Selection of long-scale oscillatory convective patterns

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(Received 9 December 1987)

A model equation describing oscillatory convective patterns is derived using the long-scale expansion under conditions of slow exchange at the containing walls. The complex amplitudes of convection waves obey on an intermediate time scale the free Schrödinger equation, and are modulated on a slower time scale, following an evolution equation that is similar to a complexified equation of long-scale stationary convection, but involves nonlinear interactions restricted by resonance conditions. The analysis of pattern selection in the vicinity of the symmetry-breaking bifurcation indicates the prevalence of a square-standing-wave pattern comprising a certain phase-locked ("antiphase") combination of perpendicular modes. The stationary antiphase pattern can be destroyed, giving way to a pattern with time-dependent amplitudes, under the influence of weak non-Boussinesq effects.

I. INTRODUCTION

Transition to oscillatory convection, leading to wave patterns in large-aspect-ratio systems, has been extensively studied in the context of double-diffusive convection, both theoretically[1] and experimentally.[2] The symmetry-breaking bifurcation with formation of wave patterns poses a particularly attractive theoretical problem, since a high degeneracy of the bifurcation in the infinite domain makes rich dynamics possible already in the analytically tractable small-amplitude region near the bifurcation point. In contrast to amplitude equations of stationary patterns, that have a dull gradient structure, amplitude equations of wave patterns are capable of generating time-dependent behavior that involves competition between propagating and standing waves and is enhanced by resonant interaction between noncollinear waveforms.[3]

So far, numerical studies of oscillatory convection have been by necessity restricted to either small-aspect-ratio systems or to one-dimensional setups that do not leave space to interaction of alternatively directed waves and selection of two-dimensional patterns. The latter problem has also remained out of reach of analytical studies, due to the difficulty of actual computation of wave interaction coefficients. Even the linear theory can be constructed analytically only at the price of either abandoning no-slip boundary conditions in favor of stress-free boundaries, or replacing the Navier-Stokes by the Darcy equation. At the same time, analytical studies of pattern selection[3] remained abstract before wave interaction coefficients could be connected with actual parameters of a viable physical model.

In this paper, we shall adopt a model of long-scale convection that allows us to circumvent computational difficulties, while retaining rich dynamics of competing patterns. It is known[4,5] that convective cells with a large aspect ratio are formed under conditions of poor heat exchange at the confining plates. Under these conditions, the "lubrication approximation" can be used to solve hydrodynamic equations easily without resorting to nonphysical stress-free boundary conditions. A remarkably simple equation describing stationary convective patterns in long-scale approximation[5] can be written in a rescaled form

$$\partial_\tau \phi + \kappa\phi - \nabla^2\phi + \nabla^4\phi - \nabla\cdot[\,|\nabla\phi|^2\nabla\phi] = 0 \ . \quad (1)$$

This equation possesses a gradient structure that rules out interesting dynamics. The gradient structure is destroyed when Eq. (1) is amended to take into account inertial effects,[6] but this does not influence behavior in the vicinity of the symmetry-breaking bifurcation.

Long-scale equations with a more complex structure than (1) can be obtained for double-diffusive systems;[7] a (structurally unstable) equation derived by Childress and Spiegel in the vicinity of bifurcation at the double zero eigenvalue, that replaces ∂_τ in Eq. (1) by ∂_τ^2, thereby changing the gradient structure to Hamiltonian, is particularly interesting.

In this paper, we shall derive and explore a long-scale equation describing oscillatory convective patterns. The long-scale expansion leads us in Sec. II to the Schrödinger equation for amplitudes of bifurcating waves. The amplitude modulation on a slower time scale is described (Sec. III) by an evolution equation that can be viewed as a complexified Eq. (1), but involves nonlinear interactions severely restricted by resonance conditions. The problem of pattern selection in the vicinity of the symmetry-breaking bifurcation is addressed in Sec. IV. In further sections, we discuss the influence of inertial and non-Boussinesq effects and apply results to a particular case of thermal convection in the presence of the Soret effect.

The principal conclusion regarding pattern selection in the particular case considered in Sec. VII is the prevalence of a standing-wave pattern comprising a certain phase-locked (antiphase) combination of perpendicular modes. This effect is purely three dimensional. In a one-dimensional setting (corresponding to two-dimensional

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flow) a propagating wave would be selected, but in a two-dimensional setting (three-dimensional flow) a standing-wave pattern is favored due to specific phase-dependent interaction of non-collinear modes. The stationary antiphase pattern can be destroyed, giving way to a pattern with time-dependent amplitudes, under the influence of weak non-Boussinesq effects.

II. LONG-SCALE EXPANSION

We shall consider convection in a fluid layer confined between two rigid horizontal plates and infinitely extended in horizontal directions, and use hydrodynamic equations in the conventional Boussinesq approximation, together with transport equations for appropriate state variables (concentrations and temperature). In the quiescent state, state variables vary linearly across the fluid layer due to externally imposed increments between the upper and lower plates, and are constant in the lateral direction. While decrements of some variables can be imposed independently, others may emerge spontaneously due to nondiagonal transport effects (e.g., concentration gradients induced by thermal gradients in the presence of the Soret effect). The symmetry-breaking bifurcation to a convective state is caused by buoyancy forces due to changes of the fluid density, that is assumed to depend linearly on state variables.

We shall use an n-component *state vector* $\mathbf{\Theta}$ to denote deviations of the composition and temperature of the fluid from their values in the quiescent state, i.e., from linear profiles in the vertical direction. Hydrodynamic and transport equations will be written in a dimensionless form, using the layer thickness as a length scale, scaling time by a characteristic diffusion time of one of the state variables, and components of the state vector, by respective decrements across the fluid layer in the quiescent state.

It is convenient to represent a solenoidal velocity field in the form $\mathbf{u}=\mathrm{curl}(\boldsymbol{\lambda}\psi)+\mathrm{curl}^2(\boldsymbol{\lambda}\chi)$, where $\boldsymbol{\lambda}$ is the unit vector along the z axis (directed against the gravity) and ψ and χ are toroidal and poloidal potentials, respectively. Hydrodynamic equations for the potentials χ and ψ are obtained by applying operators curl and curl^2 to the Navier-Stokes equation. If inertial effects are neglected (in the limit of large Prandtl numbers), source terms appear only in the equation of the poloidal potential χ, and the toroidal potential ψ can be set to zero. The equation of χ, with inertial terms omitted, takes the form

$$(\nabla^2+\partial_z^2)^2\nabla^2\chi=\mathcal{R}*\nabla^2\mathbf{\Theta}\ , \tag{2}$$

where $\mathcal{R}$ is the vector of Rayleigh numbers, and ∇ is a two-dimensional operator in the lateral plane $\mathbf{x}$ (hence ∇^2 is the two-dimensional Laplace operator). The no-slip boundary conditions are

$$\nabla^2\chi=\nabla^2\partial_z\chi=0 \quad \text{at } z=\pm\tfrac{1}{2}\ . \tag{3}$$

The convective diffusion equation takes the form

$$\mathbf{D}*(\partial_z^2+\nabla^2)\mathbf{\Theta}-\mathbf{G}\nabla^2\chi=\partial_t\mathbf{\Theta}+\nabla\mathbf{\Theta}\cdot\partial_z\nabla\chi-\nabla^2\chi\partial_z\mathbf{\Theta} +\nabla\mathbf{\Theta}\times\nabla\psi\ . \tag{4}$$

Here $\mathbf{D}$ is the matrix of diffusivities (not necessarily diagonal), and the asterisk denotes the scalar product in the state space, so that it would not be confused with the scalar product in the $\mathbf{x}$ plane, denoted by the dot; $\mathbf{G}$ is the vector of gradients of state variables across the fluid layer in the quiescent state (by scaling convention, the components of $\mathbf{G}$ can be either ± 1 or 0, and are positive when the value of the respective variable increases in the direction of gravity). The last term, containing the vector product forming a pseudoscalar from a pair of two-dimensional vectors, can be omitted at large Prandtl numbers (see, however, Sec. VI).

