Selection in the Saffman-Taylor bubble and asymmetrical finger problem

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A systematic study of the zero-surface-tension solutions for a bubble in a Hele-Shaw cell is made. Then the technique used for the Saffman-Taylor finger is applied to find how the actual solution for nonzero surface tension is selected out of this family. We find that only symmetrical bubbles are allowed. In particular, in the limit of zero surface tension we find that the velocity of the bubble is twice the velocity of the fluid at infinity. We also find in the case of Saffman-Taylor fingers that the symmetrical finger is selected against the asymmetrical ones.

I. INTRODUCTION

The Saffman-Taylor finger problem[1] is one of the simplest examples of pattern selection.[2] These fingers are produced when a low-viscosity fluid is pushed into a high-viscosity one, inside a Hele-Shaw cell, that is, a very flat and a very long channel. Saffman and Taylor[3] found that the relative width of the finger, as compared to the channel width, could take any value whereas experiment showed that a definite width was selected.

Twenty years later McLean and Saffman[4] showed that a definite width selection is obtained when the surface tension between the two fluids is taken into account. They found good agreement between experiment and the numerical solutions of their equations. However they were unable to point out the mathematical mechanism in their equations which produces selection. In particular, a perturbation expansion in the limit of low surface tension T gave no hint of selection.

This puzzle has been solved recently by various groups.[5-8] The basic principle of the answer is the following. McLean-Saffman's equations are somewhat similar to a Schrödinger equation. Here the role of $\hbar$ is played by the surface tension T, that is, T is a factor in the second derivative coming into a second-order differential equation. This kind of equation generates solutions with transcendental terms in T, analogous to the semiclassical solutions of the Schrödinger equation which do not admit an expansion in powers of $\hbar$. Now a physically acceptable solution must have a vanishingly small contribution from these terms when one goes to infinity on either side of the finger. This is analogous to requiring that for a bound state the solution of the Schrödinger equation decreases exponentially. Hence a selection for the finger width is obtained which is analogous to the energy quantification for bound states.

McLean-Saffman's equations are, however, markedly different from a Schrödinger equation. They are a set of nonlinear integro-differential equations. Nevertheless it has been shown[6] that the singularities of the $T=0$ solution play a role analogous to the turning points for the Schrödinger equation. These singularities are actually out of the physical region and McLean-Saffman's equations must be extended in the complex plane. In the $T \to 0$ limit the selection is obtained by matching the actual solution to the $T=0$ solution around these points. In this process McLean-Saffman's equations reduce to a single ordinary nonlinear differential equation with specific boundary conditions. In particular, an analytic solution can be obtained in the regime corresponding to the quasiclassical limit for the Schrödinger problem (high quantum number).

The net result is that for a fixed velocity for the finger, an infinite (denumerable) set of widths is selected. Since experiment gives a single width corresponding to the narrowest finger, it remains to explain why this width is selected. This could very well be a dynamical selection, all the fingers being unstable against the narrowest one.[9] This can be physically easily understood when one realizes that at fixed driving pressure the narrowest finger is also the fastest one, so in the growing process it overcomes[10] all the other ones.

The purpose of the present paper is to apply our techniques[6] to the more general problem of shape selection for a bubble (rather than a finger) of less viscous fluid moving in a more viscous one in a Hele-Shaw cell. This problem has been initiated by Saffman and Taylor[11] who found a two-parameter solution for zero surface tension. This has been recently generalized by Tanveer[12] into a three-parameter family. Kadanoff[13] found a four-parameter one, but as pointed out by Tanveer,[14] there is a relation between these four parameters. Here we present first a systematic method to obtain all the possible solutions for $T=0$. We find the four parameter solution obtained by Kadanoff with a somewhat more heuristic analysis, together with the relation pointed out by Tanveer. Our method allows us to introduce a new and simple representation for the bubble problem.

Then we study the selection problem which turns out to be very similar to the finger problem.[6] We find that in the limit $T \to 0$, the bubble has a velocity which is twice the velocity of the surrounding fluid far away from the bubble. This is in agreement with recent numerical calculations by Tanveer.[12] We also find another selection rule in the case of asymmetrical bubbles. Taking into account Tanveer's relation, this means that only symmetrical bubbles are allowed. The asymmetrical finger is a particular case of the asymmetrical bubble, and in this case we obtain that the symmetric finger is selected, again in agreement with Tanveer.[15]

Finally, we try to find an explanation for the loss of solution found numerically by Tanveer[12] beyond some surface tension. We show that there is a jump in the selection rule for a given value of the surface tension. This could provide an explanation. Another likely possibility is linked to the difficulty of satisfying the convergence condition at the back of the bubble and is discussed at the end of Sec. III.

II. ZERO-SURFACE-TENSION SOLUTIONS

We consider a bubble of zero-viscosity fluid moving with velocity U in a viscous fluid whose velocity V at infinity is taken as unity. We also take the half-width a of the Hele-Shaw cell to be unity. The two-dimensional velocity field $\mathbf{u}\equiv(u_x,u_y)$ of the viscous fluid satisfies Darcy's law[3]

$$\mathbf{u}=-\frac{b^2}{12\mu}\nabla p \;, \tag{1}$$

where b is the spacing between the plates of the Hele-Shaw cell, μ the viscosity, and p the pressure. The viscous fluid is considered as incompressible

$$\nabla\cdot\mathbf{u}=0 \tag{2}$$

and the velocity field derives from a complex potential $w=\phi+i\psi$ (with $\phi=-b^2p/12\mu$) which is an analytic function of the complex position $z=x+iy$. The complex velocity field $u=u_x-iu_y$ is given by $u=\partial w/\partial z$. At infinity we must have $w\approx z$ and on the channel sides $u_y=0$ implies $\psi=\pm 1$. The boundary condition on the bubble[4] is

$$p=-\frac{T}{R} \tag{3}$$

if the constant pressure inside the bubble is taken conveniently. R is the radius of curvature of the interface. For zero surface tension this gives $\phi=0$ on the bubble. In this case the condition[3] that the bubble moves as whole with velocity U translates into $\psi=U(y-y_0)$ on the bubble where y_0 is a constant.

