Expressions of various joint probability distributions of photoelectrons in terms of a photocount distribution $P(n,t_1,t_2)$

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Expressions of various joint probability distributions of photoelectrons in terms of the photocount distribution $P(n,t_1,t_2)$ in which n photoelectrons are registered between t_1 and t_2 are systematically obtained with extensive use of an introduced simple notation. The expressions obtained are used to find two pairs of expressions of the second-order and associated intensity correlation functions in terms of $P(n,t_1,t_2)$. Some applications to quantum optics are also made to obtain results in agreement with previous work.

I. INTRODUCTION

Investigation of coherence properties of an optical field was limited to effects of the second-order field correlation, such as interference and diffraction phenomena, until the experiments by Hanbury Brown and Twiss,[1] who observed the correlation between intensity fluctuations of light beams; this proves the existence of the fourth-order field correlation. Since then experiments of this type[2,3] and experiments which measure higher-order field correlation[4–6] have been performed by many investigators. A general theory of coherence which describes the statistical properties of the electromagnetic field including these higher-order effects has developed in both classical[7] and quantum terms[8] by the combination of the Maxwell wave theory with the stochastic theory and of quantum electrodynamics with the density operator.

On the other hand, statistics of an optical field are usually detected with photodetectors in which incident photons are stochastically converted into photoelectrons with a certain photodetection efficiency. Photon statistics are therefore converted to photoelectron statistics[9,10] which, in general, are not simply a scaled-down version of the original photon statistics[11,12] and can be characterized with various time distributions of photoelectrons; they may be joint probability distributions or the photocount distribution $P(n,t_1,t_2)$ in which n photoelectrons are registered between t_1 and t_2. (For stationary process this photocount distribution depends on time difference t_2-t_1 alone and can be written as $P(n,t_2-t_1)$ since then the photocount distribution does not change as a result of time displacements.) Some equations which relate joint probability distributions to $P(n,t_1,t_2)$ have been found;[13] for example, Glauber[14] obtained an expression of the joint probability distribution of time intervals between two consecutive photocounts and that of the probability distribution of waiting times in terms of the photocount distribution $P(n,t_1,t_2)$. However, no exact equation has been presented which relates the second-order intensity correlation function, which has played a fundamental role in revealing statistical properties of an optical field, to one of the most commonly used photocount distributions, $P(n,t_1,t_2)$, to the best knowledge of the author.

The main purpose of this paper is to derive systematically the expressions of various joint probability distributions of photoelectrons in terms of the photocount distribution $P(n,t_1,t_2)$. In particular, an exact relation between $P(n,t_1,t_2)$ and the second-order intensity correlation function is obtained. Some applications to quantum optics are also made, to obtain results in agreement with the previous work.

In Sec. II we first present some definitions and assumptions used in this paper. Second, on the basis of these assumptions a simple but useful notation on various joint probability distributions of photoelectrons is introduced. These distributions include the generalized forms of the probability distribution of time intervals between two consecutive photoelectrons and that of residual waiting times. Last, with extensive use of the introduced notation, various joint probability distributions are systematically expressed in terms of the photocount distribution $P(n,t_1,t_2)$. In Sec. III the expressions obtained in Sec. II are used to find two pairs of the second-order and its associated intensity correlation functions in terms of the photocount distribution $P(n,t_1,t_2)$. Two interrelations between these probability distributions are indicated. In Sec. IV some applications to quantum optics are made to demonstrate the validity of our theory.

II. BASIC RELATIONS

A. A simple but useful notation

Suppose that times when photoelectrons are emitted at a photocathode are registered with a resolving time dt_i ($i=1,2,\dots$). Here we do not set resolving times dt_i at different times t_i equal for mathematical convenience. We assume that dt_i are set much shorter than the average time interval between two consecutive photoelectrons so that the probability of more than one photoelectron being registered within dt_i ($i=1,2,\dots$) is negligible. It is easy to show that this condition is satisfied even when radiation shows photon bunching.