The boundary conditions are

$$\mathbf{D}*\partial_z\mathbf{\Theta}\pm\tfrac{1}{2}\mathbf{B}*\mathbf{\Theta}=0 \quad \text{at } z=\pm\tfrac{1}{2}\ , \tag{5}$$

where $\mathbf{B}$ is the matrix of dimensionless exchange coefficients at the containing horizontal plates (Biot numbers).

We are interested in the case of long-scale convective patterns that emerge when Biot numbers are small. We set therefore $\mathbf{B}\to\epsilon^2\mathbf{B}$, and rescale the lateral coordinates and time as $\nabla\to\epsilon^{1/2}\nabla$, $\partial_t\to\epsilon\partial_t$. Equations (2) and (4) are solved by expanding the variables in small parameter ϵ,

$$\mathbf{\Theta}=\mathbf{\Theta}_0+\epsilon\mathbf{\Theta}_1+\cdots,\quad \chi=\chi_0+\epsilon\chi_1+\cdots\ . \tag{6}$$

To allow for parametric deviations, we shall also use expansions of the vector of Rayleigh numbers and matrix of diffusivities,

$$\mathcal{R}=\mathcal{R}_0+\epsilon\mathcal{R}_1+\cdots,\quad \mathbf{D}=\mathbf{D}_0+\epsilon\mathbf{D}_1+\cdots\ . \tag{7}$$

In the zeroth order,

$$\partial_z^2\mathbf{\Theta}_0=0,\quad \nabla^2\partial_z^4\chi_0=\mathcal{R}_0*\nabla^2\mathbf{\Theta}_0,\quad \partial_z\mathbf{\Theta}_0(\pm\tfrac{1}{2})=0\ . \tag{8}$$

The zeroth-order state vector $\mathbf{\Theta}_0$ satisfying (8) is any z-independent vector $\mathbf{\Theta}_0(\mathbf{x},t)$. The χ potential is computed as

$$\chi_0=f_0(z)\mathcal{R}_0*\mathbf{\Theta}_0,\quad f_0(z)=\tfrac{1}{24}(z^4-\tfrac{1}{2}z^2+\tfrac{1}{16})\ . \tag{9}$$

In the first order, Eq. (4) reduces to

$$\mathbf{D}_0*\partial_z^2\mathbf{\Theta}_1=\partial_t\mathbf{\Theta}_0+[f_0(z)\mathbf{R}_0-\mathbf{D}_0]*\nabla^2\mathbf{\Theta}_0 +\partial_z f_0(\mathcal{R}_0*\nabla\mathbf{\Theta}_0)\cdot\nabla\mathbf{\Theta}_0\ , \tag{10}$$

where $\mathbf{R}_0$ is the outer product in the state space $\mathbf{R}_0=\mathcal{R}_0\otimes\mathbf{G}$. It is required for solvability of this equation, subject to the boundary conditions $\mathbf{D}_0*\partial_z\mathbf{\Theta}_0(\pm\frac{1}{2})=0$, that the integral of the right-hand side of (10) across the fluid layer vanish. Since the nonlinear term vanishes upon integration, this gives

$$\partial_t\mathbf{\Theta}_0=\mathbf{A}*\nabla^2\mathbf{\Theta}_0\ , \tag{11}$$

with

$$\mathbf{A}=\mathbf{D}_0-\langle f_0(z)\rangle\mathbf{R}_0\ , \tag{12}$$

where $\langle(\cdots)\rangle=\int_{-1/2}^{1/2}(\cdots)dz$, $\langle f_0(z)\rangle=\frac{1}{720}$.

Being linear, Eq. (11) can yield only trivial dynamics. If all eigenvalues of the matrix $\mathbf{A}$ have negative real parts, it describes diffusional relaxation to the quiescent state $\mathbf{\Theta}_0=0$. If the real part of at least one eigenvalue of this matrix is positive, catastrophic short-scale instability sets in, causing the breakdown of the long-scale approximation. The only interesting situation, where the long-scale expansion can be useful, is encountered in the vicinity of a bifurcation manifold, where the matrix $\mathbf{A}$ has an eigenvalue with the vanishing real part. In the following, we shall concentrate on the case when the matrix $\mathbf{A}$ has a pair of imaginary eigenvalues, that corresponds to transition to oscillatory convection. The steady-state solution of Eq. (11) is

$$\mathbf{\Theta}_0=\phi(\mathbf{x},t,\tau)\mathbf{U}+\phi^*(\mathbf{x},t,\tau)\mathbf{U}^* , \tag{13}$$

where the superscript asterisks denote complex conjugates. The eigenvector $\mathbf{U}$ satisfies

$$\mathbf{A}*\mathbf{U}=-i\omega_0\mathbf{U} , \tag{14}$$

and the complex amplitude ϕ obeys the free Schrödinger equation

$$i\partial_t\phi=\omega_0\nabla^2\phi . \tag{15}$$

The amplitude ϕ can be also modulated on an extended time scale τ. The general solution of Eq. (15) can be presented as a Fourier decomposition

$$\phi=\sum_i \phi_i(\mathbf{k}_i,\tau) ,\qquad \phi_i=\hat{\phi}(\mathbf{k}_i,\tau)\exp[i(\mathbf{k}_i\cdot\mathbf{x})+k_i^2\omega_0 t)] . \tag{16}$$

The evolution equations for Fourier amplitudes $\hat{\phi}(\mathbf{k}_i)$ on the slow time scale τ have to be obtained in the next order of the long-scale expansion. Solvability conditions in higher orders will involve averaging over the intermediate time scale t, as well as integrating across the fluid layer. Therefore only resonant terms from the Fourier decomposition (16) can contribute to solvability conditions. In particular, linear terms containing ϕ^* will not contribute to the evolution equation of ϕ, and can be omitted from the outset. A nonlinear term containing a product $\prod\phi_i\phi_j^*$ can enter the evolution equation of ϕ_l if it satisfies the resonance conditions $\sum\mathbf{k}_i-\sum\mathbf{k}_j=\mathbf{k}_l$, $\sum k_i^2-\sum k_j^2=k_l^2$.

III. EVOLUTION EQUATION

In this section, we shall derive the evolution equation on the slow time scale using the general zero-order solution (13) and (16), with ϕ_i standing for either $\phi(\mathbf{k}_i)$ or $\phi^*(\mathbf{k}_i)$, and $\lambda_i=\pm i\omega_0$ denoting eigenvalues corresponding to $\mathbf{U}_i=\mathbf{U}^*$ and $\mathbf{U}_i=\mathbf{U}$, respectively. Summation over repeated eigenfunction indices is presumed throughout.