Our method follows the original idea[3] of Saffman and Taylor: z is an analytic function of w since $u\neq 0$ everywhere, and it is more convenient to look for this function because the boundary conditions are well defined. This function is analytic inside the strip $\psi\in[-1,1]$ with $y=\pm 1$ on the border, $z\approx w$ at infinity, and corresponding to the bubble, there is a cut on the $\phi=0$ axis on which $y=\operatorname{Im}z=\lambda\psi+y_0$ with $\lambda=U^{-1}$.

In order to solve this problem we map the boundaries on the real axis through the conformal mapping

$$W=i\sinh\left[\frac{\pi}{2}w\right] . \tag{4}$$

This sends (see Fig. 1) the upper side of the channel on a cut $]-\infty,-1]$, the $\phi>0$ part corresponding to the upper rim of the cut, the $\phi<0$ part to the lower rim. Similarly, the lower side of the channel is sent on $[1,\infty[$. Finally, the bubble cut is sent on a cut $[b_1,b_2]$ with

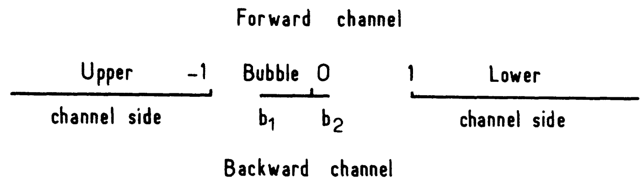

FIG. 1. The Hele-Shaw cell and the bubble in the W plane.

$|b_1|,|b_2|\leq 1$. The upper complex plane corresponds to the forward part of the channel, the lower one to the backward part.

On the cuts $]-\infty,-1]$ and $[1,\infty[$, the analytic function $Z(W)=z-w$ satisfies $\operatorname{Im}Z=0$, it goes to a constant when W goes to infinity in the upper half plane and to another constant when W goes to infinity in the lower half plane. These two constants can be chosen with a zero sum (this corresponds physically to translate the bubble). On the cut $[b_1,b_2]$,

$$\operatorname{Im}Z=\frac{2}{\pi}(1-\lambda)\arcsin W+y_0 \;. \tag{5}$$

Actually, as pointed out by Kadanoff,[13] it is more convenient to deal with the velocity field u itself. Here we consider instead of $Z(W)$

$$Z'(W)\equiv\frac{\partial Z}{\partial W}=\frac{2(u^{-1}-1)}{i\pi(1-W^2)^{1/2}} \tag{6}$$

which satisfies the same boundary conditions as $Z(w)$ except that

$$\operatorname{Im}Z'(W)=\frac{2}{\pi}\frac{1-\lambda}{(1-W^2)^{1/2}} \tag{7}$$

on the cut $[b_1,b_2]$, and $Z'(W)$ goes to zero when $W\to\infty$.

We know the imaginary part of Z' on the cuts. From the analyticity of Z' we have dispersionlike relations. Cauchy's formula gives

$$\begin{aligned} Z'(W_0)&=\frac{1}{2i\pi}\int_C dW\frac{Z'(W)}{W-W_0}\\ &=\frac{1}{2i\pi}\int dW\frac{Z'_+(W)-Z'_-(W)}{W-W_0} \;. \end{aligned} \tag{8}$$

The contour C encircles the cuts [it should also contain the circle at infinity but this one does not contribute since from Eq. (6) $WZ'\to 0$ for $W\to\infty$]. The second integral is over the intervals $]-\infty,-1]$, $[b_1,b_2]$, $[1,\infty[$ and $Z'_\pm$ are the Z' values on the upper and lower rim of the cuts. If we let W_0 go on the real axis, we obtain

$$Z'_+(W_0)+Z'_-(W_0)=-\frac{i}{\pi}P\int dW\frac{Z'_+(W)-Z'_-(W)}{W-W_0} \;. \tag{9}$$

Since $\operatorname{Im}(Z'_+-Z'_-)=0$ on the cuts, this tells us that $\operatorname{Re}(Z'_++Z'_-)=0$ on the real axis. Therefore on the cuts

$$Z'_++Z'_-=i\operatorname{Im}(Z'_++Z'_-)=2i\operatorname{Im}Z'_\pm \;, \tag{10}$$

and we know $Z'_+ + Z'_-$ on the cuts: it is zero on $]-\infty, -1]$, and $[1,\infty[$, and on $[b_1,b_2]$ it is $4i(1-\lambda)/\pi(1-W^2)^{1/2}$.

Now finding Z' when $Z'_+ + Z'_-$ is known on the cuts is a Riemann-Hilbert problem which we solve by the standard Carleman's technique.[16] We find a function $L(W)$ which satisfies $L_+/L_- = -1$ on the cuts. A convenient solution is[16]

$$L(W) = [(1-W^2)(W-b_1)(W-b_2)]^{-1/2} . \tag{11}$$

Then $Z'(W)/L(W)$ has a jump equal to

$$4(\lambda-1)[(W-b_1)(b_2-W)]^{1/2}/\pi$$

across the cut $[b_1,b_2]$. In the general case an integral representation is easily written for the solution. But here the answer is clearly

$$\frac{Z'(W)}{L(W)} = \frac{2i(1-\lambda)}{\pi}\sqrt{(W-b_1)(W-b_2)} + p(W) , \tag{12}$$

