Perturbative analytical study of long-pulse free-electron lasers: Mode competition for a high-gain Compton regime

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The mode competition in a long-pulse free-electron laser operating in a high-gain Compton regime is analyzed. Due to the strong crossed saturation, a dominant mode is able to suppress competing modes, leading to single-mode operation. This is explicitly shown by a stability analysis of the system of coupled integrodifferential equations describing the evolution of two competing modes, of which only one survives at saturation. In the course of the study the radiation field evolution in the small-signal regime is treated in a unified manner for low and high gain.

I. INTRODUCTION

In a recent paper[1] we have presented a perturbative analysis of a free-electron laser (FEL) with long pulses in the small-gain regime. Our main interest in that paper was the study of mode competition in such a FEL.

As was shown in the paper alluded to, through strong-mode competition the dominant mode is able to suppress other modes and the result is single-mode operation. The mode competition is, of course, a nonlinear phenomenon and terms up to third order in the field were taken into account. For simplicity, however, the study was restricted to small-gain devices. In the present paper, we lift that restriction and extend the analysis to include FEL's operating in a high-gain Compton regime. This is of practical interest since the first reported single-mode performance[2] was in a FEL operating in the borderline between low and high gain.

In the course of our analysis we present a formula for the small-signal amplitude which, besides containing the familiar low-gain expression, is also valid for high gain. The small-signal (linear) problem is treated by solving a Volterra integrodifferential equation by the Laplace transform method. Afterwards, with some mild approximation, the analysis is extended to high intensities for the single-mode case. The treatment is completely analytical except for two integrals that have to be calculated numerically. Up to third order the field tends to a limiting value.

The high-gain high-intensity two-mode problem is treated next. An analysis of the coupled Volterra equations shows the stable solutions to be single mode. The problem of sideband (or synchrotron) instabilities is not addressed in the present paper and will be treated analytically elsewhere. The results of this paper apply then to FEL's where sidebands do not appear either for dynamical reasons or because dispersive elements have been placed in the resonator.

The treatment in this paper is one dimensional, and the basic equations and notation are introduced in Sec. II. Section III is devoted to the derivation of a small-field expression valid for any gain regime. The saturation functions derived in Ref. 1 are summarized in Sec. IV. These saturation functions are used in Sec. V in obtaining expressions for the field which are valid in the general case including high gain and high intensities. Also for the general case, the two-mode problem is analyzed in Sec. VI.

II. ONE-DIMENSIONAL FEL EQUATIONS

For an undulator with constant period $\lambda_0=2\pi/k_0$, length L, and uniform peak field B_0, we can write the vector potential

$$\mathbf{A}_u=i\mathbf{n}B_0e^{-ik_0L\tau}/2k_0+\text{c.c.}\ , \tag{1}$$

where $\mathbf{n}$ is the polarization vector [$\mathbf{n}=\hat{\mathbf{x}}$ for linear and $\mathbf{n}=(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}})$ for circular polarization] and $\tau=z/L$ is the normalized longitudinal position coordinate. In such an undulator the electrons acquire the transverse velocity

$$\mathbf{v}_T=i\mathbf{n}(cK/2\gamma)e^{-ik_0L\tau}+\text{c.c.},\quad K=eB_0/mck_0\ . \tag{2}$$

In order to simplify the analysis we will consider the one-dimensional problem in which the radiation field can be expressed as a superposition of plane-wave modes. In such a case the electric field of the radiation can be written as

$$\mathbf{E}=\frac{\mathbf{n}mc^2}{2eL}\sum_q a_q(\tau)e^{i[k_qL\tau-\omega_q t+\phi_q(\tau)]}+\text{c.c.}\ , \tag{3}$$

where $a_q(\tau)$ is the adimensional amplitude for mode q and $k_q=\omega_q/c$ is its corresponding wave vector. In the particular case of an oscillator with angular frequency spacing between longitudinal modes $\omega_f=c\pi/L_0$, all mode frequencies are integer multiples of ω_f, as in $\omega_q=q\omega_f$.

Only Compton regime FEL's will be considered neglecting space-charge effects. In the slowly varying amplitude and phase approximation the wave equation for the vector potential (3) reduces to

$$\sum_q e^{i(k_q L\tau-\omega_q t+\phi_q)}[\partial_\tau a_q(\tau)+ia_q(\partial_\tau\phi_q+\alpha/2)] \\ +(2-p)(c.c.)=-(\mu_0 eL^2/mcp)\mathbf{n}^*\cdot\mathbf{J}\ , \quad (4)$$

where α is the mode loss per pass, and $p=|\mathbf{n}|^2$ is 1 for linear and 2 for circular polarization. $\mathbf{J}$, the transverse electron current driving the FEL, will be discussed in detail in the following.