Next, we define three basic probabilities:[15] $0(t)$, $1(t)$, and $e(t)$, each meaning the probability of zero, one, and either zero or one count being registered during t to $t+dt$, respectively. Since zero- and one-count events occur exclusively, and, by assumption, more than one count never occurs in dt, we have

$$0(t)+1(t)=e(t)=1 \ .$$

On the basis of this notation, we introduce a series of generalized joint probability distributions such as $(1(t_1),n,1(t_2))$ and $(0(t_1),n,1(t_2))$, etc. $(1(t_1),n,1(t_2))$ denotes the joint probability distribution that one photoelectron, n photoelectrons, and one photoelectron are registered during t_1 to t_1+dt_1, t_1+dt_1 to t_2, and t_2 to dt_2+dt_2, respectively. This joint probability distribution may be regarded as a generalized form of the probability distribution of time intervals between two consecutive photoelectrons. $(0(t_1),n,1(t_2))$ denotes the joint probability distribution that no photoelectron, n photoelectrons, and one photoelectron are registered during t_1 to t_1+dt_1, t_1+dt_1 to t_2, and t_2 to t_2+dt_2, respectively. It is convenient to assume that $(1(t_1),n,1(t_2))=0$ if $n<0$. In a similar way, we can introduce joint probability distributions $(1(t_1),n,e(t_2))$ and $(0(t_1),n,e(t_2))$, etc., where the symbol $e(t_2)$ denotes that either zero or one photoelectron is registered during t_2 to t_2+dt_2. In what follows we will derive basic equations which express the photocount distribution $P(n,t_1,t_2)$, its first- and second-order time derivatives in terms of the generalized joint probability distributions introduced above. Then we will solve its inverse problem, that is, we will obtain expressions of various joint probability distributions in terms of the first- and second-order time derivatives of the photocount distribution. This problem is nontrivial and of some intrinsic interest from the physical and mathematical points of view; some applications to quantum optics will be made in Sec. IV to demonstrate the validity of our theory.

B. Basic relations

The photocount distribution $P(n,t_1,t_2)$ of n photoelectrons being registered between t_1 and t_2 can be expressed as a sum of generalized joint probability distributions,

$$P(n,t_1,t_2)=(1(t_1),n-1,e(t_2))+(0(t_1),n,e(t_2)) \ .$$

The symbol $e(t_2)$ on the right-hand side of this equation appears because we impose no condition on a photocount during t_2 to t_2+dt_2. For $P(n,t_1+dt_1,t_2)$, we impose no condition on a photocount during either t_1 to t_1+dt_1 or t_2 to t_2+dt_2. Therefore we have

$$P(n,t_1+dt_1,t_2)=(1(t_1),n,e(t_2))+(0(t_1),n,e(t_2)) \ ,$$

where

$$(1(t_1),n,e(t_2))=(1(t_1),n,1(t_2))+(1(t_1),n,0(t_2)) \ .$$

In a similar manner, we have two more equations as follows:

$$\begin{aligned} P(n,t_1,t_2+dt_2)=&(1(t_1),n-2,1(t_2))\\ &+(1(t_1),n,0(t_2))\\ &+(0(t_1),n-1,1(t_2))\\ &+(0(t_1),n,0(t_2)) \ ,\end{aligned}$$

$$P(n,t_1+dt_1,t_2+dt_2)=(e(t_1),n,0(t_2))+(e(t_1),n-1,1(t_2)) \ .$$

On the other hand, the first-order time derivative of $P(n,t_1,t_2)$ with respect to t_1 can be expressed as a difference of generalized joint probability distributions,

$$\begin{aligned}\frac{\partial}{\partial t_1}P(k,t_1,t_2)dt_1&=P(k,t_1+dt_1,t_2)-P(k,t_1,t_2)\\ &=(1(t_1),k,e(t_2))\\ &\quad-(1(t_1),k-1,e(t_2)) \ .\end{aligned}$$

This equation is correct to first order in dt_1. The expression, when summed up over k, leads to the expression of the generalized joint probability distribution $(1(t_1),m,e(t_2))$ as a linear combination of time derivatives of $P(k,t_1,t_2)$, that is,

$$(1(t_1),m,e(t_2))=\sum_{k=0}^{m}\frac{\partial}{\partial t_1}P(k,t_1,t_2)dt_1 \ , \quad (1)$$

where we have used the assumption that $(1(t_1),-1,1(t_2))=0$. In a similar manner, the first-order time derivative of $P(n,t_1,t_2)$ with respect to t_2 can be expressed as

$$\begin{aligned}\frac{\partial}{\partial t_2}P(k,t_1,t_2)dt_2&=P(k,t_1,t_2+dt_2)-P(k,t_1,t_2)\\ &=[(1(t_1),k-2,1(t_2))\\ &\quad-(1(t_1),k-1,1(t_2))]\\ &\quad+[(0(t_1),k-1,1(t_2))\\ &\quad-(0(t_1),k,1(t_2))] \ .\end{aligned}$$

