{nmq}/ik_n)\nabla\times[\nabla\times\hat{\mathbf{e}}_zI_m(k_nr)e^{i(m\varphi+k_nz)}], & n\neq0\\ -id_{nmq}\nabla\times(\hat{\mathbf{e}}_zr^me^{im\varphi}), & n=0 .\end{cases} \tag{4.4b}$$

The Lagrange multiplier α has been renamed as $-\lambda\to-\lambda_{nmq}$. $I_m(k_nr)$ is the modified Bessel function, which obeys

$$\nabla^2[I_m(k_nr)e^{i(m\varphi+k_nz)}]=0 . \tag{4.5}$$

Real parts are understood to be taken throughout Eqs. (4.1) through (4.6). β_{nmq} and id_{nmq} are real numbers which are determined by imposing the boundary conditions. When the boundary conditions are all imposed, it will be seen that the only arbitrary number left in the solution (4.1) is the amplitude ζ_{nmq}.

The boundary conditions on $\mathbf{j}$ are imposed by requiring that j_φ and j_z both vanish at $r=a$, for arbitrary n,m,q. These provide a pair of simultaneous, linear, homogeneous, algebraic equations for the ζ_{nmq} and id_{nmq}. We can solve for the one in terms of the other consistently only if the determinant of the system vanishes; this is what determines the γ_{nmq} and the λ_{nmq}.

Written out by components, the $\mathbf{J}_{nmq}$ are

$$\mathbf{J}_{nmq}=\left[\hat{\mathbf{e}}_r\left[\lambda_{nmq}\frac{im}{r}J_m(\gamma_{nmq}r)+ik_n\frac{dJ_m(\gamma_{nmq}r)}{dr}\right]+\hat{\mathbf{e}}_\varphi\left[-\lambda_{nmq}\frac{dJ_m(\gamma_{nmq}r)}{dr}-\frac{mk_n}{r}J_m(\gamma_{mmq}r)\right]+\hat{\mathbf{e}}_z[\gamma_{nmq}^2J_m(\gamma_{nmq}r)]\right]\exp(im\varphi+ik_nz) . \tag{4.6}$$

The equations obtained by requiring that each j_φ and j_z vanish at $r=a$ are, respectively,

$$0=\zeta_{nmq}\left[-\lambda_{nmq}\frac{dJ_m(\gamma_{nmq}a)}{da}-\frac{mk_n}{a}J_m(\gamma_{nmq}a)\right]+\frac{im}{a}\frac{d_{nmq}}{\lambda_{nmq}}\times\begin{cases}I_m(k_na), & n\neq0\\ a^m, & n=0\end{cases} \tag{4.7a}$$

and

$$\zeta_{nmq}\gamma_{nmq}^2J_m(\gamma_{nmq}a)+\frac{ik_nd_{nmq}}{\lambda_{nmq}}I_m(k_na)+\frac{\beta_{nmq}}{\lambda_{nmq}}\delta_{n,0}\delta_{m,0}=0 . \tag{4.7b}$$

Without losing generality, we may set $id_{00q}=0$ and $\beta_{nmq}=0$, unless $n=m=0$.

The condition that the system (4.7) be soluble reduces to

$$\frac{dJ_0(\gamma_{00q}a)}{da}=0 \quad \text{for } m=0=n , \tag{4.8a}$$

$$\frac{dJ_0(\gamma_{n0q}a)}{da}=0 \quad \text{for } m=0 \text{ and } n\neq0 , \tag{4.8b}$$

$$J_m(\gamma_{0mq}a)=0 \quad \text{for } m\neq0 \text{ and } n=0, \tag{4.8c}$$

and

$$k_n\left[\lambda_{nmq}\frac{dJ_m(\gamma_{nmq}a)}{da}+\frac{mk_n}{a}J_m(\gamma_{nmq}a)\right]+\frac{m}{a}\gamma_{nmq}^2J_m(\gamma_{nmq}a)=0 , \tag{4.8d}$$

for both m and n different from zero. Equation (4.8d) can be manipulated to read

$$\gamma_{nmq}aJ_m'(\gamma_{nmq}a)+\frac{m\lambda_{nmq}}{k_n}J_m(\gamma_{nmq}a)=0 , \tag{4.9}$$

which rather interestingly guarantees that $\mathbf{J}_{nmq}\cdot\hat{\mathbf{e}}_r=0$ at $r=a$ (but not that $\mathbf{j}_{nmq}\cdot\hat{\mathbf{e}}_r=0$ there); this property is in fact shared by all cases in Eqs. (4.8).

Having determined the γ_{nmq} by Eq. (4.8), the id_{nmq} can be straightforwardly determined, in the cases in which they are nonzero, and are proportional to ζ_{nmq}. The only nonvanishing β_{nmq} are the β_{00q}, which are straightforwardly obtained from setting the z component of $\mathbf{j}_{00q}=0$ at $r=a$,

$$\beta_{00q}=-\zeta_{00q}\lambda_{00q}^3 J_0(\lambda_{00q}a) \ . \tag{4.10}$$

There are two possible signs of $\lambda_{nmq}=\pm(\gamma_{nmq}^2+k_n^2)^{1/2}$ for each $\gamma_{nmq}>0$. It will be seen in Sec. V that the sign of λ_{nmq} is fixed by the sign of $\langle \mathbf{j}\cdot\mathbf{B}\rangle$. The γ_{nmq} are ordered in q as ascending positive values, with $q=1$ always identifying the lowest nonzero γ_{nmq} for any n and m.