In keeping with our one-dimensional treatment let us consider an electron beam with an initial uniform density $\rho=\rho_L\rho_T$, i.e., ρ_L planes per unit length with ρ_T electrons per unit area in each plane. The transverse current density driving the FEL is

$$\mathbf{J}(t,x)=-e\mathbf{v}_T(\tau)\sum_j \delta^3[\mathbf{x}-\mathbf{x}_j(t)] \\ \Longrightarrow -e\rho_T\mathbf{v}_T\sum_j \delta[L\tau-z_j(t)]\ . \quad (5)$$

The $\Longrightarrow$ in Eq. (4) indicates the transition from the general three-dimensional situation to the simplified one-dimensional problem.

For $K^2\ll 1$ one can disregard harmonic radiation. In such a case the current can also be decomposed in the same way as the electric field (2), namely,

$$\mathbf{J}=-\mathbf{n}(mc/e\mu_0L^2)\sum_q J_q e^{i(k_qL\tau-\omega_q t+\phi_q)}+\text{c.c.} \quad (6)$$

With this definition, J_q are the adimensional sources for the amplitudes and phases as in

$$\partial_\tau a_q+ia_q(\partial_\tau\phi_q-\alpha/2)=J_q\ . \quad (7)$$

As Fourier coefficients in the expansion (6), the current amplitudes J_q are obtained from Eq. (4) as

$$J_q=-(ec\mu_0L^2/2pmcL_0) \\ \times\int dt\, e^{i[\omega_q t-(k_q+k_0)L\tau-\phi_q]}\mathbf{n}^*\cdot\mathbf{J} \\ =i(e^2\mu_0c\rho_TKL^2/4mc\gamma L_0) \\ \times\sum_j e^{i[\omega_q t_j(\tau)-(k_q+k_0)L\tau]}\ , \quad (8)$$

where the integration is over a fundamental period $T_f=2\pi/\omega_f$ and the sum is over planes of particles with injection times uniformly distributed over this period.

The electron that enters the undulator at time t_{j0} will reach the position $z=L\tau$ at the time

$$t_j(\tau)=t_{j0}+(L\tau/v_0)-\xi_j(\tau)L/C\ , \quad (9)$$

in which v_0 is the initial electron velocity[3] and the small time shift $\xi_j(t)$, due to the bunching, is obtained from the pendulum equation[4]

$$d^2\xi_j/d\tau=(L/c^2)d^2z/dt^2 \\ =i\Lambda\sum_q(c/L\omega_R)a_q\psi_{jq}^*e^{iR}jq+\text{c.c.}\ , \quad (10)$$

with

$$\omega_R=2\gamma^2ck_0/(1+pK^2/2)\ , \\ \mu_q=L[(k_q+k_0)-\omega_q/v_0],\quad \Lambda=\pi pKN/\gamma^2\ , \quad (11a) \\ R_{jq}=\mu_q\tau-\omega_qt_{j0}+\phi_q,\quad \psi_{jq}=\exp(-iL\omega_q\xi_j/c)\ .$$

The parameter μ_q measures the detuning from the resonant frequency ω_R. Using Eqs. (9) and (11a) we can write the current amplitudes of Eq. (8) as

$$J_q=\sum_i J_q^{(i)}=iJ_0(\omega_R/\omega_q\rho_LL_0)\sum_j\psi_{jq}(\tau)e^{-iR_{jq}(\tau)}\ , \\ J_0=\frac{e^2c\mu_0L\rho K}{4m\gamma\omega_R}\ , \quad (11b)$$

where we have indicated an expansion in powers i of the radiation field contained in the $\xi_j(\tau)$.

III. LINEAR THEORY

In the present section the one-dimensional FEL linear problem is treated exactly. The results are small-signal gain and frequency pulling formulas applicable to the whole range of gain regimes: from low to high.

The current to first order in the radiation field was obtained in Ref. 1 and it reads

$$J_q^{(1)}=(J_0/\rho_LL_0)\sum_j e^{-iR_{jq}(\tau)}(\omega_q/\omega_R)\sum_b\int_0^\tau d\tau_1\int_0^{\tau_1}d\tau_2a_b(\tau_2)\sin R_{jb}(\tau_2) \\ =ie^{i\phi(\tau)}G_0\int_0^\tau d\tau_1\int_0^{\tau_1}d\tau_2a_b(\tau_2)e^{i\phi(\tau_2)}e^{-i\mu(\tau-\tau_2)}\ , \quad (12)$$

with

$$G_0=\pi e^2\rho pK^2L^2N/2\epsilon_0mc^2\gamma^3\ . \quad (13)$$

Equation (12) is the linear source for the reduced wave equation (7) which, with $\chi\equiv a\exp i\phi$, can then be written as

$$\frac{d\chi(\tau)}{d\tau}=iG_0\int_0^\tau d\tau_1\int_0^{\tau_1}d\tau_2e^{-i\mu(\tau-\tau_2)}\chi(\tau_2)\ . \quad (14)$$