where $p(W)$ is a polynomial (it must be an analytic function over the whole complex plane and can only grow algebraically for $W\to\infty$ since $Z'\to 0$). From Eq. (6) since $u\to 1$ for $w\to\infty$, $p(W)$ must be of the form

$$p(W) = \frac{2i}{\pi}(\lambda-1)(W+C) ,$$

where C is a constant. This leads finally to

$$u^{-1} = \lambda + (1-\lambda)\frac{W+C}{\sqrt{(W-b_1)(W-b_2)}} \tag{13}$$

in agreement with Kadanoff[13] [our parameters are related to his parameters by $b_1 = -\sin\psi_-$, $b_2 = -\sin\psi_+$, and $C = \sin\psi_2$, and his w which we write as w_K is related to W by $W = i(w_K - 1/w_K)/2$]. We note that Saffman and Taylor's symmetric bubble[11] corresponds to $b_1 = -b_2$, $C = 0$, their asymmetric finger[11] to $C\neq 0$, $b_2 = -b_1 = 1$ (and their symmetric finger to $b_2 = -b_1 = 1$ and $C = 0$).

Now, as pointed out by Tanveer,[14] Eq. (13) will not lead in general to a single valued function $Z(W)$ when we integrate $Z'(W)$, unless we require that the integral of $Z'(W)$ around a contour encircling the cut $[b_1,b_2]$ is zero. This leads to the additional requirement

$$\int_{b_1}^{b_2} dW \frac{W+C}{\sqrt{(1-W^2)(b_2-W)(W-b_1)}} = 0 . \tag{14}$$

Another way to see that an additional condition is necessary is to solve directly for $Z(W)$ by the Carleman technique. If we use $L(W)$ given by Eq. (11), we obtain a solution depending on seven parameters, namely, λ, b_1, b_2, y_0, and the three (purely imaginary) coefficients of an arbitrary second-degree polynomial which can be introduced in the equation analogous to Eq. (12). But we must require that $Z(W)$ be finite for $W = \pm 1$, b_1, b_2; otherwise, the result does not correspond physically to a bubble. This gives four relations and therefore three free parameters. Alternatively, we can use $(1-W^2)L(W)$ instead of $L(W)$ in our solution. This insures automatically $Z(\pm 1) = 0$. The solution depends on λ, b_1, b_2, y_0, and an imaginary constant, these last two parameters being determined by requiring a finite $Z(W)$ for $W = b_1, b_2$. However, for our purpose, the most convenient representation of the solution is Eq. (13) together with condition Eq. (14). We will come back to this condition after having studied the selection due to surface tension.

III. SELECTION

We know that selection is produced by the nonzero surface tension. We must use now Eq. (3) as a boundary condition instead of $p = 0$. It can be rewritten as

$$\phi = \frac{\alpha}{R}, \quad \alpha = \frac{Tb^2}{12\mu} . \tag{15}$$

We introduce the velocity field by taking the derivative of this equation with respect to the arclength s along the bubble interface

$$\alpha\frac{\partial}{\partial s}\left[\frac{1}{R}\right] = \frac{\partial\phi}{\partial s} . \tag{16}$$

Then we rewrite this equation in a form convenient for analytic continuation out of the interface. If the x axis is along the channel pointing in the direction of the bubble motion and θ is the angle between the x axis and the normal to the interface, the angle between the x axis and the tangent to the interface is $\theta+\pi/2$, and we have $dz = ie^{i\theta}ds$, where dz is taken along the interface. Since $1/R = \partial\theta/\partial s$ this leads to

$$\frac{\partial}{\partial s}\left[\frac{1}{R}\right] = ie^{i\theta}\frac{\partial^2}{\partial z^2}(e^{i\theta}) . \tag{17}$$

On the other hand, we have with respect to any reference frame (X, Y):

$$\frac{\partial\phi}{\partial X} - i\frac{\partial\phi}{\partial Y} = u_X - iu_Y . \tag{18}$$

If X is taken along the normal n and Y along the tangent, this leads to

$$\frac{\partial\phi}{\partial n} - i\frac{\partial\phi}{\partial s} = u_n - iu_s = (u_x - iu_y)e^{i\theta} = ue^{i\theta} . \tag{19}$$

Since the bubble moves with velocity U we have $\partial\phi/\partial n = U\cos\theta$. Finally Eq. (16) becomes

$$\varepsilon\frac{\partial^2 f}{\partial z^2} + \frac{1}{f^2} = \frac{2u}{U} - 1 , \tag{20}$$

where $f = e^{i\theta}$ and $\varepsilon = 2\alpha/U = Tb^2/6\mu U$.

Equation (20) is the generalization for our purpose of the differential equation of McLean and Saffman. Their second equation is an integral equation which expresses the analyticity of u, but we will not need for our analysis to write this equation explicitly.

For low surface tension, ε is small, and it seems possible to solve Eq. (20) by an ε expansion. But this fails near the singularities of f where f and therefore $\partial^2 f/\partial z^2$ diverge. To zeroth order in ε, these singular points are obtained by setting in Eq. (20) $\varepsilon = 0$ and $u = u_0$, where u_0 is the $T = 0$ solution found in the preceding paragraph. Hence the points where f diverge satisfy

$$u_0=\frac{U}{2} \;. \tag{21a}$$

Since $f=e^{i\theta}$ these points are clearly not on the interface, and this is why θ and Eq. (20) must be extended analytically out of the interface.