Before solving Eq. (10), we eliminate the time derivative with the help of Eq. (11), and replace the matrix $\mathbf{R}_0$ using the identity

$$\mathbf{R}_0*\mathbf{U}_i=(\mathbf{D}_0-\boldsymbol{\lambda}_i)*\mathbf{U}_i/\langle f_0\rangle . \tag{17}$$

This reduces Eq. (10) to

$$\partial_z^2\mathbf{\Theta}_1=(f_0/\langle f_0\rangle-1)(1-\boldsymbol{\lambda}_i\mathbf{D}_0^{-1})*\mathbf{U}_i\nabla^2\phi_i + \partial_z f_0(\mathcal{R}_0*\mathbf{U}_i)(\mathbf{D}_0^{-1}*\mathbf{U}_j)\nabla\phi_i\cdot\nabla\phi_j . \tag{18}$$

It is convenient to separate a constant part of the first-order state vector $\mathbf{\Theta}_1^{(c)}$ and z-dependent even and odd parts that both vanish upon integration across the fluid layer,

$$\mathbf{\Theta}_1=\mathbf{\Theta}_1^{(c)}+\mathbf{\Theta}_1^{(\text{even})}+\mathbf{\Theta}_1^{(\text{odd})} , \tag{19}$$

$$\mathbf{\Theta}_1^{(\text{even})}=g_1(z)(1-\boldsymbol{\lambda}_i\mathbf{D}_0^{-1})*\mathbf{U}_i\nabla^2\phi_i , \tag{20}$$

$$\mathbf{\Theta}_1^{(\text{odd})}=h_1(z)(\mathcal{R}_0*\mathbf{U}_i)(\mathbf{D}_0^{-1}*\mathbf{U}_j)\nabla\phi_i\cdot\nabla\phi_j , \tag{21}$$

$$\partial_z^2 g_1=(f_0/\langle f_0\rangle-1),\quad \partial_z h_1=f_0 . \tag{22}$$

The only condition needed to fix integration constants in (22), while keeping g_1 even and h_1 odd, is $\langle g_1\rangle=0$; the conditions $\partial_z g_1(\pm\frac{1}{2})=\partial_z h_1(\pm\frac{1}{2})=0$ are satisfied automatically.

Next, the first-order flow potential is obtained by integrating the first-order hydrodynamic equation

$$\partial_z^4\nabla^2\chi_1=\mathcal{R}_0*\nabla^2\mathbf{\Theta}_1+\mathcal{R}_1*\nabla^2\mathbf{\Theta}_0-2(\nabla^2)^2\partial_z^2\chi_0 . \tag{23}$$

The solution of this is

$$\chi_1=\chi_1^{(\text{even})}+\chi_1^{(\text{odd})} , \tag{24}$$

$$\chi_1^{(\text{even})}=[f_{11}(z)\mathcal{R}_0*(1-\boldsymbol{\lambda}_i\mathbf{D}_0^{-1})*\mathbf{U}_i - f_{10}(z)\mathcal{R}_0*\mathbf{U}_i]\nabla^2\phi_i+f_0(z)\mathcal{R}_1*\mathbf{U}_i\phi_i + f_0(z)\mathcal{R}_0*\mathbf{\Theta}_1^{(c)} , \tag{25}$$

$$\chi_1^{(\text{odd})}=f_1(z)(\mathcal{R}_0*\mathbf{U}_i)(\mathcal{R}_0*\mathbf{D}_0^{-1}*\mathbf{U}_j)\nabla\phi_i\cdot\nabla\phi_j , \tag{26}$$

$$\partial_z^4 f_{11}=g_1(z),\quad \partial_z^4 f_{10}=2\partial_z^2 f_0(z),\quad \partial_z^4 f_1=h_1(z) ,$$
$$f_{11}=\partial_z f_{11}=f_{10}=\partial_z f_{10}=f_1=\partial_z f_1=0 \quad \text{at } z=\pm\tfrac{1}{2} . \tag{27}$$

The equation of the second-order state vector is

$$\mathbf{D}_0*\partial_z^2\mathbf{\Theta}_2=\partial_\tau\mathbf{\Theta}_0+\partial_t\mathbf{\Theta}_1+\mathbf{G}\nabla^2\chi_1-\mathbf{D}_0*\nabla^2\mathbf{\Theta}_1 -\mathbf{D}_1*\nabla^2\mathbf{\Theta}_0+\partial_z\nabla\chi_1\cdot\nabla\mathbf{\Theta}_0+\partial_z\nabla\chi_0\cdot\nabla\mathbf{\Theta}_1 -\nabla^2\chi_0\partial_z\mathbf{\Theta}_1 . \tag{28}$$

Using the representations (19) and (24) of the first-order functions and omitting terms that vanish upon integrating across the fluid layer reduces (28) to

$$\mathbf{D}_0*\partial_z^2\mathbf{\Theta}_1=\partial_\tau\mathbf{\Theta}_0+\partial_t\mathbf{\Theta}_1^{(c)}+[f_0(z)\mathbf{R}_0-\mathbf{D}_0]*\nabla^2\mathbf{\Theta}_1^{(c)} -\mathbf{D}_1*\nabla^2\mathbf{\Theta}_0+\mathbf{G}\nabla^2\chi_1^{(\text{even})} +\partial_z\nabla\chi_0\cdot\nabla\mathbf{\Theta}_1^{(\text{odd})}-\nabla^2\chi_0\partial\mathbf{\Theta}_1^{(\text{odd})} . \tag{29}$$

Integrating (29) across the fluid layer and using the boundary condition

$$\mathbf{D}_0 * \partial_z \mathbf{\Theta}_1 \pm \tfrac{1}{2}\mathbf{B} * \mathbf{\Theta}_0 = 0 \quad \text{at } z = \pm\tfrac{1}{2} \tag{30}$$

yields the solvability condition in the form

$$\partial_t \mathbf{\Theta}_1^{(c)} = \mathbf{A} * \nabla^2 \mathbf{\Theta}_1^{(c)} + \mathbf{F}(\mathbf{\Theta}_0) \ , \tag{31}$$

where the matrix $\mathbf{A}$ is the same as in (12), whereas the inhomogeneous part is

$$\mathbf{F}(\mathbf{\Theta}_0) = \mathbf{U}_i \partial_\tau \phi_i + \mathbf{B} * \mathbf{U}_i \phi_i - (\mathbf{D}_1 - \langle f_0 \rangle \mathbf{R}_1) * \mathbf{U}_i \nabla^2 \phi_i + [\langle f_{11} \rangle \mathbf{R}_0 * (1 - \lambda_i \mathbf{D}_0^{-1}) - \langle f_{10} \rangle \mathbf{R}_0)] * \mathbf{U}_i (\nabla^2)^2 \phi_i$$
$$- \langle f_0^2 \rangle (\mathcal{R}_0 * \mathbf{U}_i)(\mathcal{R}_0 * \mathbf{U}_k)(\mathbf{D}_0^{-1} * \mathbf{U}_j) \nabla \cdot [(\nabla \phi_i \cdot \nabla \phi_j) \nabla \phi_k] \ . \tag{32}$$