Twice integrating by parts Eq. (14), and calling $\Xi\equiv\chi'$ and $\chi_0\equiv\chi(0)$, one arrives at

$$\Xi(\tau)=iG_0\left[\chi_0 g(\tau)+\int_0^\tau d\tau_1 g(\tau-\tau_1)\Xi(\tau_1)\right], \tag{15}$$

where

$$g(\tau)=[(1+i\mu\tau)e^{-i\mu\tau}-1]/\mu^2 . \tag{16}$$

Equation (15) is a Volterra integral equation[5] for Ξ, with the kernel $g_1(\tau)$. This type of equation can be solved using the Laplace transforms[6]

$$\Xi(k)=\int_0^\infty d\tau e^{-k\tau}\Xi(\tau) ,$$
$$g(k)=\int_0^\infty d\tau e^{-k\tau}g(\tau)=\frac{1}{k(k+i\mu)^2} . \tag{17}$$

Then, the convolution property

$$\int_0^\infty dk\, e^{-k\tau}\int_0^\tau d\tau_1 g(\tau-\tau_1)\Xi(\tau_1)=g(k)\Xi(k) \tag{18}$$

reduces the integral equation (15) to the algebraic

$$\Xi(k)=iG_0[\chi_0 g(k)+g(k)\Xi(k)] . \tag{19}$$

$\Xi(k)$ is then

$$\Xi(k)=iG_0\chi_0 g(k)/[1-iG_0 g(k)] . \tag{20}$$

The inverse Laplace transform of this expression provides a solution of the integral equation (15) in the form

$$\Xi(\tau)=(G_0\chi_0/2\pi)\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dk e^{-k\tau}g(k)/[1-iG_0 g(k)] . \tag{21}$$

And a further integration in τ yields the fields

$$\chi(\tau)=(\chi_0/2\pi i)\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dk\, e^{k\tau}\frac{(k+i\mu)^2}{k(k+i\mu)^2-iG_0} . \tag{22}$$

This integral can be obtained by the method of residues. For this we need the roots of the denominator in the integrand of Eq. (22). These roots can be written as

$$k_1=-(2i/3)\sin\{\arcsin[\tfrac{1}{3}(1-27G_0/2\mu^3)]\}-2i\mu/2 ,$$
$$k_2=-(2i/3)\sin\{\arcsin[2\pi/3-\tfrac{1}{3}(1-27G_0/2\mu^3)]\}$$
$$-2i\mu/3 , \tag{23}$$
$$k_3=-(2i/3)\sin\{\arcsin[4\pi/3-\tfrac{1}{3}(1-27G_0/2\mu^3)]\}$$
$$-2i\mu/3 ,$$

where arcsin is a real angle for $\mu\geq(27G_0/4)^{1/3}$. For $|x|>1$ the analytic extension

$$\arcsin(x)=\pi/2-\ln[x+(x^2-1)^{1/2}]$$

is to be used. For $\mu<(27G_0/4)^{1/3}$ this leads to

$$k_1=\Omega_1+i\Omega_2-2i\mu/3 ,$$
$$k_2=-(1+\mu/|\mu|)\Omega_1/2$$
$$+i(3\mu/|\mu|-1)\Omega_2/2-2i\mu/3 , \tag{24}$$
$$k_2=-(1-\mu/|\mu|)\Omega_1/2$$
$$-i(3\mu/|\mu|+1)\Omega_2/2-2i\mu/3 ,$$

where

$$\Omega_1=(\sqrt{3}/2)[(G_0/2-(\mu/3)^3+\{G_0[G_0/4-(\mu/3)^3]\}^{1/2})^{1/3}-(G_0/2-(\mu/3)^3-\{G_0[G_0/4-(\mu/3)^3]\}^{1/2})^{1/3}] ,$$
$$\Omega_2\doteq\tfrac{1}{2}[(G_0/2-(\mu/3)^3+\{G_0[G_0/4-(\mu/3)^3]\}^{1/2})^{1/3}+(G_0/2-(\mu/3)^3-\{G_0[G_0/4-(\mu/3)^3]\}^{1/2})^{1/3}] . \tag{25}$$

In terms of the k_i of Eq. (23) or (24), the result of the integral (22) can be written as

$$\chi(\tau)=ae^{i(\phi_0+\delta\phi)}=\frac{k_1+i\mu}{3k_1+i\mu}e^{ik_1\tau}+\frac{k_2+i\mu}{3k_2+i\mu}e^{ik_2\tau}$$
$$+\frac{k_3+i\mu}{3k_3+i\mu}e^{ik_3\tau} . \tag{26}$$

Let us now pause to comment on the low-gain (or small-G_0) limit in which case χ is slowly varying and can be taken outside the integral in Eq. (14). In this case that expression is reduced to

$$\frac{d\chi(\tau)}{d\tau}=iG_0 g(\tau)\chi(\tau) , \tag{27}$$

with the solution

$$\chi_s(\tau)\approx\chi_0 e^{G_0 g_2(\tau)} \quad \text{(low gain)} , \tag{28}$$

where

$$g_2(\tau)=i\int_0^\tau d\tau_1 g(\tau_1)$$
$$=[2-i\mu\tau-(2+i\mu\tau)e^{-i\mu\tau}]/\mu^3 . \tag{29}$$