This expression is correct to first order in dt_2. The expression, when summed up over k, leads to the following expression:

$$(1(t_1),m-1,1(t_2))+(0(t_1),m,1(t_2))=-\sum_{k=0}^{m}\frac{\partial}{\partial t_2}P(k,t_1,t_2)dt_2 \ . \quad (2)$$

The joint probability distribution $(1(t_1),m-1,1(t_2))$ appearing as the first term on the left-hand side of Eq. (2) may be regarded as a generalized form of the probability distribution of time intervals between two consecutive photoelectrons, including the latter probability distribution as a special case $(m=1)$. This generalized joint probability distribution can be expressed as a double summation of the second-order time derivative of $P(n,t_1,t_2)$. To obtain this expression let us express the second-order time derivative of $P(n,t_1,t_2)$ in terms of generalized joint probability distributions,

$$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}P(k,t_1,t_2)dt_1dt_2$$
$$=P(k,t_1+dt_1,t_2+dt_2)-P(k,t_1+dt_1,t_2)$$
$$-P(k,t_1,t_2+dt_2)+P(k,t_1,t_2)$$
$$=[(1(t_1),k-1,1(t_2))$$
$$-(1(t_1),k,1(t_2))]$$
$$-[(1(t_1),k-2,1(t_2))$$
$$-(1(t_1),k-1,1(t_2))]\ .$$

By taking a double summation of this equation, we obtain the required expression as follows:

$$(1(t_1),n,1(t_2))$$
$$=-\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}P(k,t_1,t_2)dt_1dt_2\ . \quad (3)$$

The joint probability distribution $(0(t_1),n,1(t_2))$ can be obtained by the substitution of Eq. (3) in Eq. (2),

$$(0(t_1),n,1(t_2))=-\sum_{k=0}^{n}\frac{\partial}{\partial t_2}P(k,t_1,t_2)dt_2$$
$$+\sum_{m=0}^{n-1}\sum_{k=0}^{m}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}P(k,t_1,t_2)$$
$$\times dt_1dt_2\ . \quad (4)$$

Further, using Eqs. (3) and (4), we obtain

$$(e(t_1),n,1(t_2))=(0(t_1),n,1(t_2))$$
$$+(1(t_1),n,1(t_2))$$
$$=-\sum_{k=0}^{n}\frac{\partial}{\partial t_2}P(k,t_1,t_2)dt_2$$
$$-\sum_{k=0}^{n}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}P(k,t_1,t_2)dt_1dt_2\ . \quad (5)$$

Equations (1) and (3)–(5) are the generalized expressions of joint probability distributions of time intervals between two consecutive photoelectrons and residual waiting times and other associated joint probability distributions. The last terms in Eqs. (4) and (5) are of second order in dt_1 and dt_2, and therefore can be neglected in comparison with other terms which are of first order in these small quantities. Thus we will omit these terms in the discussion below.

III. EXPRESSIONS OF THE SECOND ORDER AND ITS ASSOCIATED JOINT PROBABILITY DISTRIBUTIONS OF PHOTOELECTRONS IN TERMS OF $P(n,t_1,t_2)$

Now, let us use the obtained equations to find expressions of the second-order and its associated joint probability distributions of photoelectrons in terms of $P(n,t_1,t_2)$. In general, correlation functions of all orders are needed for the complete description of statistical properties of the fluctuating system. However, considerable attention has been focused upon the second-order and its associated intensity correlation functions, since they have played a fundamental role in revealing the statistical properties of an optical field in the fluctuating system; it discriminates pseudothermal[2] and gas discharge[3] light from coherent light and is used for the study of laser operation near threshold,[16–18] photon bunching,[19] antibunching,[20] and squeezed states.[21] The second-order and its associated correlation functions can be characterized with the following twofold and associated joint probability distributions.