Since the homogeneous part of (31), coinciding with (11), has a nontrivial steady-state solution, the solvability conditions of Eq. (31) are

$$\int \int \exp\{ -i [(\mathbf{k}_l \cdot \mathbf{x}) + k_l^2 \omega_0 t)]\} (\mathbf{U}^\dagger \cdot \mathbf{F}) d^2\mathbf{x}\, dt = 0 \ , \tag{33}$$

where $\mathbf{U}^\dagger$ is the eigenvector of the transposed matrix $\mathbf{A}^\dagger$ with the eigenvalue $-i\omega_0$. Using Eq. (16), we obtain the evolution equation for the Fourier amplitude $\hat{\phi}_l$ of a mode with the wave-number vector $\mathbf{k}_l$ in the form

$$\partial_\tau \phi_l + \kappa \phi_l - \mu k^2 \phi_l + \nu k^4 \phi_l - \sum \sigma_{ijkl} (\mathbf{k}_i \cdot \mathbf{k}_j)(\mathbf{k}_l \cdot \mathbf{k}_k) \phi_i \phi_j \phi_k = 0 \ , \tag{34}$$

where summation is carried over all modes satisfying the resonance conditions

$$\pm \mathbf{k}_i \pm \mathbf{k}_j \pm \mathbf{k}_k = \mathbf{k}_l, \quad \pm k_i^2 \pm k_j^2 \pm k_k^2 = k_l^2 \ , \tag{35}$$

with the positive or negative sign, respectively, when ϕ_i stands for $\phi(\mathbf{k}_i)$ or $\phi^*(\mathbf{k}_i)$. Carets over the symbols of Fourier amplitudes are omitted. The coefficients of Eq. (34) are

$$\kappa = \mathbf{U}^\dagger * \mathbf{B} * \mathbf{U}, \quad \mu = \mathbf{U}^\dagger * (\langle f_0 \rangle \mathbf{R}_1 - \mathbf{D}_1) * \mathbf{U} \ , \tag{36}$$

$$\nu = \mathbf{U}^\dagger * \mathbf{R}_0 * [\langle f_{11} - f_{10} \rangle + i\omega_0 \langle f_{11} \rangle \mathbf{D}_0^{-1}] * \mathbf{U} \ , \tag{37}$$

$$\sigma_{ijk} = \langle f_0^2 \rangle (\mathcal{R}_0 * \mathbf{U}_i)(\mathcal{R}_0 * \mathbf{U}_k)(\mathbf{U}^\dagger * \mathbf{D}_0^{-1} * \mathbf{U}_j) \ . \tag{38}$$

The averages can be easily computed after integrating (22) and (27),

$$\langle f_{11} \rangle / \langle f_0 \rangle = -\tfrac{5}{462} \ , \quad \langle f_{11} - f_{10} \rangle / \langle f_0 \rangle = \tfrac{17}{462} \ , \quad \langle f_0^2 \rangle / \langle f_0 \rangle^2 = \tfrac{10}{7} \ . \tag{39}$$

IV. SINGLE-WAVELENGTH PATTERNS

In this section, we shall explore solutions of Eq. (34) in the case when wave numbers of interacting waves have the same absolute value $k = |\mathbf{k}|$ but different orientations. This situation arises quite naturally near a primary symmetry-breaking transition, when waves with a certain preferred wavelength are excited, but no preferred orientation exists due to the rotational symmetry in the horizontal plane.

Physically, one can expect that the linear terms of zero and fourth order in k both act in a stabilizing way, suppressing both long- and short-scale waveforms (the formal condition insuring this action is that κ and ν both have positive real parts). Unlike κ and ν, the coefficient at the quadratic term, μ, depends on parametric deviations $\mathbf{D}_1, \mathbf{R}_1$, which can be chosen in such a way that $\mathrm{Re}(\mu) > 0$ to induce linear instability in an intermediate range of k. The preferred absolute value of the wave-number vector, corresponding to the extremum of the dispersion relation of Eq. (34) is given by

$$k_0^2 = \mathrm{Re}(\mu) / 2\, \mathrm{Re}(\nu) = \bar{\mu} / 2\bar{\nu} \ . \tag{40}$$

The last expression uses the convenient shorthand notation $\mu = \bar{\mu} + i\tilde{\mu}$, etc. Setting $k = k_0$, we obtain the linear growth coefficient in (34) with real and imaginary parts

$$\bar{\lambda} = -\bar{\kappa} + \bar{\mu}^2 / 4\bar{\nu}, \quad \tilde{\lambda} = -\tilde{\kappa} + \bar{\mu}\tilde{\mu} / 2\bar{\nu} - \tilde{\nu}\bar{\mu}^2 / 4\bar{\nu}^2 \ . \tag{41}$$

Cubic interaction terms involving modes with the identical absolute value but different orientations of the wave-number vectors, that satisfy the resonance conditions (35), can be of two different kinds. Interaction terms of the first kind, that are insensitive to phases of respective amplitudes, are obtained by setting ϕ_i, ϕ_j, and ϕ_k in (34) to ϕ_l, ϕ_j, and ϕ_j^*, and summing over all possible permutations of these amplitudes. The interaction coefficient depends on the angle θ between the two modes, $\cos\theta = (\mathbf{k}_l \cdot \mathbf{k}_j) / k_0^2$, and is defined as

$$\sigma_{jl} = \sigma_1(\theta_{jl}) = -\sum \sigma_{ijkl} (\mathbf{k}_i \cdot \mathbf{k}_j)(\mathbf{k}_l \cdot \mathbf{k}_k) \ , \tag{42}$$

where summation is carried out over all permutations as stated. Denoting

$$\alpha = \langle f_0^2 \rangle \, |\mathcal{R}_0 * \mathbf{U}|^2 (\mathbf{U}^\dagger * \mathbf{D}_0^{-1} * \mathbf{U}) \ , \quad \beta = \alpha + \langle f_0^2 \rangle (\mathcal{R}_0 * \mathbf{U})^2 (\mathbf{U}^\dagger * \mathbf{D}_0^{-1} * \mathbf{U}^*) \ , \tag{43}$$

we obtain from (38)

$$\sigma_1(\theta) = k_0^4 [\beta + (\beta + 2\alpha) \cos^2\theta] \ . \tag{44}$$

Another kind of interaction, that turns out to be phase dependent, involves pairs of oppositely directed waves (that can combine to form a standing wave). Denoting by ϕ_{-j} the amplitude of the wave with the wave-number vector $-\mathbf{k}_j$, we obtain the interaction term of the second kind by setting ϕ_i, ϕ_j, and ϕ_k in (34) to ϕ_j, ϕ_{-j}, and ϕ_{-l}^*, with all possible permutations. The respective interaction coefficient is computed, analogous to (42), as

$$\sigma_2(\theta)=2k_0^4(\alpha+\beta\cos^2\theta) \ . \quad (45)$$

As we shall presently see, interaction of this kind favors formation of standing waves with suitably adjusted phases.

Collecting all interaction terms and noting that, for combinatorial reasons, the self-interaction coefficient is $\sigma_0=\frac{1}{2}\sigma_1(0)$, we rewrite the amplitude equation (34) in the form

$$\partial_\tau\phi_l=\lambda\phi_l-\phi_l\left[\sigma_0|\phi_l|^2+2\sigma_0|\phi_{-l}|^2+\sum\sigma_1(\theta_{jl})|\phi_j|^2\right]-\sum\sigma_2(\theta_{jl})\phi_j\phi_{-j}\phi_{-l}^* \ . \quad (46)$$

The first summation is carried over both positive and negative indices $\pm j$.