Two functions have been traditionally associated with the gain per pass. One is $\Gamma=2\ln|\chi^{(1)}/\chi_0|$ and the other is the fractional power variation $G_f=|\chi^{(1)}/\chi_0|^2-1$. For low gain both are approximately equal to

$$G=2G_0\mathrm{Re}g_2$$
$$=2G_0\{2[1-\cos(\mu)]-\mu\sin(\mu)\}/\mu^3 \quad \text{(low gain)} . \tag{30}$$

As is well known, the maximum gain given by this expression is equal to $G=0.27G_0$.

Figure 1 (2) shows plots of Γ (G_f) normalized to 1, as a function of the detuning μ. The plots in both figures were obtained from the exact small-signal amplitude (26) for different values of G. It is seen that for low gain ($G\sim0.1$) the result is the familiar antisymmetric curve that follows from Eq. (30). For low gain the maximum is at 2.6. At higher gains the maximum shifts towards lower values of μ with a minimum of $\mu=1.55$ for $G_0\sim25$. After that, the maximum shifts slowly towards higher values of μ, and then remains at $\mu=1.70$ up to high values of G (~1000). This behavior is shown in Fig. 3. Figure 4 is a plot of Γ as a function of G. For any FEL

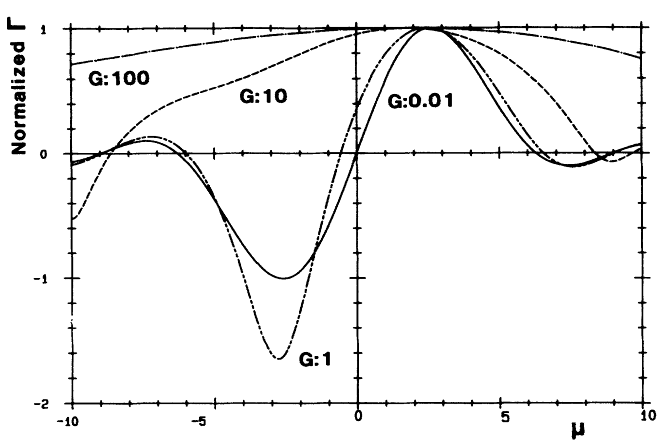

FIG. 1. Γ normalized to a maximum of 1 as a function of the detuning μ, for different values of G. The positive gain peak broadens as G increases. The actual value of Γ is obtained by multiplying the ordinate by the following factors: $G=0.01\rightarrow 9.979\times 10^{-3}$, $G=1\rightarrow 0.822$, $G=10\rightarrow 3.798$, $G=10^2 \rightarrow 10.319$.

operating in the Compton regime with a cold electron beam, this plot can be used to find the actual gain in terms of $G=0.27G_0$.

For very high gain ($G_0\geq 80$, $G\geq 20$) the field and gain expressions effectively reduce to

$$\chi_h=\frac{1}{3}\left[1+\frac{2i\mu\Omega_2/3}{\Omega_1+i(\Omega_2-\mu/3)}\right]e^{\Omega_1\tau+i(\Omega_2-2\mu/3)\tau} , \tag{31a}$$

$$G=\frac{1}{9}\left[1+\frac{(4\mu/3\sqrt{3})(\Omega_1+\mu^2/9G_0^{1/3})}{\Omega_1^2+(\Omega_1-\mu/\sqrt{3}+\mu^2/9G_0^{1/3})^2/3}\right]e^{2\Omega_1} , \tag{31b}$$

with

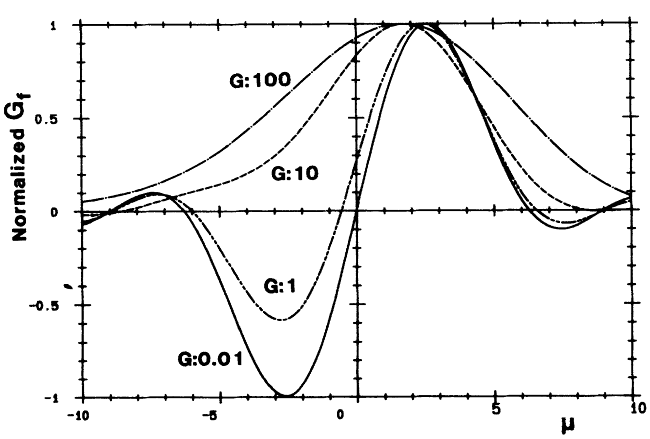

FIG. 2. Fractional gain per mass G_f normalized to a maximum of 1 as a function of the detuning μ, for different values of G. The actual value of G_f is obtained by multiplying the ordinate by the following factors: $G=0.01\rightarrow 9.98\times 10^{-3}$, $G=1\rightarrow 1.274$, $G=10\rightarrow 43.626$, $G=10^2\rightarrow 3.030\times 10^4$.