The determination of the possible $\mathbf{j}_{nmq}$ which satisfy all the boundary conditions is now complete, up to the overall multiplicative amplitude ζ_{nmq}. The constraint of constant helicity supply rate will be expressed as $\langle \mathbf{j}_{nmq}\cdot\mathbf{B}_{nmq}\rangle=\text{const}$ where $\mathbf{B}_{nmq}$ is the magnetic field to be determined in Sec. V.

V. ASSOCIATED MAGNETIC FIELD PROFILES

For each $\mathbf{j}_{nmq}$ of the form of Eq. (4.1), there is a $\mathbf{B}_{nmq}$ for which

$$\nabla\times\mathbf{B}_{nmq}=\mathbf{j}_{nmq}=\mathbf{J}_{nmq}+\frac{\nabla s_{nmq}}{\lambda_{nmq}} \ . \tag{5.1}$$

It is useful to write ∇s_{nmq} in the form of Eq. (4.4b) and identify

$$\mathbf{B}_{nmq}=\frac{\zeta_{nmq}\mathbf{J}_{nmq}}{\lambda_{nmq}}+\mathbf{B}_0+\nabla\chi_{nmq}+\begin{Bmatrix}(d_{nmq}/ik_n)\nabla\times\hat{\mathbf{e}}_z I_m(k_n r)e^{i(m\varphi+k_n z)}\\ -id_{0mq}\hat{\mathbf{e}}_z r^m e^{im\varphi}\end{Bmatrix}\ \text{for}\ \begin{cases}n\neq 0\\ n=0\end{cases}$$
$$+\frac{\beta_{00q}r}{2\lambda_{00q}}\delta_{n,0}\delta_{m,0}\hat{\mathbf{e}}_\varphi \ , \tag{5.2}$$

where some of the symbols remain to be defined. $\mathbf{B}_0$ is a constant, uniform magnetic field in the z direction, $B_0\hat{\mathbf{e}}_z$, and accounts for all the toroidal flux $B_0\pi a^2$, since all the other functions are "fluxless." χ_{nmq} is a solution of Laplace's equation $\nabla^2\chi_{nmq}=0$, and is written

$$\chi_{nmq}=\begin{cases} ib_{nmq}I_m(k_n r)e^{i(m\varphi+k_n z)}, & n\neq 0\\ ib_{0mq}r^m e^{im\varphi}, & n=0 \text{ and } m\neq 0\\ 0, & n=0=m \ .\end{cases} \tag{5.3}$$

The real numbers b_{nmq} and b_{0mq} are determined by the boundary condition $\mathbf{B}\cdot\hat{\mathbf{n}}=0$ at $r=a$. The last term of (5.2) is a poloidal contribution whose curl leads to the average toroidal current contribution that follows from (4.10).

All terms in (5.2) are thus completely determined except the overall amplitude ζ_{nmq}. Since the ib_{nmq} will be proportional to the id_{nmq}, notice that every term in (5.2) *except* $\mathbf{B}_0$ contains a multiplicative factor of ζ_{nmq}. β_{00q} is responsible for all the toroidal current, and there is none for modes with $n^2+m^2>0$. $\mathbf{B}_0$ is responsible for the total toroidal flux, which is always $B_0\pi a^2$ regardless of m and n.

VI. SEARCH FOR THE MINIMUM-DISSIPATION STATE

The only undetermined number in $\mathbf{j}_{nmq}$ and $\mathbf{B}_{nmq}$ is now the overall amplitude ζ_{nmq}, given n, m, and q. The problem has been reduced to finding the minimum value of $\langle \mathbf{j}_{nmq}^2\rangle$, subject to a given value of $K\equiv\langle \mathbf{j}_{nmq}\cdot\mathbf{B}_{nmq}\rangle$, with $\mathbf{j}_{nmq}$ and $\mathbf{B}_{nmq}$ given by (4.1) and (5.2). It is possible to prove a very useful relation between these volume averages, which we now demonstrate.

First, consider the case $n^2+m^2>0$. Substitute (4.1) into $\int \mathbf{j}_{nmq}\cdot\mathbf{B}_{nmq}d^3x$ to get (real parts understood *before* multiplication)

$$\int \mathbf{j}_{nmq}\cdot\mathbf{B}_{nmq}d^3x =\int d^3x\left[\zeta_{nmq}\mathbf{J}_{nmq}+\frac{\nabla s_{nmq}}{\lambda_{nmq}}\right]\cdot\mathbf{B}_{nmq} \ . \tag{6.1}$$

The last term in (6.1) vanishes upon conversion to a surface integral and using the boundary conditions; for the rest, we may write

$$\int d^3x\, \zeta_{nmq}\mathbf{J}_{nmq}\cdot\mathbf{B}_{nmq} =\int d^3x\frac{(\nabla\times\zeta_{nmq}\mathbf{J}_{nmq})}{\lambda_{nmq}}\cdot\mathbf{B}_{nmq} =\int d^3x\frac{1}{\lambda_{nmq}}\nabla\times\left[\zeta_{nmq}\mathbf{J}_{nmq}+\frac{1}{\lambda_{nmq}}\nabla s_{nmq}\right]\cdot\mathbf{B}_{nmq} \ . \tag{6.2}$$