If one momentarily abstracts from the last term in Eq. (46), which is activated only when pairs of waves propagating in opposite directions are present, this amplitude equation would not significantly differ from its counterpart in the case of stationary convection. All terms, besides the last one, are phase independent, and the truncated equation can be converted into an equation for real amplitudes $\rho_l=|\phi_l|$ involving only real parts of interaction coefficients,

$$\partial_\tau\rho_l=\bar{\lambda}\rho_l-\rho_l\left[\bar{\sigma}_0\rho_l^2+\sum\bar{\sigma}_1(\theta_{jl})\rho_j^2\right] \ . \quad (46a)$$

The angular dependence of the interaction coefficients $\bar{\sigma}_1(\theta)$ is very simple, with the extremum at $\theta=\pi/2$. This favors, in the case $\bar{\sigma}_0>\bar{\sigma}_1(\pi/2)$, or, due to (44), $\bar{\alpha}>0$, formation of a square pattern comprising a pair of perpendicular modes (the same is true in the case of long-scale stationary convection[5]). At $\bar{\alpha}<0$, it is necessary to have $\bar{\beta}>-\bar{\alpha}$ to insure a supercritical bifurcation; in this case a single-mode (roll) pattern is favored.

A hexagonal pattern comprising three modes at mutual angles $\pi/3$ or $2\pi/3$ is also possible, provided $\bar{\beta}<2\bar{\alpha}$. The latter condition insures that oblique modes do not decay, leaving behind a roll pattern that would consequently change into a square pattern due to the excitation of a cross wave. The direct transition from a hexagonal to a square pattern requires a strong perturbation, since it would involve excitation of a mode directed at the angle $\pi/6$ to existing modes, that is suppressed due to strong interaction at this angle. Equation (46a) has a gradient structure, and the hexagonal pattern can be characterized as metastable since it corresponds to a higher minimum of the potential, compared to the square pattern. Formation of other patterns is ruled out due to the strong mutual damping at oblique angles.

The new feature characteristic of oscillatory convection patterns is the competition between propagating and standing waves. If the roll pattern is chosen (at $\bar{\alpha}<0$), the wave must be propagating, since the interaction coefficient at $\theta=\pi$ is twice as large as the self-interaction coefficient and a counter wave is strongly damped whenever the bifurcation is supercritical. A more interesting situation arises when two perpendicular modes are present, due to the specific interaction of pairs of standing waves. For the remainder of this section, we shall concentrate on this problem, assuming that the interaction coefficients satisfy $\bar{\alpha}>0$, $\bar{\alpha}+\bar{\beta}>0$ and restricting to the favored square pattern.

Since the attention can be now restricted to four perpendicular modes $\phi_{\pm1},\phi_{\pm2}$, we can rewrite Eq. (46) setting $\theta=\pi/2$ and rescaling the amplitudes by k_0^2;

$$\dot{\phi}_1=\lambda\phi_1-\phi_1[(\alpha+\beta)|\phi_1|^2+2(\alpha+\beta)|\phi_{-1}|^2+\beta(|\phi_2|^2+|\phi_{-2}|^2)]-2\alpha\phi_2\phi_{-2}\phi_{-1}^* \ , \quad (47)$$

and symmetrically for other ϕ_j. Interaction of noncollinear waves described by the same system of amplitude equations has been studied earlier in a more abstract setting.[3] Representing the amplitudes in the polar form $\phi_j=\rho_j\exp(i\psi_j)$ and separating the real and imaginary parts reduces Eq. (47) to

$$\dot{\rho}_1=\rho_1[\bar{\lambda}-(\bar{\alpha}+\bar{\beta})\rho_1^2-2(\bar{\alpha}+\bar{\beta})\rho_{-1}^2-\bar{\beta}(\rho_2^2+\rho_{-2}^2)]-2\,\mathrm{Re}(\alpha e^{i\psi})\rho_2\rho_{-2}\rho_{-1} \ , \quad (48)$$

$$\dot{\psi}_1=\tilde{\lambda}-(\tilde{\alpha}+\tilde{\beta})\rho_1^2-2(\tilde{\alpha}+\tilde{\beta})\rho_{-1}^2-\tilde{\beta}(\rho_2^2+\rho_{-2}^2)-2\,\mathrm{Im}(\alpha e^{i\psi})\rho_2\rho_{-2}\rho_{-1}/\rho_1 \ . \quad (49)$$

Equations of other real amplitudes and phases are symmetric, with $-\psi$ replacing ψ in equations of $\rho_{\pm2},\psi_{\pm2}$. This system depends upon a single phase combination $\psi=\psi_2+\psi_{-2}-\psi_1-\psi_{-1}$. The symmetry to translations of other three phase variables corresponds to translational symmetries in two spatial directions and in time. Introducing new variables

$$\begin{aligned} q&=\tfrac{1}{2}(\rho_1^2+\rho_{-1}^2+\rho_2^2+\rho_{-2}^2) \ ,\\ p&=\tfrac{1}{2}(\rho_1^2+\rho_{-1}^2-\rho_2^2-\rho_{-2}^2) \ ,\\ r&=\tfrac{1}{2}(\rho_1^2-\rho_{-1}^2+\rho_2^2-\rho_{-2}^2) \ ,\\ s&=\tfrac{1}{2}(\rho_1^2-\rho_{-1}^2-\rho_2^2+\rho_{-2}^2) \ , \end{aligned} \quad (50)$$

we arrive at the system of amplitude equations

$$\dot{q}=2\bar{\lambda}q-(3\bar{\alpha}+5\bar{\beta})q^2-(3\bar{\alpha}+\bar{\beta})p^2+(\bar{\alpha}+\bar{\beta})(r^2+s^2)-2\bar{\alpha}R\cos\psi \ , \quad (51)$$

$$\dot{p}=2\bar{\lambda}p-6(\bar{\alpha}+\bar{\beta})pq+2(\bar{\alpha}+\bar{\beta})rs+2\tilde{\alpha}R\sin\psi \ , \quad (52)$$

$$\dot{r}=2\bar{\lambda}r-2(\bar{\alpha}+2\bar{\beta})qr-2\bar{\alpha}ps \ , \quad (53)$$

$$\dot{s}=2\bar{\lambda}s-2(\bar{\alpha}+2\bar{\beta})qs-2\bar{\alpha}pr \ , \quad (54)$$

$$\dot{\psi}=2(3\tilde{\alpha}+\tilde{\beta})p+2R^{-1}\{\bar{\alpha}\sin\psi[q(q^2-p^2-r^2-s^2)+2prs]+\tilde{\alpha}\cos\psi[p(q^2-p^2-r^2-s^2)+2qrs]\} \ , \quad (55)$$

where $R=\{[(q+p)^2-(r+s)^2][(q-p)^2-(r-s)^2]\}^{1/2}$. It is clear that the phase variable can be relevant only when counterpropagating waves are present. If there are only two waves propagating at the right angle (which corresponds in new variables to $q=r$, $p=s$), Eq. (55) becomes singular, and the variable ψ can be assigned arbitrary values. This implies that stability of a propagating pattern to excitation of counterwaves should be checked at a "most dangerous" value of ψ. It is more convenient to use for this purpose Eqs. (48) and to linearize them in the vicinity of a stationary solution $\rho_1^2=\rho_2^2=\bar{\lambda}/(\bar{\alpha}+2\bar{\beta})$, $\rho_{-1}=\rho_{-2}=0$. The stability conditions of this state are $\bar{\alpha}+\bar{\beta}>0$ (which coincides with the condition of supercritical bifurcation) and

$$(\bar{\alpha}+\bar{\beta})^2>4\,|\,\alpha\,|^2\cos(\arg\alpha-\psi)\ .$$

The most dangerous value of the phase variable is $\psi=\arg\alpha$; thus the latter condition reduces to $|\,\bar{\alpha}+\bar{\beta}\,|>2\,|\,\alpha\,|$. The square propagating wave pattern is destabilized due to the excitation of counterwaves whenever phase-dependent interaction of the second kind is sufficiently strong. The absolute value, rather than the real part of the respective interaction coefficient, is important, since the argument of this coefficient affects only phases of excited counterwaves.