We note that at these points nothing special happens to the velocity field u_0 itself. On the other hand, from Eq. (20) with $\varepsilon=0$ we have

$$u_0=\frac{U}{2}(1+e^{-2i\theta_0}) \;, \tag{21b}$$

where θ_0 is the function corresponding to the $T=0$ solution. This results directly from the fact that for $T=0$ the velocity on the interface is along the normal, and its modulus $|u_0|$ is $U\cos\theta_0$ which gives

$$u_0=|u_0|e^{-i\theta_0}=\frac{U}{2}(1+e^{-2i\theta_0}) \;. \tag{22}$$

But the complex conjugate of the velocity field is given on the interface by

$$u_0^*=|u_0|e^{i\theta_0}=\frac{U}{2}(1+e^{2i\theta_0}) \;, \tag{23}$$

and its analytic continuation is singular when $e^{i\theta_0}$ diverges (note that the analytic continuations of u_0 and u_0^* are not complex conjugate in general since the continuation of θ_0 is not necessarily real).

Now our analysis closely follows Ref. 6. In the limit $\varepsilon\to 0$ we want a solution f which matches the $T=0$ solution f_0 far away from the singular point. This will not be possible in general because of transcendental divergent terms generated by the singularity. To lowest order in ε we can replace u in Eq. (20) by its $T=0$ value u_0 given by Eq. (13). In the same way we can use the $T=0$ relation between z and W obtained implicitly in the preceding paragraph.

Equation (20) is basically identical to the equation studied in Ref. 6. In the same way complete matching will not be possible for $\varepsilon\to 0$ except if the singular points go to $z=\infty$ which corresponds to a singular point of the differential equation (20) itself. For small ε we have therefore only to consider large values of z which corresponds also to large W. A convenient variable is then $\zeta=[i\sinh(\pi z/2)]^{-1}$. For large W we have from Eq. (13)

$$\frac{\pi^2}{4}\varepsilon\zeta\frac{d}{d\zeta}\left[\zeta\frac{df}{d\zeta}\right]+\frac{1}{f^2}=2\lambda-1-2\zeta\lambda(1-\lambda)b_s-\zeta^2\lambda(1-\lambda)[\tfrac{1}{4}(b_1-b_2)^2+(b_1+b_2)b_s-4(1-\lambda)b_s^2] \;, \tag{24}$$

where $b_s=C+(b_1+b_2)/2$ and we have used

$$\frac{1}{W}\approx\zeta-(1-\lambda)b_s\zeta^2 \tag{25}$$

(valid to second order in ζ) which results from

$$\frac{\partial w}{\partial z}=u\approx 1-\frac{(1-\lambda)b_s}{W} \;. \tag{26}$$

We note that to lowest order we merely have $\zeta=1/W$. The correct expression Eq. (25) is only needed to get the proper coefficient of ζ^2 in Eq. (24), but this turns out to be irrelevant. The higher-order terms in ζ which we have neglected in Eq. (24) turn out to be irrelevant after rescaling.

As in Ref. 6 we study the matching problem by a proper rescaling. As for the finger problem f scales as $\varepsilon^{-1/3}$ and ζ as $\varepsilon^{1/3}$. Therefore as for Saffman-Taylor fingers, $2\lambda-1$ behaves as $\varepsilon^{2/3}$ for low surface tension. Moreover b_s is proportional to $\varepsilon^{1/3}$. This can be used to simplify Eq. (24) into

$$\frac{\pi^2}{4}\varepsilon\zeta\frac{d}{d\zeta}\left[\zeta\frac{df}{d\zeta}\right]+\frac{1}{f^2}=2\lambda-1-\frac{b_s\zeta}{2}-\frac{(b_1-b_2)^2}{16}\zeta^2 \;. \tag{27}$$

By rescaling $f=|2\lambda-1|^{-1/2}Q$ and

$$\zeta=4ix\,|2\lambda-1|^{1/2}/|b_1-b_2|$$

we obtain

$$\frac{1}{a}x\frac{d}{dx}\left[x\frac{dQ}{dx}\right]+Q^{-2}=\operatorname{sgn}(2\lambda-1)-i\beta x+x^2 \tag{28}$$

with $a=4|2\lambda-1|^{3/2}/\varepsilon\pi^2$ and

$$\beta=2b_s/|b_1-b_2|\,|2\lambda-1|^{1/2} \;.$$

As in Ref. 6 the surface tension has disappeared after rescaling and the solution of the matching problem will give pure numbers for the parameters a and β.

The situation is now the following. We start from one side of the bubble, and we require that when we follow the interface, no transcendental divergent term arises when we reach the other side of the bubble. This must be true whether we follow the forward or the backward part of the bubble. In the complex plane of the x variable the bubble is far away on the imaginary axis, one side towards $i\infty$, the other towards $-i\infty$. In the x plane, going from one side to the other corresponds to going from $-i\infty$ to $i\infty$ by a semicircle at infinity, $x>0$ corresponding to the forward path, $x<0$ to the backward one. Now the transcendental terms are generated by the singularities of the $T=0$ solution Q_0:

$$Q_0=[\operatorname{sgn}(2\lambda-1)-i\beta x+x^2]^{-1/2} \;, \tag{29}$$

that is, the zeros of the right-hand side of Eq. (28). They are near the origin and we will obtain the transcendental terms by deforming the above paths to make them pass near the singularities. That is, we go from $-i\infty$ to $i\infty$ along the imaginary axis either on the $x>0$ or on the $x<0$ side. If we start with $Q\approx Q_0$ on one side, we want to have $Q\approx Q_0$ on the other side. More generally we want $Q\approx Q_0$ everywhere, except near the singularities, and in particular for $x\to+\infty$.

We consider first the case $2\lambda-1>0$. For $\beta=0$ the problem Eq. (28) is identical to the one dealt with in Ref. 6. We will only treat the large-a situation which can be handled analytically by a WKB method. We will not consider the case where a is of order unity as in Ref. 6. Since it has been shown in Ref. 6 that even in this case the WKB result is a very good approximation, we expect the same to be true here.