The larger parentheses in (6.2) is $\mathbf{j}_{nmq}$, so we have proved that

$$\int \mathbf{j}_{nmq}\cdot\mathbf{B}_{nmq}d^3x=\frac{1}{\lambda_{nmq}}\int d^3x\,(\nabla\times\mathbf{j}_{nmq})\cdot\mathbf{B}_{nmq} =\frac{1}{\lambda_{nmq}}\int\nabla\cdot(\mathbf{j}_{nmq}\times\mathbf{B}_{nmq})d^3x +\frac{1}{\lambda_{nmq}}\int\nabla\times\mathbf{B}_{nmq}\cdot\mathbf{j}_{nmq}d^3x =\frac{1}{\lambda_{nmq}}\int d^3x\ \mathbf{j}_{nmq}^2 \ , \tag{6.3}$$

because the surface integral of $(\mathbf{j}\times\mathbf{B})\cdot\hat{\mathbf{n}}$ vanishes, so that

$$\int \mathbf{j}_{nmq}^2 d^3x = \lambda_{nmq}\int \mathbf{j}_{nmq}\cdot\mathbf{B}_{nmq}d^3x, \quad n^2+m^2>0 \ . \tag{6.4}$$

The relation (6.4) applies to all nmq except those with $n=0=m$, which are a special case. For those, the surface integral contribution from $00q$ does not vanish, because of the $\beta_{00q}z$ term in s_{00q}. The extra term gives, upon using (4.10), and a similar development,

$$\int \mathbf{j}_{00q}^2 d^3x = \lambda_{00q}\int \mathbf{j}_{00q}\cdot\mathbf{B}_{00q}d^3x + \zeta_{00q}\pi a^2 L_z B_0 \lambda_{00q}^3 J_0(\lambda_{00q}a) \ . \tag{6.5}$$

If it were not for the extra term in (6.5), the problem would be trivial. Equation (6.4) shows that (for $m^2+n^2>0$)

$$\frac{\langle \mathbf{j}_{nmq}^2\rangle}{\langle \mathbf{j}_{nmq}\cdot\mathbf{B}_{nmq}\rangle}=\lambda_{nmq} \ , \tag{6.6}$$

so that the minimum-dissipation state would be simply the state of minimum λ_{nmq}. [Equation (6.6) shows that the sign of λ_{nmq} is always that of $\langle \mathbf{j}\cdot\mathbf{B}\rangle$, which we take hereafter to be positive.] It is known that, given Eqs. (4.8) and (4.9), the minimum λ_{nmq} is given by

$$(\lambda_{n,\pm1,1}a)_{\min}\cong 3.11 \ , \tag{6.7}$$

and is achieved when $(k_n a)^2\cong 1.52$. If the discrete k_n do not permit this value to be achieved exactly, then the closest k_n's to this value are candidates for the minimum.

In contrast to (6.7), $\lambda_{001}a$ is $\cong 3.83$, and all the other axisymmetric states have higher λ. The question of importance is now, under what circumstances can the extra term in (6.5) pull $\int \mathbf{j}_{00q}^2 d^3x$ below the (fixed, given) value

$$(\lambda_{n,\pm1,1})_{\min}\int \mathbf{j}_{n,\pm1,1}\cdot\mathbf{B}_{n,-1,1}d^3x = (\lambda_{n,\pm1,1})_{\min}\int \mathbf{j}_{00q}\cdot\mathbf{B}_{00q}d^3x \ , \tag{6.8}$$

if any? The answer, as we shall now see, is that for low enough $\langle \mathbf{j}\cdot\mathbf{B}\rangle$, the 001 state is always the minimum-dissipation state.

To determine the effect of the extra term in (6.5), we need to solve for ζ_{00q} in terms of $K\equiv\langle \mathbf{j}_{00q}\cdot\mathbf{B}_{00q}\rangle = (\pi a^2 L_z)^{-1}\int d^3x\, \mathbf{j}_{00q}\cdot\mathbf{B}_{00q}$. We have

$$\mathbf{j}_{00q}=\zeta_{00q}\mathbf{J}_{00q}-\zeta_{00q}\lambda_{00q}^2 J_0(\lambda_{00q}a)\hat{\mathbf{e}}_z \tag{6.9}$$

and

$$\mathbf{B}_{00q}=\frac{\zeta_{00q}\mathbf{J}_{00q}}{\lambda_{00q}}-\frac{\zeta_{00q}\lambda_{00q}^2 r}{2}J_0(\lambda_{00q}a)\hat{\mathbf{e}}_\varphi+\mathbf{B}_0 \ . \tag{6.10}$$

Dotting $\mathbf{j}_{00q}$ with $\mathbf{B}_{00q}$ and integrating over the basic volume, we find that all the various definite integrals can be done in closed form. The result is

$$\frac{\int \mathbf{j}_{00q}\cdot\mathbf{B}_{00q}d^3x}{\pi a^2 L_z}\equiv\langle \mathbf{j}_{00q}\cdot\mathbf{B}_{00q}\rangle\equiv K=\zeta_{00q}^2 A_q+\zeta_{00q}B_q \ , \tag{6.11}$$

where the coefficients A_q and B_q are

$$A_q\equiv\lambda_{00q}^3\{2J_0^2(\lambda_{00q}a)+[J_1'(\lambda_{00q}a)]^2\}>0 \ , \tag{6.12a}$$

$$B_q\equiv-\lambda_{00q}^2 B_0 J_0(\lambda_{00q}a) \ . \tag{6.12b}$$

In particular, $B_1=\lambda_{001}^2 B_0\,|J_0(\lambda_{001}a)|>0$, since $J_0(\lambda_{001}a)=J_0(\gamma_{001}a)<0$. In (6.12a) and elsewhere, primes on Bessel functions mean differentiation with respect to the argument.