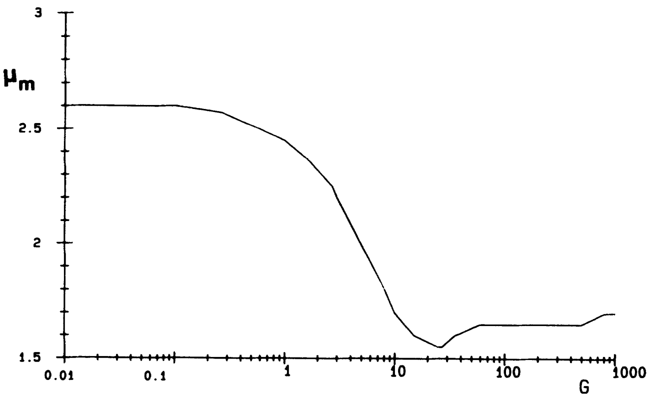

FIG. 3. Maximum gain detuning μ_m as a function of the strength parameter G.

$$\begin{aligned}\Omega_1&\approx\sqrt{3}[G_0^{1/3}-(\mu^2/9G_0^{1/3})(1+\mu/9G_0^{1/3})]/2 ,\\ \Omega_2&\approx[G_0^{1/3}+(\mu^2/9G_0^{1/3})(1-\mu/9G_0^{1/3})]/2 .\end{aligned} \tag{32}$$

Also for very high gain, the frequency shift $\delta\phi(\tau)$ is given by

$$\delta\phi=(\Omega_2-2\mu/3)\tau+\tan^{-1}\left[\frac{2\mu}{3}\,\frac{\Omega_2(\Omega_2-\mu/3)^2}{\Omega_1^2+\Omega_2^2+(\mu/3)^2}\right] . \tag{33}$$

At this point a comment on the meaning of high gain might be in order. Even when $G=100-1000$ corresponds to quite high gains in practical terms, mathematically, however, we are still outside the asymptotic region. We reach that asymptotic region by keeping the μ range fixed ($-10\leq\mu\geq 10$, as in our figures, say) and taking the limit $G_0\rightarrow\infty$. The denominator of Eq. (22) reduces to k^3-iG_0 in that limit, which shows that the dominant root tends to

$$k_1\rightarrow G_0^{1/3}(\sqrt{3}+i)/2 ,$$

independent of the detuning μ. Thus in the asymptotic

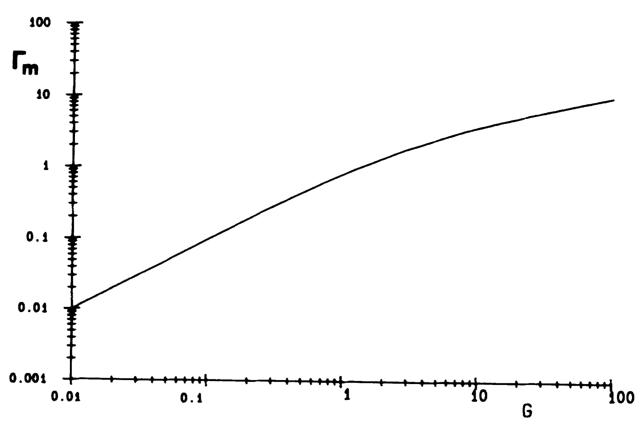

FIG. 4. Maximum gain Γ_m as a function of the parameter G for a FEL operating in the Compton regime with a cold electron beam.

region the gain curve is flat over a wide range of μ. This is consistent with the trend in Figs. 1 and 2 showing a gradual flattening of the gain curves as G (or G_0) increases.

For a FEL operating in the Compton (as opposed to the Raman) regime and for small signal, Eq. (26) provides an exact description of the field $\chi(\tau)$ at any position along the undulator. It is worth emphasizing that this formula yields, in the appropriate limits, the usual low- and high-gain field expressions.

An expression which explicitly looks like Eq. (28) at low gain and like Eq. (31b) at high gain can be obtained as follows. Let us start by introducing a modified coupling

$$G_\rho = 1/(1+\rho G_0) \quad (\rho > 1) , \tag{34}$$

such that $G_\rho \leq (1/\rho)$. Moreover, the field is divided into a small-gain part χ_s and the remainder x_R as in

$$\chi \equiv \chi_s + \chi_R , \quad \chi_s = \chi_{s0} e^{G_\rho g_2} . \tag{35}$$

For sufficiently large ρ, χ_s is slowly varying and satisfies an equation like (24). On the other hand, an equation for χ_R can be obtained by plugging Eq. (35) into the exact linear evolution equation (14). In that way we obtain

$$\frac{d\chi_R(\tau)}{d\tau} = \rho G_0 (d\chi_s/d\tau) + iG_0 \int_0^\tau d\tau_1 g(\tau-\tau_1)\chi_R(\tau_1) , \tag{36}$$

which translates into the Laplace transform equation

$$\chi_R(k) = \frac{\chi_{R0} + \rho G_0 [k\chi_s(k) - \chi_{s0}]}{k - iG_0 g(k)} . \tag{37}$$