Let us recall how the $\beta=0$ case has been treated. We have two singularities at $x=\pm i$. We arrive from $-i\infty$ and approach $-i$. In the vicinity of this point we set $x=-i(1-y_-)$ and after the rescaling $Q=(a/4)^{1/7}F$ and $y_-=(a/4)^{-2/7}(r_-/2)$, Eq. (28) becomes

$$\frac{d^2F}{dr_-^2}+\frac{1}{F^2}=r_- \ . \tag{30}$$

When one looks for exponential corrections to the asymptotic solution $F_0=1/\sqrt{r_-}$ (corresponding to $Q=Q_0$), one finds Stokes lines at $\arg(r_-)=-6\pi/7$ and $\arg(r_-)=-2\pi/7$. We want a solution which satisfies $F\approx F_0$ in the two Stokes wedges $-10\pi/7<\arg(r_-)<-6\pi/7$ and $-6\pi/7<\arg(r_-)<-2\pi/7$. It has been shown in Ref. 6 that this solution behaves as

$$F\approx\gamma(r_-)^{-3/8}\exp(\tfrac{4}{7}\sqrt{2}r_-^{7/4}),\quad \arg(\gamma)=\frac{4\pi}{7} \tag{31}$$

in the wedge $-2\pi/7<\arg(r_-)<2\pi/7$. Actually this is improper since this term is supposed to be a correction whereas being divergent it dominates F_0. It is in fact the analytic continuation of the exponentially small corrections arising in the wedges where $F\approx F_0$ and we could stay in these regions. But it is more convenient to work on the $\arg(r_-)=0$ axis with the analytic continuation (the same remark applies to our WKB expression below).

Away from the singularities we may solve Eq. (28) by a $1/a$ expansion around Q_0. We obtain a linearized differential equation. The possible transcendental corrections h are solutions of the homogeneous part of this equation

$$\frac{1}{a}x\frac{d}{dx}\left[x\frac{dh}{dx}\right]-2Q_0^{-3}h=0\ , \tag{32}$$

and they can be obtained by the WKB approximation. The solution which matches Eq. (31) is

$$h\approx\gamma\left[\frac{a}{4}\right]^{1/28}Q_0^{3/4}\exp\left[-(2a)^{1/2}\int_{-i}^{x}\frac{dx}{x}Q_0^{-3/2}\right]. \tag{33}$$

Then it was required that h is real when x is on the real axis which leads to

$$a=2\left[n+\frac{\arg(\gamma)}{\pi}\right]^2. \tag{34}$$

But we would get the same result[6] if we keep going to the singularity at $x=i$, turn around it, and require that there is no transcendental correction when we go toward $i\infty$.

Now when $\beta\neq0$ the only change is Q_0. Let us call is_+ and $-is_-$ the position of the two singularities (with $s_+,s_->0$). They are the roots of $x^2-i\beta x+1=0$. In the vicinity of $-is_-$ we set $x=-i(s_--y_-)$ and Eq. (28) becomes

$$\frac{s_-^2}{a}\frac{d^2Q}{dy_-^2}+\frac{1}{Q^2}=Sy_-\ , \tag{35}$$

where $S=s_++s_-$. After the rescaling $Q=(a/S^2s_-^2)^{1/7}F_-$ and $y_-=(a/S^2s_-^2)^{-2/7}r_-/S$, we obtain exactly Eq. (30). Since our boundary conditions are unchanged the solution is still given by Eq. (31).

Then between the two singularities the transcendental correction is still the solution of Eq. (32) and is of the form Eq. (33), except that in order to match Eq. (31) the prefactor must be changed. We find

$$h\approx\gamma\left[\frac{a}{S^2s_-^2}\right]^{1/28}Q_0^{3/4}\times\exp\left[-(2a)^{1/2}\int_{-is_-}^{x}\frac{dx}{x}Q_0^{-3/2}\right]. \tag{36}$$

In the same way we set $x=i(s_+-y_+)$ in the vicinity of is_+ and after the rescaling $Q=(a/S^2s_+^2)^{1/7}F_+$ and $y_+=(a/S^2s_+^2)^{-2/7}r_+/S$ we find again Eq. (30). But now the boundary conditions of convergence for $x\to+\infty$ and $x\to i\infty$ are translated into a convergence condition in the two wedges $2\pi/7<\arg(r_+)<6\pi/7$ and $6\pi/7<\arg(r_+)<10\pi/7$. These are just complex conjugates of the conditions for F_-. The solution is $[F_-(r_-^*)]^*$ as seen by taking the complex conjugate of Eq. (30). Therefore we find that F_+ behaves as

$$F_+\approx\gamma^*(r_+)^{-3/8}\exp(\tfrac{4}{7}\sqrt{2}r_+^{7/4}) \tag{37}$$

in the wedge $-2\pi/7<\arg(r_+)<2\pi/7$. The WKB solution of Eq. (32) which matches this behavior is

$$h\approx\gamma^*\left[\frac{a}{S^2S_+^2}\right]^{1/28}Q_0^{3/4}\times\exp\left[-(2a)^{1/2}\int_{is_+}^{x}\frac{dx}{x}Q_0^{-3/2}\right]. \tag{38}$$

Since this WKB solution is also given by Eq. (36) we end up with the selection condition

$$\tfrac{1}{14}\ln\frac{s_+}{s_-}+2i\arg(\gamma)+2in\pi=(2a)^{1/2}\int_{-s_-}^{s_+}\frac{ds}{s}(1+\beta s-s^2)^{3/4}\ , \tag{39}$$

where we have set $x=is$. The imaginary part of Eq. (39) reduces to our preceding condition Eq. (34) but the real part, which is trivially satisfied for $s_+=s_-$, gives an additional condition

$$\tfrac{1}{14}\ln\frac{s_+}{s_-}=(2a)^{1/2}P\int_{-s_-}^{s_+}\frac{ds}{s}(1+\beta s-s^2)^{3/4}\ . \tag{40}$$