We will also need the integral [from (6.5)]

$$\frac{\int \mathbf{j}_{00q}^2 d^3x}{\pi a^2 L_z}\equiv\langle \mathbf{j}_{00q}^2\rangle=\lambda_{00q}K-C_q\zeta_{00q} \ , \tag{6.13}$$

where

$$C_q\equiv-\lambda_{00q}^3 B_0 J_0(\lambda_{00q}a) \ . \tag{6.14}$$

In particular, $C_1=\lambda_{001}^3 B_0\,|J_0(\lambda_{001}a)|>0$.

The minimum dissipation will clearly occur for $q=1$; hereafter we concentrate on this case. Solving (6.11),

$$\zeta_{001}=-(2A_1)^{-1}[B_1\mp(B_1^2+4A_1K)^{1/2}] \ . \tag{6.15}$$

Only the upper sign (positive ζ_{001}) is physically relevant, and we consider it hereafter.

The solution (6.15) plugged into (6.13) can be rearranged to give $(\lambda_{001}=\gamma_{001}>0)$

$$\langle \mathbf{j}_{001}^2\rangle=\lambda_{001}K\{1+\Pi_B[1-\sqrt{1+(1/\Pi_B)}]\} \ . \tag{6.16}$$

The dimensionless parameter Π_B determines all the important physical ratios,

$$\Pi_B=\left[\frac{B_0^2}{aK}\right]\frac{(\lambda_{001}a)J_0^2(\lambda_{001}a)}{2\{2J_0^2(\lambda_{001}a)+[J_1'(\lambda_{001}a)]^2\}} = \frac{(\lambda_{001}a)}{6}\left[\frac{B_0^2}{aK}\right]\cong 0.64\left[\frac{B_0^2}{aK}\right] \ , \tag{6.17}$$

since $\lambda_{001}a\cong 3.83$.

The 001 state will be the minimum-dissipation state when the right-hand side of (6.16) is less than $(\lambda_{n,\pm1,1})_{\min}K\cong 3.11K/a$. The crossover occurs when Π_B drops to the value for which

$$1+\Pi_B[1-\sqrt{1+(2/\Pi_B)}]=\frac{(\lambda_{n,\pm1,1})_{\min}}{\lambda_{001}}\cong\frac{3.11}{3.83} \ , \tag{6.18}$$

and the solution to (6.18) is $\Pi_B\cong 0.022$. Thus, for helicity supply rates K for which $K<29B_0^2/a$, the 001 state is the minimum–energy-dissipation state. Above that critical value, the present calculation suggests only that serious complications set in, since there is no net toroidal current in the $n^2+m^2>0$ solutions.

A predicted F-Θ diagram may be expressed in terms of Π_B by solving (6.10) for the field-reversal parameter $F\equiv B_z(a)/B_0$ and the pinch parameter $\Theta=B_\varphi(a)/B_0$. We have

$$F=\frac{\zeta_{001}\gamma_{001}J_0(\gamma_{001}a)}{B_0}+1$$

$$=+1+\frac{J_0^2(\gamma_{001}a)[1-\sqrt{1+(2/\Pi_B)}]}{2\{2J_0^2(\gamma_{001}a)+[J_1'(\gamma_{001}a)]^2\}}$$

$$=1+\tfrac{1}{6}[1-\sqrt{1+(2/\Pi_B)}] \ . \qquad (6.19)$$

This follows from

$$\mathbf{B}_{001}(r)\cdot\hat{\mathbf{e}}_z/B_0$$

$$=1+(\tfrac{1}{6})[1-\sqrt{1+2/\Pi_B}]\cdot J_0(\gamma_{001}r)/J_0(\gamma_{001}a) \ ;$$

also

$$\Theta=\frac{(a\lambda_{001})J_0^2(\lambda_{001}a)}{4\{2J_0^2(\lambda_{001}a)+[J_1'(\lambda_{001}a)]^2\}}$$

$$\times[-1+\sqrt{1+(2/\Pi_B)}]$$

$$\cong 0.32[-1+\sqrt{1+(2/\Pi_B)}] \ . \qquad (6.20)$$

Combining these two expressions gives

$$F\cong 1-0.52\Theta \ . \qquad (6.21)$$

Field reversal occurs at $\Theta\cong 1.9$ (instead of 1.2, as in the "minimum-energy" formulation), and the minimum-energy-dissipation state ceases to be the 001 state at

$$\Theta\cong 0.32[-1+\sqrt{1+2/(0.022)}]\simeq 2.75$$

(instead of 1.6, as in the minimum-energy formulation).

The toroidal current density of the 001 state is

$$\mathbf{j}_{001}\cdot\hat{\mathbf{e}}_z=\zeta_{001}\gamma_{001}^2[J_0(\gamma_{001}r)-J_0(\gamma_{001}a)] \ , \qquad (6.22)$$

which remains always positive and goes smoothly to zero at $r=a$. The toroidal magnetic field is given by

$$\mathbf{B}_{001}\cdot\hat{\mathbf{e}}_z=\zeta_{001}\gamma_{001}J_0(\lambda_{001}r)+B_0 \ . \qquad (6.23)$$

Equation (6.23) is the magnetic field of the "Taylor state" plus a a constant. Whether or not it reverses before $r=a$ depends upon Π_B (or Θ), as we have already seen. Up to an additive constant, (6.22) is also the toroidal current associated with the "Taylor state;" the constant is the constant necessary to bring $j_z(r)$ smoothly to zero at $r=a$.