Solutions of Eqs. (48) and (49) or (51)–(55) belonging to the class of standing waves can be obtained by setting in (51)–(55) $r=s=0$. The system of amplitude equations restricted to standing waves is

$$\dot{q}=2\bar{\lambda}q-3(\bar{\alpha}+\bar{\beta})(q^2+p^2)-2(\bar{\beta}+\bar{\alpha}\cos\psi)(q^2-p^2)\ , \tag{56}$$

$$\dot{p}=2\bar{\lambda}p-6(\bar{\alpha}+\bar{\beta})pq-2\tilde{\alpha}(q^2-p^2)\sin\psi\ , \tag{57}$$

$$\dot{\psi}=2\bar{\alpha}q\,\sin\psi+2(3\tilde{\alpha}+\tilde{\beta}-2\tilde{\alpha}\cos\psi)p\ . \tag{58}$$

Two easily computable stationary solutions of (56)–(58) correspond to symmetric synphase and antiphase patterns,

$$q=2\bar{\lambda}/5(\bar{\alpha}+\bar{\beta}),\quad p=0,\quad \psi=0\ , \tag{59}$$

$$q=2\bar{\lambda}/(\bar{\alpha}+5\bar{\beta}),\quad p=0,\quad \psi=\pi\ . \tag{60}$$

Stability of these solutions to disintegrating into propagating waves can be checked with the help of Eqs. (53) and (54). The stability condition for the synphase pattern is $3\bar{\alpha}+\bar{\beta}<0$, which never holds under assumed conditions of supercritical bifurcation. For the antiphase pattern, the stability condition is $\bar{\alpha}>\bar{\beta}$. Recalling the stability condition for a pair of propagating waves, $|\,\bar{\alpha}+\bar{\beta}\,|>2\,|\,\alpha\,|$, and replacing $|\,\alpha\,|$ by $\bar{\alpha}$, we see that the latter condition is violated at $\bar{\alpha}>\bar{\beta}$. Thus the propagating wave pattern and the antiphase standing wave are mutually incompatible, while the synphase pattern is never realized.

The antiphase state can also be destabilized within the class of standing-wave solutions, giving way to a time-dependent standing-wave pattern. The stability condition, that can be obtained by linearizing Eqs. (57) and (58) in the vicinity of the stationary solution (60), is

$$5\,|\,\alpha\,|^2+\mathrm{Re}(\alpha^*\beta)>0\ . \tag{61}$$

The point of marginal instability corresponds to bifurcation of a stationary solution of Eqs. (56)–(58) with $\sin\psi\neq 0$. This solution yields rotating phases ψ_i, rather than a stationary standing-wave pattern, so that behavior becomes persistently time dependent. We shall return to this question after computing interaction coefficients for a particular convection problem in Sec. VII.

V. NON-BOUSSINESQ EFFECTS

It is known that non-Boussinesq effects, that destroy the symmetry of the problem to reflection in the plane $z=0$, strongly influence the selection of convective patterns; in particular, they are responsible for transition from roll to hexagonal patterns in the classical problem of stationary thermal convection in a layer between perfectly conducting plates. The problem of long-scale convection at low Biot numbers is quite sensitive to non-Boussinesq effects. Indeed, $O(1)$ terms lacking the reflectional symmetry would contribute a quadratic term to Eq. (11) and make further arguments pointless. A consistent theory, leading to a nonsymmetric amplitude equation replacing Eq. (34), can be, however, constructed for the case when non-Boussinesq terms are restricted to $O(\epsilon)$. To be definite, we shall introduce a single non-Boussinesq term reflecting dependence of viscosity on state variables, and express the dimensionless viscosity as $(1+\epsilon\mathbf{M}*\mathbf{\Theta})$. The additional term, being $O(\epsilon)$, makes its first appearance when the first-order flow potential is computed, and a new term,

$$-\mathbf{M}*\nabla\cdot\partial_z^2(\mathbf{\Theta}_0\nabla\partial_z^2\chi_0)\ ,$$

should be added to the right-hand side (rhs) of Eq. (23). The resulting contribution to the expression (25) for the even part of the first-order flow potential is

$$-(\mathbf{M}*\mathbf{U}_i)(\mathcal{R}*\mathbf{U}_j)\nabla^{-2}[\nabla\cdot(\phi_i\nabla\phi_j)]\ .$$

The operator ∇^{-2}, standing for the resolvent of the two-dimensional Laplace operator, cancels when this new term is added to Eq. (29). As a result, we arrive at the amended amplitude equation (34),

$$\partial_\tau\phi_l+\kappa\phi_l-\mu k^2\phi_l+\nu k^4\phi_l+\sum\gamma_{ij}(\mathbf{k}_l\cdot\mathbf{k}_j)\phi_i\phi_j-\sum\sigma_{ijk}(\mathbf{k}_i\cdot\mathbf{k}_j)(\mathbf{k}_l\cdot\mathbf{k}_k)\phi_i\phi_j\phi_k=0\ , \tag{62}$$

where

$$\gamma_{ij}=(\mathbf{M}*\mathbf{U}_i)(\mathcal{R}*\mathbf{U}_j)(\mathbf{U}^\dagger*\mathbf{G})=(\mathbf{M}*\mathbf{U}_i)(\mathbf{U}^\dagger*\mathbf{R}*\mathbf{U}_j)\ , \tag{63}$$

and the quadratic terms under summation satisfy the resonance conditions

$$\pm\mathbf{k}_i\pm\mathbf{k}_j=\mathbf{k}_l,\quad \pm k_i^2\pm k_j^2=k_l^2\ . \tag{64}$$

Turning to single-wavelength patterns considered in Sec. IV, we observe that quadratic interaction of a pair of perpendicular waves leads to the excitation of a wave with double frequency directed at the angle $\pi/4$. This is the only possibility compatible with the resonance conditions (64). Four such oblique waves would appear as a result of interaction between four waves constituting the square pattern. Viewing the shorter oblique waves as small-amplitude "slaved" modes, we can use Eq. (62) (with cubic terms omitted) to compute $\phi_{12} = -(\hat{\gamma}/\hat{\lambda})k_0^2\phi_1\phi_2$, where

$$\hat{\gamma}=2(\mathbf{M}*\mathbf{U})(\mathbf{U}^\dagger*\mathbf{R}*\mathbf{U}), \quad \hat{\lambda}=\kappa-2\mu k_0^2+4\nu k_0^4 \ .$$

The amplitudes ϕ_{-12}, ϕ_{1-2}, and ϕ_{-1-2} are expressed in a similar way.