We show now that this condition is satisfied only if $s_+=s_-$. We study the two sides of Eq. (40) as a function $\rho=s_+/s_-$ (or equivalently β since $\beta=s_+-s_-$ and $s_+s_-=1$). They are both zero for $\rho=1$ and change sign for $\rho\to1/\rho$, so we can consider only the case $\rho\geq1$. If we calculate the derivative of the right-hand side with respect to ρ, the contribution from the boundaries disappears because the integrand vanishes for these boundaries. The resulting integral is easily calculated by a change of variables. This derivative is

$$(2a)^{1/2}\tfrac{3}{4}(s_+ + s_-)^{1/2}B(\tfrac{3}{4},\tfrac{3}{4})\frac{d\beta}{d\rho}\ , \qquad (41)$$

where $B(x,y)$ is the Eulerian β function. Since $\rho = s_+^2 = 1/s_-^2$ we obtain finally for this derivative, with $B(\frac{3}{4},\frac{3}{4}) = \Gamma^2(\frac{3}{4})/\Gamma(\frac{3}{2}) \approx 1.69$,

$$0.63(2a)^{1/2}\frac{1}{\rho}\left[\sqrt{\rho}+\frac{1}{\sqrt{\rho}}\right]^{3/2} > \frac{0.63(2a)^{1/2}}{\rho^{1/4}}\ . \qquad (42)$$

On the other hand, the derivative of the left-hand side is $\frac{1}{14}\rho$. From Eq. (34) of Ref. 6 we have always $(2a)^{1/2} > 1$ for all the possible values of a. Therefore, the right-hand side of Eq. (40) is always growing faster than the left-hand side. Since they are equal for $\rho = 1$, there is no other possible solution.

Therefore, we find that our asymmetry coefficient $b_s = C + (b_1 + b_2)/2$ must be zero when we take into account the selection due to surface tension. But Eq. (14) gives an additional relation between C, b_1, and b_2. If we set $C = -(b_1 + b_2)/2$ in Eq. (14), it is easily seen that the only solution is $b_1 + b_2 = 0$. This is done by the change of variable $W = (b_1 + b_2)/2 + v(b_2 - b_1)/2$ and reducing the range of intergration on v to [0,1]. The quantity to be integrated is easily shown to be positive unless $b_1 + b_2 = 0$. Therefore, we come to the conclusion that, after surface tension is taken into account, only symmetrical bubbles $b_1 + b_2 = C = 0$ can exist in agreement with Tanveer's numerical calculations.[12] On the other hand, we have seen that the velocity $U = \lambda^{-1}$ is also selected. Finally, the surface of the bubble is fixed (if we assume that the fluid inside is incompressible); the overall result is that there is no free parameter left anymore. In the case of the asymmetrical finger we have $b_2 = -b_1 = 1$ and therefore we obtain the selection $C = 0$: only the symmetrical finger survives after selection by surface tension is taken into account. This is in agreement with Tanveer's result.[15]

Gathering our results we find that the velocity of the bubble is selected for low surface tension according to

$$\frac{2}{U} - 1 = \left[\frac{\pi^2 T b^2}{24\mu}\right]^{2/3}(n + \tfrac{4}{7})^{4/3}\ . \qquad (43)$$

This is identical to the result found for the Saffman-Taylor fingers. In particular in the limit of zero surface tension we obtain that the bubble moves at twice the velocity of the surrounding fluid at infinity: $U = 2$. This agrees with the numerical results[12] of Tanveer.[17]

Finally, we have not considered yet the case $2\lambda - 1 < 0$ but this is handled as in Ref. 6. Indeed for $\beta < 2$ we have two singularities $2x = i\beta \pm (4 - \beta^2)^{1/2}$, one in the $x > 0$ region and one with $x < 0$. In our path from $-i\infty$ to $i\infty$ in the $x > 0$ region we will go only around the single $x > 0$ singularity. But if we have $Q \approx Q_0$ in two Stokes wedges we will have divergent corrections as Eq. (31) in the third one with no way to get rid of them since there is no other singularity available on this side. Therefore the convergence condition cannot be satisfied. For $\beta > 2$ the two singularities are again on the imaginary axis, but this time on the same side of the origin. Then the path going from one singularity to the other will not pass near the origin and imaginary term $i\pi(2a)^{1/2}$ produced by the origin in the right-hand side of condition Eq. (39) will not be present here. Again the imaginary part of this condition cannot be satisfied. In conclusion there is no solution for $2\lambda - 1 < 0$ as for the finger case.

We notice that, although as we said our selection condition should be enforced when following the forward part as well as the backward part of the bubble, we have actually considered, say, the forward part. The reason for this is that, at first sight, the condition for the backward part is then automatically satisfied. Indeed,[6] far away from the singularities we have three Stokes sectors $-\pi < \arg(x) < -\pi/3$, $-\pi/3 < \arg(x) < \pi/3$, and $\pi/3 < \arg(x) < \pi$. This can be seen from WKB behavior as $\exp(\pm x^{3/2})$ of the transcendental corrections [see Eq. (36) for large x]. Our condition insures that, when $Q \approx Q_0$ in the first two sectors, the same is also true in the third one. Then the whole complex plane is covered and there is no need for another convergence condition.