Some typical profiles of the 001 minimum-dissipation state are illustrated in Figs. 1–5. Figures 1–5 refer to the case $B_0=0.5$, $a=\pi/2$, $\Pi_B=0.025$. This is a state close to the upper limit in Θ of the "window" in which the minimum-dissipation state is the field-reversed 001 state. Figures 1–5 are obtained by numerically evaluating Eqs. (6.9) and (6.10) for $q=1$, and plotting the evaluated quantities versus r.

Figure 1 is $B_z(r)$ as a function of r. Figure 2 is $j_z(r)$ as a function of r. Figure 3 is the radial component of $\mathbf{j}\times\mathbf{B}$ as a function of r, and is the only nonvanishing component of that vector. Note that the magnitude of $\mathbf{j}\times\mathbf{B}$ is much less than $|\mathbf{j}|\cdot|\mathbf{B}|$, indicating proximity to a force-free state. Figure 3 is a direct measure of the radial pressure gradient that the minimum-dissipation state will support. Figure 4 is a plot of the "alignment cosine"

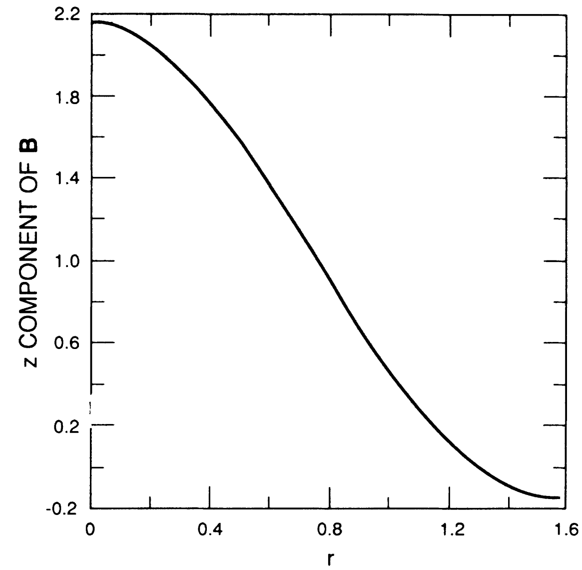

FIG. 1. $B_z(r)$ vs r, for the case $a=\pi/2$, $B_0=\frac{1}{2}$, $\Pi_B=0.025$ and $\Theta\cong 2.56$. [All graphs are in the dimensionless units of Eqs. (2.1)–(2.4).]

$\mathbf{j}\cdot\mathbf{B}/jB$, which measures the departure from the force-free condition. Note the presence of a force-free core surrounded by a region which is less so centered just above $r=1$. Finally, Fig. 5 is a plot of the ratio $\mathbf{j}\cdot\mathbf{B}/B^2$, which varies considerably more than the alignment cosine. Figures 4 and 5 are in qualitative agreement with the results of dynamical computations.

VII. SUMMARY

We have explored the consequences of the assumption that time-averaged magnetic profiles in a current-carrying conducting magnetofluid may be determined by minimizing the rate of energy dissipation. For the case of

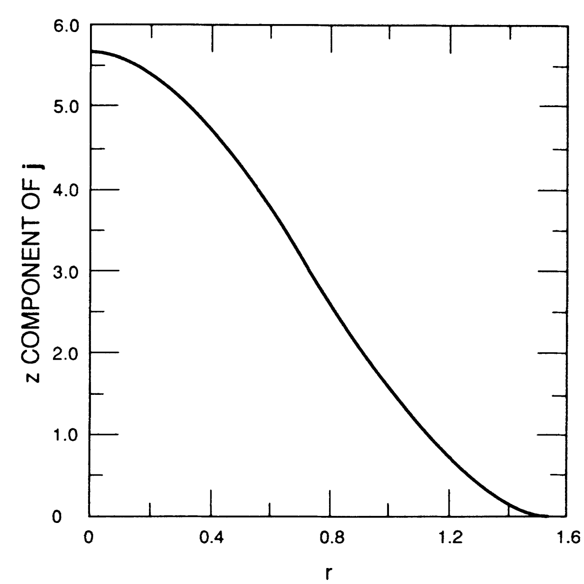

FIG. 2. $j_z(r)$ vs r, same case as Fig. 1.

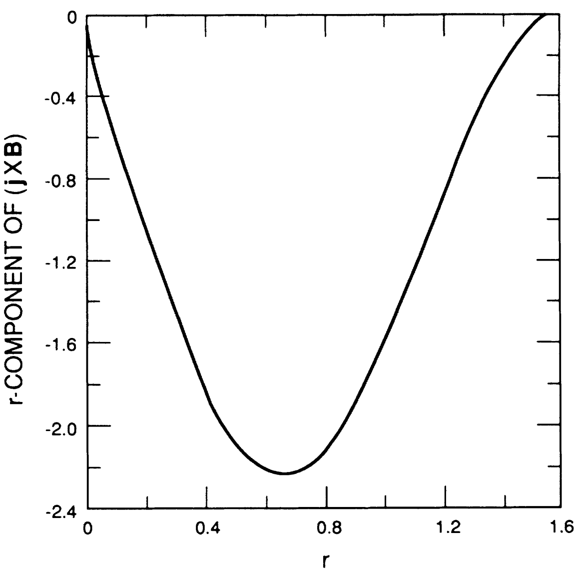

FIG. 3. $(\mathbf{j}\times\mathbf{B})_r$ vs r, same case as Fig. 1.