(1) $P_{s11}(t_1,t_2)dt_2$: the probability of the second photoelectron's being registered during t_2 to t_2+dt_2, given that the first photoelectron was registered at t_1.

(2) $P_{se1}(t_1,t_2)dt_2$: the probability of the first photoelectron's being registered during t_2 to t_2+dt_2 after an arbitrary time t_1.

(3) $P_{c11}(t_1,t_2)dt_2$: the probability of another photoelectron's being registered during t_2 to t_2+dt_2, given that a photoelectron was registered at t_1.

(4) $P_{ce1}(t_1,t_2)dt_2$: the probability of a photoelectron's being registered during t_2 to t_2+dt_2 after an arbitrary time t_1.

$P_{se1}(t_1,t_2)$ is sometimes called the probability distribution of the residual waiting time, or forward recurrence time.[22] $P_{c11}(t_1,t_2)$ is proportional to the second-order intensity correlation function. By comparing $P_{s11}(t_1,t_2)$ with $P_{se1}(t_1,t_2)$ or $P_{c11}(t_1,t_2)$ with $P_{ce1}(t_1,t_2)$ we can obtain information on the second-order correlation of photoelectrons. These probabilities can be expressed in terms of the generalized joint probability distributions introduced in Sec. II. To show this, let us consider the joint probability distribution $(1(t_1),0,1(t_2))$. This probability distribution is the product of two independent probability distributions, that is, (i) the probability $w(t_1)dt_1$ of an initial count being registered during t_1 to t_1+dt_1, where $w(t_1)$ is the average number of counts at time t_1 per unit time, and (ii) the probability $P_{s11}(t_1,t_2)dt_2$ of the second count being registered during t_2 to t_2+dt_2, given that the first count was registered at t_1,

$$w(t_1)dt_1P_{s11}(t_1,t_2)dt_2\ .$$

By equating this probability distribution to $(1(t_1),0,1(t_2))$ and dividing both sides by $w(t_1)dt_1dt_2$, we obtain

$$P_{s11}(t_1,t_2)=\frac{1}{w(t_1)dt_1dt_2}(1(t_1),0,1(t_2))\ .$$

In a similar manner, we can express the remaining three probability distributions in terms of the generalized joint probability distributions,

$$P_{se1}(t_1,t_2)=\frac{1}{dt_2}(e(t_1),0,1(t_2))\ ,$$

$$P_{c11}(t_1,t_2)=\frac{1}{w(t_1)dt_1dt_2}\sum_{n=0}^{\infty}(1(t_1),n,1(t_2)) ,$$

$$P_{ce1}(t_1,t_2)=\frac{1}{dt_2}\sum_{n=0}^{\infty}(e(t_1),n,1(t_2)) .$$

The right-hand sides of these equations, when replaced by Eqs. (3) and (5), yield the desired expressions in terms of the time derivatives of $P(n,t_1,t_2)$,

$$P_{s11}(t_1,t_2)=-\frac{1}{w(t_1)}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}P(0,t_1,t_2) , \tag{6}$$

$$P_{se1}(t_1,t_2)=-\frac{\partial}{\partial t_2}P(0,t_1,t_2) , \tag{7}$$

$$P_{c11}(t_1,t_2)=-\frac{1}{w(t_1)}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\times P(k,t_1,t_2) , \tag{8}$$

$$P_{ce1}(t_1,t_2)=-\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\partial}{\partial t_2}P(k,t_1,t_2) , \tag{9}$$