However, this argument is not completely correct because of the finite size of the bubble. The Stokes line $\arg(x) = \pm\pi$ for $x \to -\infty$ results from the fusion between the Stokes line starting from $x = -i$ with $\arg(r_-) = -10\pi/7$ and its symmetrical starting from $x = i$. But this fusion is only complete at $x = -\infty$, which is never reached because of the finite size of the bubble. Therefore, there is a small Stokes sector, located between these two preceding Stokes lines, which reaches the bubble. This comes in addition to the three above Stokes sectors. But we have no way to enforce convergence in this new sector and we would come to the paradoxical conclusion that there is no solution to the selection problem.

Actually it is easier to see this point qualitatively when one works directly in the physical plane of the Hele-Shaw cell channel. There are then four singular points, which are far away from the bubble. Two of them are in the forward part of the channel, located on the upper and lower side, respectively. The two other singular points are located in the backward part symmetrically from the two first ones. There are then two relevant Stokes lines starting from the upper forward singularity. The first one, starting with an angle $-2\pi/7$ with respect to the channel axis, goes to the lower forward singularity. In our x representation it corresponds to the Stokes line starting from $x = -i$ with $\arg(r_-) = -2\pi/7$ and going to $x = i$. The second one [corresponding to $\arg(r_-) = -6\pi/7$] starts with an angle $-6\pi/7$ with respect to the channel axis. From the behavior of the field $(2u/U) - 1$, which is easy to draw qualitatively, one can see that this line must end up on the bubble. This is obtained from the relation $\alpha_s + (3/4)\alpha_u = -3\pi/2$, where α_s and α_u are, respectively, the local angles of the Stokes line and of the $(2u/U) - 1$ field with respect to the channel axis. This relation shows that this Stokes line cannot cross the channel axis, which leads to our conclusion. Finally, there is a relevant Stokes line starting from the backward upper singularity [corresponding to our troublesome $\arg(r_-)=-10\pi/7$ Stokes line] with an angle $-3\pi/7$ with respect to the channel axis This line satisfies also $\alpha_s+(\frac{3}{4})\alpha_u=-3\pi/2$ and again can be shown to end up also on the bubble.

A possible way out of the preceding paradoxical conclusion is the following. Our WKB analysis is strictly speaking only valid in the $a\to 0$ limit. For small but nonzero a, there are no sharply defined Stokes lines (except near the singularities) and the transition from one Stokes sector to the next is progressive. The Stokes lines must be thought of as fuzzy. Since our divergent Stokes sector is very small on the bubble, this fuzziness may make it completely disappear, thereby restoring the solution. This will happen for small $2\lambda-1$, that is, small surface tension, because in this case the singularities are (in the channel plane, that is, with the variable z or equivalently W) very far away from the bubble and the divergent sector is very small. This disappearance requires also that the order n of the solution [see Eq. (43)] is rather low in order that the Stokes lines are fuzzy enough. If at fixed n we increase the surface tension T, the singularities come nearer to the bubble and the divergent Stokes sector gets too large to be destroyed by fuzziness. The solution could disappear and this might explain the loss of solution found numerically by Tanveer.[12] Similarly, if we increase n at fixed T, the Stokes lines get sharper (and also the singularities get nearer) and again the divergent Stokes sector will no longer disappear, implying perhaps a loss of solution. However, we have not proved that this indeed happens and it could well be that the corrections in the "divergent" sector are small enough to cause no problem to the existence of the solution.

As a final remark, we note that we have found in our problem a strong asymmetry between the forward and the backward regions of the bubble. This is probably a real feature of the actual solution for the bubble shape. Indeed, since pressure in front of the bubble is smaller than in the rear, the curvature will be slighty more in the front than in the rear.

IV. SMALL SYMMETRICAL BUBBLE

In this last section we suggest for the loss of solutions found numerically[12] by Tanveer for bubbles an analytical explanation different from the one considered at the end of the last section. We consider the following possibility. We have assumed precedingly that the two singularities are on the real W axis and correspond to $|W|\gg 1$. If, however, they satisfy $|W|<1$ while still on the real axis, our selection condition might be changed because we have branching points at $W=\pm 1$ in our conformal mapping. This is not a technical point: physically the singularities would be in the surrounding fluid, between the bubble and the sides of the channel, instead of being on the channel sides.

But we meet a technical difficulty. For a symmetric bubble ($C=0$, $b_2=-b_1=b$), the singularities [i.e., the zeros of the right-hand side of Eq. (20)] are located at

$$W=\pm b\lambda/(2\lambda-1)^{1/2}=\pm R\ .$$

Our approach works in the limit of low surface tension, which implies according to Eq. (43) a small $2\lambda-1$, and the singularities correspond to large W except in the case of small bubbles $b\ll 1$. Therefore, we will only consider this case and hope that our results extend qualitatively to other bubbles for which the numerical calculations have been performed. Another practical reason which requires $b\ll 1$ is that in our method we match the solution of an inner rescaled problem Eq. (28) to the solution of an outer problem $Q_0^{-1/2}$ which is merely the $T=0$ solution far away from the bubble. In order that the singularities at $|W|\approx 1$ are "far away" from the bubble, we need the bubble to be small again.

Starting from Eq. (20) with $b\ll 1$, we can perform an expansion of the right-hand side which is still valid for $W\approx 1$

$$\frac{2u}{U}-1\approx 2\lambda-1-\frac{1}{4}\frac{b^2}{W^2}\ , \tag{44}$$

where we have used $\lambda\approx\frac{1}{2}$. The inner region corresponds now to $W\approx 1$, and there is no need to rescale this variable. As before, we rescale f by $f=(2\lambda-1)^{-1/2}Q$ which gives

$$\frac{1}{\eta}\frac{d^2Q}{dz^2}+\frac{1}{Q^2}=1+x^2\ , \tag{45}$$

where we have set $R/W=ix$ and $\eta=(2\lambda-1)^{3/2}/\varepsilon$. As before we will work in the WKB regime where η is large, and hope that our result extends to the low-order branches qualitatively.