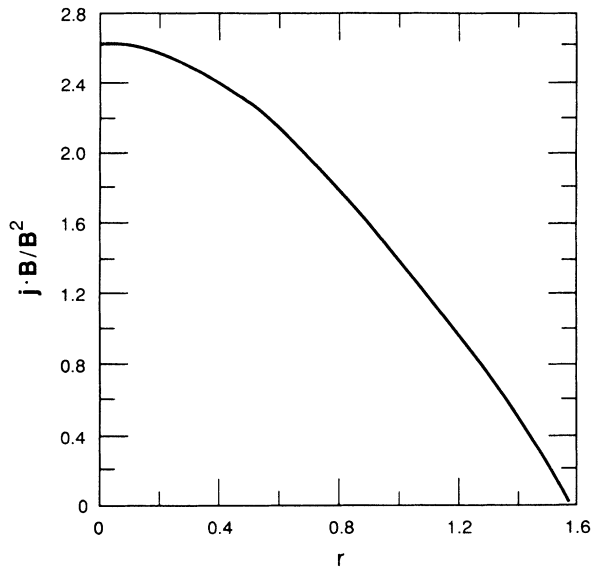

FIG. 5. Plot of $\Lambda \equiv \mathbf{j}\cdot\mathbf{B}/B^2$ vs r, same case as Fig. 1.

a uniform, incompressible magnetofluid with constant dissipation coefficients, to which these calculations are limited, this is equivalent to the principle of minimum entropy production, if the primitive definition of entropy is adopted.

The variational equation which results, for perfectly conducting boundaries with the current density unrestricted at the wall, or normal to the wall, is Eq. (3.3), with s a solution of Laplace's equation. The range of possible solutions to (3.3) is wide. For the case of the current unrestricted at the wall, the implications of the theory seem to be those of Taylor's "minimum-energy" theory. For the case in which the tangential component of the current vanishes at the wall, the results differ in several respects. It seems likely that any other possible boundary conditions will result in predictions which differ in other

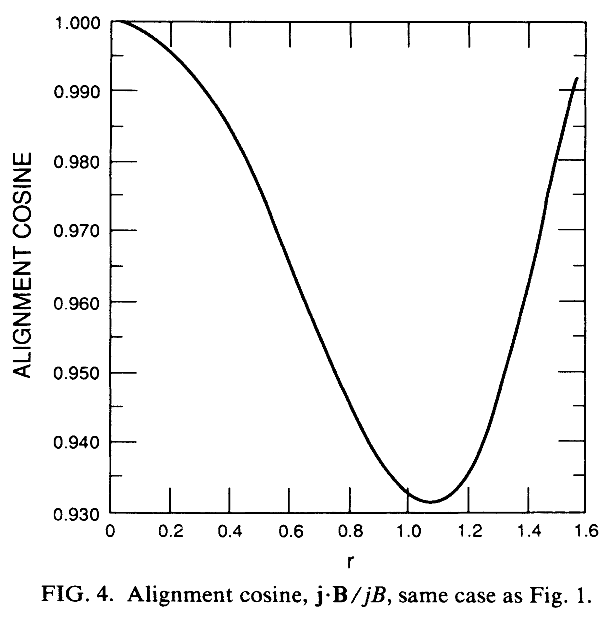

FIG. 4. Alignment cosine, $\mathbf{j}\cdot\mathbf{B}/jB$, same case as Fig. 1.

ways. One result that the current boundary conditions do yield is a finite value of $(\mathbf{j}\times\mathbf{B})\cdot\hat{\mathbf{e}}_r$, providing some confinement.

A time-averaged pressure, to go with each magnetic profile, may be inferred from Eq. (2.1). One takes the divergence *before* time averaging, and solves the resulting Poisson equation for p. If $\overline{\mathbf{j}\times\mathbf{B}}\neq 0$, $\nabla\bar{p}$ will in general be nonzero. It is interesting that below the value of Θ for which field reversal occurs, $(\mathbf{j}\times\mathbf{B})\cdot\hat{\mathbf{e}}_r$ develops a positive "hook" below $r=a$, which is an expelling, rather than a confining, force.

The principle itself is in the not entirely satisfactory position of having been proved, but under less than general conditions, that do not fully cover the cases to which we wish to apply it. This issue has been worried about by others[23,24] in the hydrodynamic context, and we do not expect an easy or early resolution to it. For purposes of this paper, it stands only as a conjecture, though potentially a numerically testable one. (See also Jaynes.[25])

Notice that the principle as given does not demand that the profiles calculated be stable or even time stationary. All that is required is that the fluctuations about them be fractionally small. There is also no reliance upon inverse cascade processes, or their close relatives, "selective decay" and "dynamic alignment."

The axisymmetric state, 001 [Eqs. (6.9) and (6.10)], is predicted up to a value of the pinch parameter $\Theta\cong 2.75$, and the toroidal magnetic field [Eq. (6.23)] reverses above $\Theta\cong 1.9$, for the $\mathbf{j}\times\hat{\mathbf{n}}=\mathbf{0}$ boundary conditions. The resulting F-Θ diagram, $F\cong 1-0.52\Theta$, lies above points which have been observed in several experiments (cf. Taylor,[12] for example, Figs. 3, 5, and 6). The experimental points, however, consistently lie well above the F-Θ curve predicted by the minimum-energy principle, whose "window" of axisymmetric reversed-field operation lies[9-12] between $\Theta\simeq 1.2$ and 1.6. It seems likely that the region between the two curves is accessible to more exotic boundary conditions. The toroidal current [Eq. (6.22)] never reverses.