where we have omitted terms of higher order in dt_i. It should be noted that $P_{ce1}(t_1,t_2)$ does not, in general, equal the average number of photoelectrons since the process we are considering is not necessarily stationary. For stationary processes whose statistical properties are not changed by time displacements, time distributions depend on time intervals alone. All corresponding relations in stationary processes can be obtained if only we replace $P(n,t_1,t_2)$ by $P(n,t_2-t_1)$ and $w(t_1)$ by the total average number of counts w per unit time. As special cases, we derive time distributions of doubly correlated photoelectrons and other associated distributions in stationary processes to demonstrate that our theory leads to results consistent with the previous work. For stationary processes $P_{s11}(t_1,t_2)$, $P_{se1}(t_1,t_2)$, $P_{c11}(t_1,t_2)$, $P_{ce1}(t_1,t_2)$, and $P(n,t_1,t_2)$ depend only on time difference t_2-t_1 and can be written as $P_{s11}(\tau)$, $P_{se1}(\tau)$, $P_{c11}(\tau)$, $P_{ce1}(\tau)$, and $P(n,\tau)$, respectively, where $\tau\equiv t_2-t_1$. The expressions of these time distributions in terms of $P(n,\tau)$ can be obtained from Eqs. (6)–(9) by replacing $\partial/\partial t_1$ by $-\partial/\partial\tau$, $\partial/\partial t_2$ by $\partial/\partial\tau$, $w(t_1)$ by w, and $P(n,t_1,t_2)$ by $P(n,\tau)$,

$$P_{s11}(\tau)=\frac{1}{w}\frac{d^2}{d\tau^2}P(0,\tau) , \tag{10}$$

$$P_{se1}(\tau)=-\frac{d}{d\tau}P(0,\tau) , \tag{11}$$

$$P_{c11}(\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{w}\frac{d^2}{d\tau^2}P(k,\tau) , \tag{12}$$

$$P_{ce1}(\tau)=\sum_{m=0}^{\infty}\sum_{k=0}^{m}-\frac{d}{d\tau}P(k,\tau) . \tag{13}$$

Equations (10) and (11) are identical to those obtained by Glauber.[14] Since the twofold joint probability distribution $P_{c11}(\tau)$ is proportional to the second-order intensity correlation function, Eq. (12) shows an exact expression of the second-order intensity correlation function in terms of $P(n,\tau)$.

We end this section by indicating two interrelations between $P_{s11}(\tau)$ and $P_{c11}(\tau)$, and between $P_{se1}(\tau)$ and $P_{ce1}(\tau)$. From Eqs. (10) and (12) we obtain

$$P_{c11}(\tau)=P_{s11}(\tau)+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\frac{1}{w}\frac{d^2}{d\tau^2}P(k,\tau) . \tag{14}$$

From this equation we see that the two joint probability distributions $P_{s11}(\tau)$ and $P_{c11}(\tau)$ do not, in general, include the same information on photoelectron statistics. From Eqs. (11) and (13) we obtain an interrelation between $P_{se1}(\tau)$ and $P_{ce1}(\tau)$ as follows:

$$P_{ce1}(\tau)=P_{se1}(\tau)+\sum_{m=1}^{\infty}\sum_{k=0}^{m}-\frac{d}{d\tau}P(k,\tau) . \tag{15}$$

From this equation we also see that the two joint probability distributions $P_{se1}(\tau)$ and $P_{ce1}(\tau)$ do not, in general, include the same information on photoelectron statistics.

IV. APPLICATIONS TO QUANTUM OPTICS

We apply Eqs. (10)–(13) to the fundamental statistics, that is, the Poisson and Bose-Einstein statistics, to demonstrate the validity of our theory. For the Poisson statistics, the probability distribution $P(n,\tau)$ in which n photoelectrons are registered during τ is given by

$$P(n,\tau)=\exp(-w\tau)\frac{(w\tau)^n}{n!} , \tag{16}$$

where w is the average number of photoelectrons per unit time. Substituting (16) in (10)–(13), we obtain

$$P_{s11}(\tau)=P_{se1}(\tau)=w\exp(-w\tau) , \tag{17}$$

$$P_{c11}(\tau)=P_{ce1}(\tau)=w . \tag{18}$$

The expressions of $P_{s11}(\tau)$ and $P_{c11}(\tau)$ are consistent with the experimental results of Arecchi *et al.*[2] The facts that $P_{s11}(\tau)$ and $P_{c11}(\tau)$ are equal to $P_{se1}(\tau)$ and $P_{ce1}(\tau)$, respectively, indicate that no two photoelectrons in the Poisson process are correlated.