We are back to the same inner problem again except that the relation between z and W cannot be approximated by its large-W expression. Away from the singularities at $x=\pm i$, the transcendental corrections to Q_0 are again obtained by the WKB approximation and following the same path as in the preceding paragraph, this leads to

$$(2\eta)^{1/2}\int_{x=-i}^{x=i}dz(1+x^2)^{3/4} \tag{46}$$

which replaces the right-hand side of Eq. (39). Since the bubble is very small we can approximate to lowest order in b the velocity field by $u\approx 1$. This gives

$$\partial z/\partial W=\partial w/\partial W=2/i\pi(1-W^2)^{1/2}$$

and expression (46) can be replaced by

$$(2\eta)^{1/2}\frac{2}{\pi}\int_{-i}^{i}\frac{dx}{x}\frac{(1+x^2)^{3/4}}{(1+x^2/R^2)^{1/2}}\ . \tag{47}$$

This reduces to the right-hand side of Eq. (39) in the limit $R\gg 1$ (with $a=4\eta/\pi^2$).

As long as $R>1$ nothing is changed compared to the situation $R\gg 1$. Indeed for a symmetric bubble only the imaginary part of expression (47) comes in. Since it comes only from the vicinity of $x=0$, it is unchanged. Moreover in the vicinity of $x=\pm i$ we obtain the same nonlinear problem Eq. (30), γ is unchanged and we find again Eq. (43).

If $R<1$, the square root becomes imaginary when x is between $\pm iR$ and $\pm i$ and this leads to an additional contribution to the imaginary part of (47) which is

$$\frac{4}{i\pi}(2\eta)^{1/2}\int_R^1 \frac{ds}{s}\frac{(1-s^2)^{3/4}}{[(s^2/R^2)-1]^{1/2}}=-2i(2\eta)^{1/2}\phi(R) \tag{48}$$

since the contribution from $[-i,-iR]$ and $[iR,i]$ add together. We note that the actual cut, corresponding physically to the sides of the channel, is between $-iR$ and iR and we work with the determination corresponding to the forward side of the bubble. In fact the other determination never appears because we have no WKB expression valid both for the forward and the backward regions.

One may also worry because of the integrable singularities at $x=\pm iR$. However, there is no real singularity here, this problem appears only because of our conformal mapping. If we go back to the physical variable z, these points correspond to the two points on the sides of the channel facing the bubble. Clearly there is nothing particular around these points. It can be checked directly by considering the nonlinear problem that there is no additional contribution from these points.

Finally we consider the nonlinear problem around $x=\pm i$. We set again $x=-i(1-y_-)$ which gives

$$-\frac{\pi^2}{4\eta}(1-R^2)\frac{d^2Q}{dy_-^2}+\frac{1}{Q^2}=2y_- \; . \tag{49}$$

With the rescaling

$$Q=[\eta/\pi^2(1-R^2)]^{1/7}F$$

and

$$y_-=[\eta/\pi^2(1-R^2)]^{-2/7}(r_-/2)\; ,$$

we have

$$-\frac{d^2F}{dz_-^2}+\frac{1}{F^2}=r_- \tag{50}$$

with the asymptotic behavior $F_0=(r_-)^{-1/2}$ as in Eq. (30). But here, because of the change of sign in front of the second derivative, the Stokes lines [corresponding to the exponential corrections to F_0 obtained by linearizing Eq. (50)] are for $\arg(r_-)=0$ and $\arg(r_-)=-4\pi/7$. Our boundary conditions require convergence in the two wedges

$$-12\pi/7<\arg(r_-)<-8\pi/7$$

and

$$-8\pi/7<\arg(r_-)<-4\pi/7$$

(we must have divergence in the wedge $-4\pi/7<\arg(r_-)<0$ in order to match the known WKB divergent behavior when we start from $-iR$ and go along $[-iR,0[$).

Instead of solving directly this new nonlinear problem, we map it on the old one by $F=e^{i\pi/7}F'$ and $r_-=e^{-2i\pi/7}r'_-$. The equation for F' and the boundary conditions are just the same as Eq. (30). Therefore the solution for F' is again Eq. (31) which gives for F

$$F\approx e^{i\pi/28}\gamma(r_-)^{-3/8}\exp(-i\tfrac{4}{7}\sqrt{2}r_-^{7/4})\; . \tag{51}$$

Thus $\arg(\gamma)$ is replaced by $\arg(\gamma)+\pi/28$, and we end up with the new selection condition

$$a=\frac{2}{1-\phi(R)}\left[n+\frac{\arg(\gamma)}{\pi}+\frac{1}{28}\right]^2 \tag{52}$$

instead of Eq. (34).

In conclusion we find that, when R crosses the value $R=1$ (which means when $2\lambda-1$ goes beyond $b^2\lambda^2\approx b^2/4$), there is a sudden jump in the selection condition and therefore a discontinuity in the selected solution. This might explain the disappearance of solutions found by Tanveer. We note, however, that this jump is small, and it is not obvious that it would lead to a problem in the numerical procedure (note that we do not find disappearing solutions, we just suggest that the discontinuity might produce difficulties in the numerical calculation). But we cannot have a definite conclusion since we worked in the limit of large order n and small bubble b. It is conceivable that for low orders and large bubble the jump is larger. In order to check our proposal it would be interesting to perform numerical calculations for large order and small bubble.

Between this explanation and the one considered at the end of Sec. III, we do not know which one is the good one, if any. Let us stress, however, that in the case considered in this last section, where the singularities are in the liquid, it can be seen by looking at the large x limit that there is no longer any divergent Stokes sector on the back of the bubble. The troublesome Stokes lines close on each other instead of reaching the back of the bubble. Therefore, one should always find solutions in this regime.

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