We also intend to explore the case of a radially varying $\eta(r)$, a feature both of laboratory experiments and the driven computations[19] which have been done. The Euler-Lagrange equations for this case are a generalization of Eq. (3.3),

$$\nabla\times(\eta\mathbf{j})+\alpha\eta\mathbf{j}+(\alpha/2)\nabla\eta\times\mathbf{B}+\nabla s=\mathbf{0}\ , \tag{7.1}$$

which is a considerably more difficult relation than (3.3), and may require numerical treatment.

With perfectly conducting boundaries and $\mathbf{j}\times\hat{\mathbf{n}}=0$, the present theory is not much of a guide as to what happens above $\Theta\cong2.75$, when the axisymmetric (001) state ceases to be the state of minimum dissipation. Unlike the minimum-energy theory [or Eq. (3.3) with $\mathbf{j}(r=a)$ unrestricted], there is no continuous variation of the eigenvalues λ_{nmq}, and we have not seen our way through to a formulation in terms of "mixed" states. The helical states are currentless and are thus not good candidates for states of operation above $\Theta\cong2.75$. What seems most likely to us is that above this threshold and in the vicinity of it, the fractional variations about the time-averaged quantities are no longer negligible, and the whole theory breaks down: Eq. (3.3) itself no longer applies. This would seem to be qualitatively consistent with the numerical results[17–19] (which are for the not entirely comparable case of a square, rather than a circular, channel).

A final interesting observation is the necessity of a small but nonzero (in fact, first order in the resistivity) fluid velocity that is required in the steady state. This can be seen by time averaging Eq. (2.3) to get $0=\nabla\times(\overline{\mathbf{v}\times\mathbf{B}})+\eta\nabla^2\overline{\mathbf{B}}$. As long as $\nabla\times(\nabla\times\overline{\mathbf{B}})=\nabla\times\overline{\mathbf{j}}\neq0$, a velocity is *required* for a time-averaged steady state. Its contribution to Eqs. (2.8) and (B2) may be regarded as of higher order in the dissipation coefficients, so that its contribution can be neglected in the variational calculations. This may not be the case in other flows, in which a significant amount of cross helicity ($\langle\mathbf{v}\cdot\mathbf{B}\rangle$) is present; these also seem to fall within the scope of the present method. We are not unaware of the possible extensions of the method to other hydrodynamic and magnetohydrodynamic cases, and plan to explore some of these in the future.

ACKNOWLEDGMENTS

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APPENDIX A: HYDRODYNAMIC EXAMPLE

Consider a one-dimensional Navier-Stokes shear flow, bounded by infinite parallel planes at $y=-a$ and $y=+a$. Let the velocity field be $\mathbf{v}=u(y)\hat{\mathbf{e}}_x$ with the boundary conditions that $u(\pm a)=0$. The energy dissipation rate is (ν is the viscosity)

$$R=\nu\int d^3x\,[du(y)/dy]^2\ , \tag{A1}$$

where ν is the viscosity and the region of integration is a large volume between the parallel planes. Periodic boundary conditions may be assumed in the x and z directions. The region of integration is a large volume between $y=-a$ and $y=+a$.

The minimum value of R is constrained away from zero by the requirement that a constant average flow rate be maintained. This can be represented as

$$\int u(y)d^3x=\text{const}\ . \tag{A2}$$

The constraint may be taken into account by a Lagrange multiplier 2α. After an integration by parts, the minimization of R leads to the Euler-Lagrange equation

$$\nu\frac{d^2u(y)}{dy^2}+\alpha=0\ . \tag{A3}$$

The general solution to Eq. (A3) is

$$u(y)=-(\alpha/2\nu)y^2+c_1y+c_2\ , \tag{A4}$$

where c_1 and c_2 are constants of integration. Imposing the boundary conditions at $y=\pm a$ gives

$$u(y)=U_0(a^2-y^2)\ , \tag{A5}$$

the familiar parabolic Poiseuille profile. U_0 can easily be expressed in terms of α and ν.

This is probably the simplest case in which the minimum-dissipation principle leads to a known flow profile. The problem may be worked assuming less at the outset, without changing the answer; for example $[du(y)/dy]^2$ may be replaced in Eq. (A1) by the vorticity of a general three-dimensional velocity field squared, $(\nabla\times\mathbf{v})^2$. The reader may readily convince himself that the same method leads to the correct flow profile for plane Couette flow, rotating Couette flow or pipe flow. All these have the property that the curl of the vorticity is the gradient of a scalar. It can also be made to give Hartmann flow, which does *not* satisfy the conditions of the proof.

The intuitive content of the minimum–dissipation-rate principle is that spatial gradients of the fields tend to smooth themselves out as much as they can. They are prohibited, however, from becoming uniform by boundary conditions and constraints. The extent to which this becomes true when gradients exceed instability thresholds is an open (and interesting) question.

APPENDIX B: MINIMUM MHD DISSIPATION RATE

It is painful to admit that the minimum–disspation-rate principle, like the minimum-energy principle, is a conjecture and not a theorem. It can be proved, following Lamb,[8] and the proof is given below, but the conditions the proof requires are stronger than those which apply to the current profiles in Sec. V: They are that the curls of the current density and vorticity are gradients of scalars.