On the other hand, for the Bose-Einstein statistics two photoelectrons are correlated or bunched if their time interval is comparable to or shorter than the coherence time of light. This so-called "bunching" effect of photoelectrons can be visualized by comparison of $P_{s11}(\tau)$ to $P_{ce1}(\tau)$ or $P_{c11}(\tau)$ to $P_{ce1}(\tau)$. The photocount distribution $P(n,\tau)$ for the Bose-Einstein statistics is given by[9]

$$P(n,\tau)=\frac{(w\tau)^n}{(1+w\tau)^{n+1}} , \tag{19}$$

where τ is required to be much shorter than the coherence time of light. The substitution of Eq. (19) into Eqs. (10) and (11) yields the former pair of the joint probability distributions,

$$P_{se1}(\tau)=\frac{w}{(1+w\tau)^2} , \tag{20}$$

$$P_{s11}(\tau)=\frac{2w}{(1+w\tau)^3} . \tag{21}$$

These equations were first obtained by Glauber.[14] For short time intervals such that $w\tau \ll 1$, $P_{s11}(\tau)$ is twice as large as $P_{se1}(\tau)$, which shows that photons which obey the Bose-Einstein statistics tend to be registered in bunches. The latter pair of the joint probability distributions can be obtained by the substitution of Eq. (19) into Eqs. (12) and (13). After some algebraic manipulation, we obtain

$$P_{c11}(\tau)=2w \ , \tag{22}$$

$$P_{ce1}(\tau)=w \ . \tag{23}$$

$P_{c11}(\tau)$ is twice as large as $P_{ce1}(\tau)$ so long as Eq. (19) holds,

$$P_{c11}(\tau)=2P_{ce1}(\tau) \ . \tag{24}$$

Equation (24) is another expression of photon bunching in a Gaussian field well known in the semiclassical theory,[23]

$$P_{c11}(\tau \ll \tau_c)=2P_{c11}(\tau' \gg \tau_c) \ , \tag{25}$$

where τ is taken much shorter than the coherence time τ_c of light, while τ' is taken much longer than τ_c. It is noted that there is an apparent difference between Eqs. (24) and (25)—that is, Eq. (24) connects two probability distributions for the same time interval under different initial conditions, while Eq. (25) connects two probability distributions under the same initial condition but for different time intervals. However, both equations express the same content from the different points of view: from the standpoint of conditional probability $P_{c11}(\tau)$ is an initially conditioned probability distribution which can carry phase information on the two relevant detected photons, while $P_{ce1}(\tau)$ is not initially conditioned so that it cannot carry the phase information. From the standpoint of time, this situation can be restated as follows: two photons can carry information on phase correlation if the time interval between the two photons is not much longer than the coherence time of light.

Finally, let us derive the expressions of time distribution $P_{s11}(\tau)$, $P_{se1}(\tau)$, $P_{c11}(\tau)$, and $P_{ce1}(\tau)$ in terms of the cycle-averaged intensity of light $I(t)$ using Eqs. (6)–(9). Since these expressions are well known,[19,24] this demonstration makes it clear that our theory is consistent with the previous work. The probability $P(1,t,t+dt)$ of a photoelectron's being registered during t to $t+dt$ is proportional to the cycle-averaged intensity of light at time t,[25]

$$P(1,t,t+dt)=\alpha I(t)dt \ , \tag{26}$$

where α is the detection efficiency. By a simple algebraic manipulation,[26] we find that Eq. (26) leads to the Poisson distribution in the number of photoelectrons,

$$P(n,t_1,t_2)=\exp\left[-\alpha\int_{t_1}^{t_2} I(t)dt\right]\frac{\left[\alpha\int_{t_1}^{t_2} I(t)dt\right]^n}{n!} \ . \tag{27}$$

However, this is not the probability distribution that can be obtained experimentally,[9] for the intensity $I(t)$ generally fluctuates at random. The ensemble average of the right-hand side of Eq. (27) is an observable,

$$P(n,t_2-t_1)=\left\langle \exp\left[-\alpha\int_{t_1}^{t_2} I(t)dt\right]\frac{\left[\alpha\int_{t_1}^{t_2} I(t)dt\right]^n}{n!}\right\rangle \ , \tag{28}$$

where the symbol $\langle\ \rangle$ denotes the operation of the ensemble average over the integrated intensity distribution $\int_{t_1}^{t_2} I(t)dt$, and we assume the stationarity of the field. Substituting Eq. (28) in Eqs. (6)–(9), and replacing t_2-t_1 by τ, we obtain