Using the amplitudes of oblique waves in quadratic terms of the amplitude equations of $\phi_{\pm 1}, \phi_{\pm 2}$, we observe that the quadratic resonance conditions (64) can be satisfied by setting, say, $l=1$, $i=12$, $j=2$, and entering the latter mode with the negative sign, i.e., taking the complex conjugate of the second mode. This leads to the amplitude equations of the same form as (45) and (46) but with the interaction coefficient between perpendicular waves, β, replaced by $\beta-\gamma$, where

$$\gamma=2\hat{\lambda}^{-1}k_0^4\,|\,\mathbf{M}*\mathbf{U}\,|^2(\mathbf{U}^\dagger*\mathbf{R}*\mathbf{U})^2 \ . \tag{65}$$

Other interaction coefficients remain unchanged. The stability conditions of alternative patterns discussed in Sec. IV are, of course, modified when the above correction to the interaction coefficient is introduced, but, as before, the choice is between either standing or propagating roll and square-wave patterns. Quadratic interaction between waves of identical wavelength directed at an angle other than right is incompatible with the resonance conditions (64).

VI. INERTIAL EFFECTS

Inertial effects are, generally, expected to enhance complexity of behavior due to the generation of vertical vorticity by nonlinear interactions. The hydrodynamic equations to be used, instead of Eq. (2), when the inverse Prandtl number $\mathcal{P}^{-1}\neq 0$, are

$$(\nabla^2+\partial_z^2)^2\nabla^2\chi = \mathcal{R}*\nabla^2\Theta+\mathcal{P}^{-1}[(\nabla^2+\partial_z^2)\nabla^2\partial_t\chi+\nabla\cdot\partial_z(\nabla^2\chi\nabla\partial_z^2\chi)] \ , \tag{66}$$

$$(\nabla^2+\partial_z^2)\nabla^2\psi=\mathcal{P}^{-1}(\partial_t\nabla^2\psi+\nabla\nabla^2\chi\times\nabla\partial_z^2\chi) \ . \tag{67}$$

Other nonlinear terms, that would be relevant only in higher orders, are omitted. The boundary conditions for the toroidal potential ψ are, analogous to (3),

$$\nabla^2\psi=0 \quad \text{at } z=\pm\tfrac{1}{2} \ . \tag{68}$$

The additional terms do not have to be taken into account before the first-order flow field is computed with the help of Eq. (23), which has to be now replaced by

$$\partial_z^4\nabla^2\chi_1=\mathcal{R}_0*\nabla^2\Theta_1+\mathcal{R}_1*\nabla^2\Theta_0-2(\nabla^2)^2\partial_z^2\chi_0 +\mathcal{P}^{-1}[\partial_z^2\nabla^2\partial_t\chi_0+\nabla\cdot\partial_z(\nabla^2\chi_0\nabla\partial_z^2\chi_0)] \ , \tag{69}$$

$$\partial_z^2\nabla^2\psi_1=\mathcal{P}^{-1}\nabla\nabla^2\chi_0\times\nabla\partial_z^2\chi_0 \ . \tag{70}$$

The additional term entering the expression (25) for $\chi_1^{(\text{even})}$ is computed using (9), (11), and (13) as

$$\tfrac{1}{2}\lambda_i\mathcal{P}^{-1}f_{10}(z)\mathbf{U}_i\nabla^2\phi_i \ . \tag{71}$$

The nonlinear term in Eq. (69) contributes to the expression (25) for $\chi_1^{(\text{odd})}$, but odd terms do not enter Eq. (29) and are therefore irrelevant for our purpose.

The toroidal potential, that can differ from zero only in the presence of inertial effects, is computed using Eq. (70) as

$$\psi_1=\mathcal{P}^{-1}\hat{f}_1(z)(\mathcal{R}_0*\mathbf{U}_i)(\mathcal{R}_0*\mathbf{U}_j)\nabla^{-2}(\nabla\nabla^2\phi_i\times\nabla\phi_j) \ . \tag{72}$$

Here ∇^{-2} stands for the resolvent of the two-dimensional Laplacian and the function $\hat{f}_1(z)$ satisfies

$$\partial_z^2\hat{f}_1=f_0\partial_z^2f_0, \quad \hat{f}_1=0 \quad \text{at } z=\pm\tfrac{1}{2} \ . \tag{73}$$

Using (70)–(73) in the equation of the second-order state vector (29), supplemented by the term $\nabla\Theta_0\times\nabla\psi_1$, we arrive at the amplitude equation (34) containing additional terms of the type

$$\hat{\sigma}_{ijkl}[\mathbf{k}_i^2/\,|\,\mathbf{k}_i+\mathbf{k}_j\,|^2](\mathbf{k}_i\times\mathbf{k}_j)(\mathbf{k}_k\times\mathbf{k}_l)\phi_i\phi_j\phi_k \ , \tag{74}$$

with

$$\hat{\sigma}_{ijkl}=\mathcal{P}^{-1}\langle\hat{f}_1\rangle(\mathcal{R}_0*\mathbf{U}_i)(\mathcal{R}_0*\mathbf{U}_j)(\mathbf{U}^\dagger*\mathbf{D}_0^{-1}*\mathbf{U}_k) \ . \tag{75}$$

Integrating Eq. (73) yields $\langle\hat{f}_1\rangle/\langle f_0\rangle^2=\frac{15}{7}$. As before, the interacting modes should satisfy the resonance conditions (35).

The correction to the coefficient ν in (38) due to Eq. (71) is purely imaginary, $-\frac{1}{2}i\omega_0\langle f_{10}\rangle\mathcal{P}^{-1}$, and therefore does not affect the preferred wavelength of bifurcating waves. The additional interaction terms cancel when the wave numbers in Eq. (74) have the same absolute value; therefore, inertial effects do not affect pattern selection in the vicinity of the primary bifurcation.

VII. THERMAL CONVECTION IN THE PRESENCE OF THE SORET EFFECT

As a simple example, we consider thermal convection in the presence of the Soret effect that is responsible for coupling of thermal and diffusional fluxes. The heat and mass fluxes are defined as

$$-j_T=\kappa\nabla T, \quad -j_C=\kappa_S\nabla T+D\nabla C \ , \tag{76}$$

where κ is the thermal diffisivity, D is the molecular diffusivity, κ_S is the Soret coefficient; the conjugate Dufour effect is neglected. The fluid density is assumed to vary linearly with both temperature T and concentration C,

$$\rho=\rho_0[1-\alpha_T(T-T_0)+\alpha_C(C-C_0)] \ . \tag{77}$$

It is advantageous to use a "rotated" state vector, replacing concentration C by the linear combination $\eta=C-C_0+(\kappa_S/D)(T-T_0)$, which is constant in the quiescent state.[8] We shall scale the vertical coordinate z by the layer thickness h, horizontal coordinates $\mathbf{x}$ by $l=h/\epsilon^{1/2}$, time by the characteristic horizontal conduction time l^2/κ, and χ potential by $h\kappa$. The small parameter ϵ is identified with the square root of the thermal Biot number. The temperature deviation from the linear profile in the quiescent state is scaled by the temperature decrement across the layer ΔT, and the concentration deviation, or the combination η, by $\hat{\alpha}\Delta T/\alpha_c$ where $\hat{\alpha}=\alpha_T(1+S)$; $S=\kappa_S\alpha_C/D\alpha_T$ is the separation ratio. In this formulation, the problem contains dimensionless parameters $\mathcal{R}=\hat{\alpha}gh^3\Delta T/\kappa\nu$ (Rayleigh number), $L=D/\kappa$ (inverse Lewis number), and $\psi=S/L(1+S)$. Inertial effects are neglected. The matrices used in the general formulation of the problem in Sec. II are identified as

$$\mathbf{D}=\begin{bmatrix}1 & 0\\ \psi L & L\end{bmatrix},\quad \mathbf{G}=\begin{bmatrix}1\\0\end{bmatrix},\quad \boldsymbol{\mathcal{R}}=\begin{bmatrix}\mathcal{R}\\-\mathcal{R}\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}. \tag{78}$$