We dot Eq. (2.1) with $\mathbf{v}$, (2.3) with $\mathbf{B}$, add the results, and perform some integrations by parts. Invoking no boundary conditions yet, the result is

$$\frac{d}{dt}\int_V d^3x\left[\frac{v^2}{2}+\frac{B^2}{2}\right] = \int_S d\sigma\cdot\left[-\mathbf{v}\left[p+\frac{v^2}{2}\right]-\nu\omega\times\mathbf{v}\right]-\int_S d\sigma\cdot(\mathbf{E}\times\mathbf{B}) - \int_V d^3x\,(\nu\omega^2+\eta j^2)\,, \tag{B1}$$

where $\mathbf{E}=-\mathbf{v}\times\mathbf{B}+\eta\mathbf{j}$ is the electric field. In Eq. (B1), which is a statement of conservation of energy, the first two integrals are surface integrals over the closed surface S bounding the volume V. They represent the rates of feeding mechanical and magnetic energy into the system. The last integral is the "dissipation function" $R=R(\mathbf{v},\mathbf{B})$:

$$R(\mathbf{v},\mathbf{B})\equiv\int_V d^3x(\nu\omega^2+\eta j^2) \tag{B2}$$

is the rate of energy dissipation, viscous and resistive.

In a steady state, which we now represent by subscript zero ($\mathbf{v}_0,\mathbf{B}_0,p_0,j_0,\ldots$), the left-hand side of (B1) vanishes, and

$$\int_S d\sigma\cdot\left[\mathbf{v}_0\left[p_0+\frac{v_0^2}{2}\right]+\nu\omega_0\times\mathbf{v}_0\right] + \int_S d\sigma\cdot(\mathbf{E}_0\times\mathbf{B}_0)=R(\mathbf{v}_0,\mathbf{B}_0)\,. \tag{B3}$$

Now imagine any other possible solution of (1.1)–(1.4) which is close to $\mathbf{v}_0,\mathbf{B}_0,\ldots$, and obeys the same boundary conditions. That is, let $\mathbf{v}_0\to\mathbf{v}_0+\delta\mathbf{v}_1$, $\mathbf{B}_0\to\mathbf{B}_0+\delta\mathbf{B}_1$, etc., where the perturbed fields $\delta\mathbf{v}_1,\delta\mathbf{B}_1,\ldots$ are zero over the boundary (in our case, $r=a$ and $z=0$ and L_z). The surface integrals are unaffected, and the dissipation function becomes

$$R(\mathbf{v}_0+\delta\mathbf{v}_1,\ \mathbf{B}_0+\delta\mathbf{B}_1) = R(\mathbf{v}_0,\mathbf{B}_0)+2\int_V d^3x\,(\nu\omega_0\cdot\delta\omega_1+\eta\mathbf{j}_0\cdot\delta\mathbf{j}_1) + R(\delta\mathbf{v}_1,\delta\mathbf{B}_1)\,. \tag{B4}$$

If the cross terms involving integrals of $\omega_0\cdot\delta\omega_1$ and $\mathbf{j}_0\cdot\delta\mathbf{j}_1$ in (B4) can be shown to be zero, the theorem is proved, since $R(\delta\mathbf{v}_1,\delta\mathbf{B}_1)>0$. The physical solution is always, then, one which makes R a minimum.

If it is possible to write $\nabla\times\mathbf{j}_0=\nabla\pi_0$ and $\nabla\times\omega_0=\nabla\Xi_0$, we may write

$$\int_V d^3x\,\mathbf{j}_0\cdot\delta\mathbf{j}_1=\int_V d^3x\,\mathbf{j}_0\cdot\nabla\times\delta\mathbf{B}_1 = \int_V \nabla\cdot(\delta\mathbf{B}_1\times\mathbf{j}_0)d^3x + \int_V(\nabla\times\mathbf{j}_0)\cdot\delta\mathbf{B}_1 d^3x\,, =0\,, \tag{B5}$$

since $\delta\mathbf{B}_1=0$ over S. An identical proof suffices for $\int_V\omega_0\cdot\delta\omega_1 d^3x$, and the cross terms then drop out of Eq. (B4). Since R is positive definite, the theorem follows.

In a time-dependent steady state in which products of fluctuations are negligible compared to products of time averages, an equation [Eq. (B3)] applies to the time averages, and one may demonstrate a similar theorem among time averages.

We have invested considerable effort in proving the theorem without requiring that the curls of $\mathbf{j}_0$ and ω_0 be gradients of scalars, without quite succeeding. If only perturbed fields $\delta\mathbf{v}_1,\delta\mathbf{B}_1,\ldots$ are allowed which are compatible with the constraints $\int\mathbf{j}\cdot\mathbf{B}\,d^3x=$const, then to *first order* in the perturbations

$$\int(\mathbf{j}_0\cdot\delta\mathbf{B}_1+\delta\mathbf{j}_1\cdot\mathbf{B}_0)d^3x=0=2\int\mathbf{j}_0\cdot\delta\mathbf{B}_1 d^3x\,.$$

Using Eq. (3.3), this is

$$\int_V\delta\mathbf{B}_1\cdot\left[\frac{\nabla\times\mathbf{j}_0}{\lambda}+\frac{\nabla s_0}{\lambda}\right]d^3x=\frac{1}{\lambda}\int_V(\nabla\times\delta\mathbf{B}_1)\cdot\mathbf{j}_0 d^3x=0 =\lambda^{-1}\int_V\delta\mathbf{j}_1\cdot\mathbf{j}_0 d^3x=0\,, \tag{B6}$$

and, of course, $\omega_0=0$ is enough for our purposes. Unfortunately, a complete proof seems to require dealing with the second-order terms in $\int\mathbf{j}\cdot\mathbf{B}\,d^3x$ also, which we have been unable to do. Finally, note that one solution of (3.3), with $\mathbf{j}$ unrestricted, *does* satisfy all the conditions of the theorem, namely, the uniform current density solution $\mathbf{j}=\text{const}\times\hat{\mathbf{e}}_z$.

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