$$P_{s11}(\tau)=\frac{\alpha\left\langle I(t)I(t+\tau)\exp\left[-\int_t^{t+\tau} I(t')dt'\right]\right\rangle}{\langle I(t)\rangle} \ , \tag{29}$$

$$P_{se1}(t)=\alpha\left\langle I(t)\exp\left[-\int_t^{t+\tau} I(t')dt'\right]\right\rangle \ , \tag{30}$$

$$P_{c11}(t)=\frac{\alpha\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle} \ , \tag{31}$$

$$P_{ce1}(t)=\alpha\langle I(t)\rangle=w \ . \tag{32}$$

All these results are in agreement with the previous work,[19,24] which shows the validity of our theory.

V. CONCLUSIONS

We present a systematic theory which relates various joint probability distributions of photoelectrons to one of the most commonly used probability distributions, $P(n,t_1,t_2)$, in which n photoelectrons are registered between t_1 and t_2. The expressions obtained are used to find two pairs of expressions of the second-order and its associated intensity correlation functions in terms of $P(n,t_1,t_2)$. Two interrelations between $P_{s11}(\tau)$ and $P_{c11}(\tau)$, and between $P_{se1}(\tau)$ and $P_{ce1}(\tau)$ are indicated. Some applications to quantum optics are also made, to obtain results in agreement with the previous work.

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[1]R. Hanbury Brown and R. Q. Twiss, Nature (London) **177**, 27 (1956); **178**, 1046 (1956); **178**, 1447 (1956); Proc. R. Soc. London, Ser. A **242**, 300 (1957); **243**, 291 (1957); **248**, 199 (1958); **248**, 222 (1958).

[2]F. T. Arecchi, E. Gatti, and A. Sona, Phys. Lett. **20**, 27 (1966).

[3]B. L. Morgan and L. Mandel, Phys. Rev. Lett. **16**, 1012 (1966).

[4]F. Davidson and L. Mandel, Phys. Lett. **27A**, 579 (1968).

[5]F. Davidson, Phys. Rev. **185**, 446 (1969).

[6]M. Corti and V. Degiorgio, Opt. Commun. **11**, 1 (1974); Phys. Rev. A **14**, 1475 (1976).
[7]E. Wolf, in *Proceedings of the Symposium on Optical Masers, Brooklyn, 1963* (Polytechnic, New York, 1963), p. 29.
[8]R. J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).
[9]L. Mandel, Proc. Phys. Soc. London **72**, 1037 (1958); **74**, 233 (1959); **81**, 1104 (1963).
[10]P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).
[11]M. Scully, in *Quantum Optics,* edited by R. J. Glauber (Academic, New York, 1969), p. 586.
[12]F. Ghielmetti, Nuovo Cimento B **35**, 243 (1976).
[13]B. E. A. Saleh, *Photoelectron Statistics* (Springer, Berlin, 1978).
[14]R. J. Glauber, in *Proceedings of the Symposium on Modern Optics, Brooklyn, 1967,* edited by J. Fox (Polytechnic, New York, 1967), p. 1.
[15]Note that these probabilities implicitly include a time interval during which each event occurs. We use such nonstandard notation to make mathematical expressions on joint probability distributions concise.
[16]A. W. Smith and J. A. Armstrong, Phys. Rev. Lett. **16**, 1169 (1966).
[17]F. T. Arecchi, G. S. Rodari, and A. Sona, Phys. Lett. **25A**, 59 (1967).
[18]C. Freed and H. A. Haus, IEEE J. Quantum Electron. **QE-2**, 90 (1966).
[19]L. Mandel, in Ref. 14, p. 143.
[20]D. F. Walls, Nature (London) **280**, 451 (1979).
[21]D. F. Walls, Nature (London) **306**, 141 (1983).
[22]B. E. A. Saleh, J. Phys. A **6**, L161 (1973).
[23]C. L. Mehta and E. Wolf, Proc. Phys. Soc. London, Sect. A **134**, 1143 (1964); **134**, 1149 (1964).
[24]G. J. Troup, Proc. Phys. Soc. London **87**, 361 (1966).
[25]L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc. London **84**, 435 (1964).
[26]R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1973).