The condition of oscillatory instability is

$$\mathcal{R}=\mathcal{R}_0=720(1+L)\ , \tag{79}$$

corresponding to the onset of oscillatory convection with the frequency

$$\omega_0=L\,|\,1+\psi(1+L^{-1})\,|^{1/2}\ . \tag{80}$$

The normalized eigenvectors $\mathbf{U},\mathbf{U}^\dagger$, of $\mathbf{A}$, $\mathbf{A}^\dagger$ with the eigenvalue $-i\omega_0$ are

$$\mathbf{U}=\begin{bmatrix}L+i\omega_0\\ -\psi L\end{bmatrix},\quad \mathbf{U}^\dagger=-\frac{1}{2}\begin{bmatrix}i/\omega_0\\ 1/L\psi+i/\omega_0\psi\end{bmatrix}\ . \tag{81}$$

It is convenient to parametrize the locus of Hopf bifurcation by the frequency ω_0 and to eliminate ψ by inverting Eq. (80),

$$\psi=-(L^2+\omega_0^2)/L(L+1)\ . \tag{82}$$

The value of ψ is negative, in accordance with a well-known fact that the Soret effect must be anomalous (with the light component concentrating in the cold region) to allow for oscillatory convection.[1,2] Computation of the parameters of the amplitude equation yields

$$\kappa=\frac{1}{2}\left[1+\frac{i}{\omega_0}[L(L+1)+\omega_0^2]\right],$$
$$\mu=\tfrac{1}{1440}\mathcal{R}_1\left[1-\frac{i}{\omega_0}\frac{L-\omega_0^2}{L+1}\right], \tag{83}$$
$$\nu=\frac{1}{924}\left[\left[1+\frac{1}{L}\right](17L+5\omega_0^2)+\frac{i}{\omega_0}(-17L+12\omega_0^2+5\omega_0^4/L)\right].$$

The real parts of κ and ν are always positive, insuring stabilization on both large and small scales. The symmetry-breaking bifurcation occurs at

$$\mathcal{R}_1=\left[\frac{2}{231}\left[1+\frac{1}{L}\right](17L+5\omega_0^2)\right]^{1/2}, \tag{84}$$

yielding waves with the wave number

$$k_0=\left[\frac{1}{462}\left[1+\frac{1}{L}\right](17L+5\omega_0^2)\right]^{-1/4}. \tag{85}$$

The interaction coefficients entering Eq. (47) are

$$\alpha=\frac{1}{2}\left[1+\frac{1}{L}\right][L^2+\omega_0^2(1+L^2)+\omega_0^4]+\frac{i}{2L\omega_0}[L^2+\omega_0^2(L+\omega_0^2)(1+L+L^2)\omega_0^6]\ , \tag{86}$$

$$\beta=\frac{i}{\omega_0}[L^2+\omega_0^2(1+L^2)+\omega_0^4]\ . \tag{87}$$

The interaction coefficient of two perpendicular waves β turns out to be purely imaginary. As follows from the discussion in Sec. IV, this implies that propagating patterns are unstable. The antiphase standing pattern is stable both to disintegration into propagating waves and to perturbations within the class of standing waves. The latter is seen immediately, since the imaginary parts of both α and β are positively definite, so that the condition (61) always holds. Thus the antiphase pattern of standing waves is selected.

Non-Boussinesq effects are qualitatively significant in this case, inasmuch as they contribute a nonvanishing real part to the interaction coefficient of perpendicular waves. If we assume that viscosity depends on temperature only, but not on concentration, i.e., set $\mathbf{M}=|\,m_1,0\,|$, the interaction coefficient (65) is computed as

$$\gamma=720m_1^2(L^2+\omega_0^2)[(1+L)^2+\omega_0^2)]^{-1}\times\{[(1+4L+2L^2)-\omega_0^2(3+2L)]+(i/\omega_0)[-\omega_0^4+\omega_0^2(3+5L+L^2)-L(1+L)]\}\ . \tag{88}$$

The real part of γ is positive at $\omega_0^2<(1+4L+2L^2)/(3+2L)$. Under these conditions, perpendicular waves actually enhance one another, and bifurcation becomes subcritical when the value of m_1 is sufficiently high. The stationary amplitude, Eq. (60), of the antiphase pattern, is modified to $q=2\overline{\lambda}/(\overline{\alpha}+5\overline{\beta}-2\overline{\gamma})$, and the transition to a subcritical bifurcation manifests itself as a high-amplitude runaway at large $\overline{\gamma}$. On the contrary, at higher frequencies, increasing temperature dependence of viscosity leads to mutual damping of perpendicular waves, and a propagating roll pattern becomes favorable at large values of m_1.

Another possibility is the destabilization of the stationary antiphase pattern due to the violation of the condition (61), which is now modified to

$$5\,|\,\alpha\,|^2+\mathrm{Re}(\alpha^*\beta)+2\,\mathrm{Re}(\alpha^*\gamma)>0\ . \tag{89}$$

This condition is violated, at a sufficiently high value of m_1, if the last term is negative. Inspecting Eq. (88) we see that this is indeed the case at sufficiently large frequencies when both the real and imaginary parts of γ are negative. Instability is also possible at low frequencies, when the imaginary part of γ is negative and large. Under these conditions, non-Boussinesq effects induce time-dependent behavior by destabilizing the antiphase pattern of standing waves that is prevalent in their absence.

VIII. HIGHER DEGENERACIES

Higher degeneracies can be arranged by fixing values of additional parameters of the matrix **A**, i.e., restricting to a higher co-dimension manifold in the parametric space of the original problem. With the dimension of state space $n=2$ (double-diffusive convection), the only possible degeneracy is the double zero eigenvalue. If $n=3$, one can obtain another co-dimension two degeneracy, when the matrix **A** has a zero and a pair of imaginary eigenvalues. In addition, a co-dimension three degeneracy—triple zero eigenvalue—becomes possible. The list can be continued at higher n. When the degeneracy is algebraic, but not geometric, an amplitude ϕ_i can be assigned to each distinct eigenvector $\mathbf{U}_i$, and the solution expressed as

$$\Theta_0 = \sum \phi_i \mathbf{U}_i \ , \tag{90}$$

with the amplitudes ϕ_i either constant on the $O(1/\epsilon)$ time scale (for zero eigenvalues), or obeying on this scale the Schrödinger equation with a suitable frequency (for imaginary eigenvalues). The resonance conditions analogous to (35) govern the structure of amplitude equations.

When the degeneracy is geometric as well as algebraic, the expansion procedure has to be modified by using multiple scaling of different amplitudes.[7] Higher degeneracies are generally expected to yield a rich, and as yet totally unexplored variety of dynamic patterns.

ACKNOWLEDGMENTS

This work was carried out during the author's leave of absence at the Division of Applied Mathematics of Brown University, which was made possible by the U.S. Defense Advanced Projects Agency–Universities Research Initiative (DAPRA-URI) Grant No. N00014-86-K-0754.

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