The initial values χ_{s0} and χ_{R0} have to add up to χ_0. It is convenient to choose them as in

$$\chi_{s0} = \chi_0/(1+\rho G_0), \quad \chi_{R0} = \chi_0 \rho G_0/(1+\rho G_0) . \tag{38}$$

Using this in the inverse Laplace transform of Eq. (37) and adding χ_s of Eq. (35) we obtain the linear field valid for any gain regime

$$\chi(\tau) = \chi_s(\tau)/(1+\rho G_0) + \rho G_\rho \{1 + iG_\rho g(\tau)[\chi_s(\tau)/\chi_0]\}\chi_h , \tag{39}$$

with G_ρ as in Eq. (34), χ_h as in Eq. (31a), and a convenient value for ρ could be, for instance, $\rho = 2$. Equation (39) has the correct limits, i.e., $\chi \approx \chi_s$ for $G_0 \ll (1/\rho)$ and $\chi \approx \chi_h$ for $G_0 \gg (1/\rho)$. In closing this section let us repeat that Eq. (39) is an approximation while Eq. (26) is exact.

IV. SATURATION TERMS

In our previous work we have shown that the saturation terms can be written as

$$-\tfrac{1}{2} s(\mu_q, \tau) a_q \left[a_q^2 + 2 \sum_{b \neq q} a_b^2 \right] , \tag{40}$$

where $s = s_R + is_I$, simply related to the saturation function $S = S_R + iS_i$ to be introduced later, is given by

$$\begin{aligned} s_R(\mu,\tau) &= \sigma[3x + (1/8 - 1.25x^2)\sin x + (1.5 - x^2/8)\sin 2x \\ &\quad - x(3.13 + x^2/24)\cos x - \cos 2x]/\mu^6 , \\ s_I &= \sigma[-3 - 0.75x^2 + (1.5 - 1.25x^2)\cos x \\ &\quad - (3.88 + x^2/24)x \sin x + x \sin 2x \\ &\quad + (1.5 - x^2/8)\cos 2x]/\mu^6 , \\ x &= \mu\tau, \quad \sigma = \frac{e^2 \rho K L}{\epsilon_0 m c \gamma \omega_q} \left[\frac{L\Lambda\omega}{c} q \right]^3 . \end{aligned} \tag{41}$$

Adding Eq. (40) to the linear part (14) we will treat the nonlinear single-mode problem in Sec. V and the two-mode problem in Sec. VI.

V. NONLINEAR HIGH-GAIN SINGLE-MODE PROBLEM

When only a single mode is present, the second term inside the large parentheses in Eq. (40) is, of course, absent. Adding the remaining term to Eq. (14) one ends up with the single-mode evolution equations which, up to third order in the radiation field, read

$$\frac{d\chi(\tau)}{d\tau} = iG_0 \int_0^\tau d\tau_1 (\tau-\tau_1) e^{-i\mu(\tau-\tau_1)} \chi(\tau_1) - \tfrac{1}{2} s \chi(\tau)^2 \chi^*(\tau) . \tag{42}$$

Actually, in the nonlinear term the χ should be inside the integrals calculated in obtaining the saturation function s. However, the approximation is well justified since (1) for small fields the nonlinear term is negligible and it does not make much difference whether the χ are kept inside integrals or taken out, and (2) on the other extreme of very high fields the gain is small due to saturation effects, which makes χ slowly varying.

The nonlinear term in Eq. (42) can be rewritten as

$$\begin{aligned} \tfrac{1}{2} s \chi^2 \chi^* &= \tfrac{1}{2} s \chi^3 e^{-i2(\phi_0 + \delta\phi + \Delta\phi)} \\ &= \tfrac{1}{2} s \chi^3 e^{-i2(\phi_0 + \delta\phi)} + O(\chi^4) , \end{aligned} \tag{43}$$

where $\delta\phi$ is the linear frequency shift given by Eq. (33), the nonlinear frequency shift $\Delta\phi$ together with χ^3 yields a contribution of fourth order which will not be considered. That is, we will only retain the third-order nonlinear term and end up with the approximate equation

$$\frac{d\chi(\tau)}{d\tau} = iG_0 \int_0^\tau d\tau_1 (\tau-\tau_1) e^{-i\mu(\tau-\tau_1)} \chi(\tau_1) - \tfrac{1}{2} s e^{-i2(\phi_0+\delta\phi)} \chi(\tau)^3 . \tag{44}$$

As a first step towards solving this equation it is convenient to transform to the field Ψ defined by $\Psi \exp(-2\Omega\tau) = \chi^{-2}$, where $\Omega = \Omega_1 + i(\Omega_2 - 2\mu/3)$ is the dominant exponent in Eq. (28) and Ψ is slowly varying in comparison with the exponential. With this, Eq. (44) turns into

$$\frac{d\Psi(\tau)}{d\tau}=2\Omega\Psi-2iG_0\Psi^{3/2}(\tau)\int_0^\tau d\tau_1(\tau-\tau_1)e^{-(\Omega+i\mu)(\tau-\tau_1)}\Psi^{-1/2}(\tau_1)+s(\tau)e^{2\Omega\tau-i2(\phi_0+\delta\phi)} . \quad (45)$$

Being slowly varying $\Psi^{-1/2}(\tau_1)$ inside the integral can be approximated by

$$\Psi^{-1/2}(\tau_1)\approx\Psi^{-1/2}(\tau)+\tfrac{1}{2}(\tau-\tau_1)\Psi^{-3/2}(\tau)(d\Psi/d\tau) . \quad (46)$$

With this, Eq. (45) is transformed into the linear

$$\frac{d\Psi(\tau)}{d\tau}=f_1(\tau)\Psi(\tau)+f_0(\tau)e^{-i2\phi_0} , \quad (47)$$

where

$$f_0=\frac{s(\tau)(\Omega+i\mu)^3e^{2\Omega\tau-i2\delta\phi}}{(\Omega+1\mu)^3+2iG_0-iG_0)\{1+[1+(\Omega+i\mu)\tau]^2\}e^{-(\Omega+i\mu)\tau}} , \quad (48)$$

$$f_1=\frac{2(\Omega+i\mu)\{\Omega(\Omega+i\mu)^2-iG_0+iG_0[1+(\Omega+i\mu)\tau]e^{-(\Omega+i\mu)\tau}}{(\Omega+i\mu)^3+2iG_0-iG_0\{1+[1+(\Omega+i\mu)\tau]^2\}e^{-(\Omega+i\mu)\tau}} . \quad (49)$$

The solution of Eq. (47) is readily obtained. Written in terms of the square of the original field $\mathcal{X}$ it is

$$\mathcal{X}^2(\tau)=\frac{a_0^2e^{2i\phi_0}e^{2\Omega\tau+f_2(\tau)}}{1+a_0^2S(\tau)} , \quad (50)$$

with

$$f_2(\tau)=-\int_0^\tau d\tau_1 f_1(\tau_1) , \quad (51)$$

$$S(\tau)\equiv|S|e^{i\Sigma}=\int_0^\tau d\tau_1 f_0(\tau_1)e^{f_2(\tau_1)} . \quad (52)$$

A brief comment is in order now. Looking at Eq. (43), the appearance of $-2i\phi_0$ in the saturation term might seem rather unsatisfactory. In the end result of Eq. (50), however, that $-2i\phi_0$ canceled with the exponent of $\mathcal{X}_0^2$.

In general, the gain per pass $G=[a(1)/a_0]^1-1$ can be calculated in terms of

$$[a(\tau)/a_0]^2=\frac{e^{2\Omega_1\tau+\mathrm{Re}f_2(\tau)}}{[1+a_0^4|S(\tau)|^2+2a_0^2|S(\tau)|\cos(\Sigma)]^{1/2}} , \quad (53)$$

while the phase as position $z=L\tau$ is given by

$$\phi(\tau)=\phi_0+(\Omega_2-2\mu/3)\tau+\tfrac{1}{2}\mathrm{Im}f_2(\tau)-\tfrac{1}{2}\tan^{-1}\left[\frac{a_0^2|S|\sin(\Sigma)}{1+a_0^2|S|\cos(\Sigma)}\right] . \quad (54)$$

In the particular case of a small-gain situation, $S(\tau=1)$ is given in Ref. 1. At saturation the second term in the denominator of Eq. (50) is much larger than 1 and the field tends towards a limiting value according to

$$\mathcal{X}^2(\tau)\Longrightarrow\frac{e^{2i\phi_0}e^{2\Omega\tau+f_2(\tau)}}{S(\tau)} . \quad (55)$$

VI. TWO-MODE PROBLEM

We have previously[1] studied the two- (neighboring-) mode problem in a low-gain regime and showed that the long-pulse FEL dynamics leads to single-mode operation. In the present section we will extend the two-mode analysis to include high-gain regimes.

Let us consider a situation with two modes having almost equal detuning μ and gain-minus-losses Γ_R. Due to the interaction of Eq. (40) the fields satisfy the system of coupled equations

$$\mathcal{X}_i(\tau)=\mathcal{X}_{i0}+iG_0\int_0^\tau d\tau_1(\tau-\tau_1)^2e^{-i\mu(\tau-\tau_1)}\mathcal{X}_i(\tau_1)-s_2(\tau)\mathcal{X}_i(\tau)[\mathcal{X}_i^2+2\mathcal{X}_j^2] , \quad (56)$$

with $i,j=b,c$, $j\neq i$, $\mathcal{X}_{i0}\equiv\mathcal{X}_i(0)$ and

$$s_2(\tau)=\tfrac{1}{2}\int_0^\tau d\tau_1 s(\tau_1)e^{-i2[\phi_0+\delta\phi(\tau_1)]} \quad (57)$$

In Eq. (56) $\mathcal{X}_b$ and $\mathcal{X}_c$ are strongly coupled since the crossed saturation is twice as large as the self-saturation. It is convenient to work with $\mathcal{X}_\pm=(\mathcal{X}_b\pm\mathcal{X}_c)$ and $\mathcal{X}_{\pm0}\equiv(\mathcal{X}_{b0}\pm\mathcal{X}_{c0})$ which satisfy the weakly coupled equations

$$\mathcal{X}_\pm(\tau)=\mathcal{X}_{\pm0}+iG_0\int_0^\tau d\tau_1(\tau-\tau_1)^2e^{-i\mu(\tau-\tau_1)}\mathcal{X}_\pm(\tau_1)-s_2\mathcal{X}_\pm(\tau)[(1-\epsilon)\mathcal{X}_\pm^2+\epsilon\mathcal{X}_\mp^2] \quad (\epsilon=\tfrac{1}{4}) . \quad (58)$$

Taking advantage of the smallness of the parameter ϵ let us look for perturbative solutions of the form

$$\mathcal{X}_\pm=\Psi_\pm+\epsilon\,{}^1\Psi_\pm+\cdots , \quad (59)$$

where $\Psi_\pm$ are the solutions of Eq. (58) with $\epsilon=0$. Since the uncoupled equations are completely equivalently to the single-mode Eq. (44), $\Psi_\pm^2$ are given by expressions like (50). In terms of these solutions, the perturbations ${}^1\Psi_\pm$ have to satisfy

$${}^1\Psi_\pm(\tau)=[1+3s_2\Psi_\pm^2(\tau)]^{-1}\times\left\{s_2\Psi_\pm[\Psi_\pm^2(\tau)-\Psi_\mp^2(\tau)]+iG_0\int_0^\tau d\tau_1(\tau-\tau_1)^2e^{-i\mu(\tau-\tau_1)}\,{}^1\Psi_\pm(\tau_1)\right\} . \quad (60)$$

The initial conditions implied by this integral equation are ${}^1\Psi_{\pm}(\tau=0)=0$. Being solutions of a single-mode evolution equation the $\Psi_{\pm}^2$ behave at saturation as in Eq. (55), which means that Ψ_+ and Ψ_- tend to the same limiting value Ψ_∞. For times $\tau_>$, after the unperturbed Ψ_+ and Ψ_- reach their limiting value, Eq. (60) for the perturbed field quantities reduces to

$$ {}^1\Psi_{\pm}(\tau_>)=iG_0\int_0^{\tau>} d\tau(\tau_> -\tau)^2 e^{-i\mu(\tau_> -\tau)}\,{}^1\Psi_{\pm}(\tau)\ . \qquad (61)$$

This is now a homogeneous Volterra equation which, as is well known, has only trivial solutions, i.e., ${}^1\Psi_{\pm}(\tau_>)=0$. Thus the perturbed field quantities ${}^1\Psi_+$ and ${}^1\Psi_-$ tend to the same limiting value, namely, zero. A similar reasoning can be applied to the higher-order terms in the ϵ expansion with the net result that the perturbed fields vanish to all orders at saturation and the total fields $\chi_{\pm}$ tend to the unperturbed ${}^1\Psi_{\pm}$ which, as we saw, are equal at saturation. This means that only one of the modes, which field χ_1 say, survives at saturation while the other disappears ($\chi_2 \rightarrow 0$).

The analysis we have presented is applicable to competition of nearby modes since we have neglected the difference in detuning parameters. This difference is

$$\begin{aligned}\delta\mu=\mu_b-\mu_q&=L[(k_b-k_q)(1-c/v_0)]\\ &\approx(\delta\omega/\delta_R)/(1/2\pi N)\ ,\end{aligned} \qquad (62)$$

where N is the number of periods of the wiggler (or undulator). Keeping in mind that the modes with highest gain have detuning of order unity, our approximation of neglecting the detuning difference is valid for modes with $\delta\omega=\omega_b-\omega_q$ such that $(\delta\omega/\delta_R)\ll(1/2\pi N)$.

For low-gain, long-pulse Compton FEL's, the analysis of coupled differential equations in Ref. 1 has shown that stable operation is single mode. The coupled integrodifferential equation (56) leads to the conclusion that single-mode operation is also preferred in the case of high gain. What happens is that the strong crossed saturation among modes (for nearby modes it is twice the self-saturation), results in an intense competition whereby the dominant mode is able to reduce the effective gain of other nearby modes to the point of extinction. The outcome is operation at a single mode.

For FEL's operating inside the deep saturation region, terms of higher order (than the third) in the radiation field have to be taken into account. These, as well as the effects they produce (sideband instabilities,[7-13] for instance), are outside the scope and intent of the present paper.

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