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Klauder's continuous representations and fuzzy coordinate variables. II

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Based on Klauder's continuous representations in Hilbert space, we discuss a fundamental fuzziness of the coordinate and momentum variables for a quantum particle. These fuzzy dynamical variables are intimately related to a very small radical length R ($\lesssim 10^{-18}$ cm) and a very large cosmological length S ($\gtrsim 10^{10}$ light years). The uncertainty Δq for the measurement of a particle's position alone is assumed to be restricted by $S \gtrsim \Delta q \gtrsim R$. Physical properties of fuzzy base states $(\mathbf{p}|$ and $(\mathbf{q}|$ and their general cosmological and very-high-energy implications are discussed. In particular, a new density of states and its possible effects on decay rates and scattering cross sections at large momenta $p \gtrsim \hbar/R \sim 10^4$ GeV/c (c is the velocity of light) are considered. We also discuss the superposition of waves in the presence of R and S.

I. INTRODUCTION

In a previous paper[1] we discussed and explored fundamental properties of the position and the momentum operators, **Q** and **P**, and their states on the basis of Klauder's continuous representations in Hilbert space,[2] which excludes the usual eigenstates $\langle q|$ and $\langle p|$. We note that the eigenstates $\langle q|$ and $\langle p|$ have infinite lengths and cannot be normalized to have an elementary probabilistic interpretation. Dirac observed that the infinite length of the ket vectors corresponding to these eigenstates may be connected with their unrealizability and that all realizable states correspond to ket vectors that can be normalized and that form a Hilbert space.[3,4] Thus, in the bra-ket formalism of quantum mechanics, one gives up the fundamental idea of probabilistic interpretation of a state for the sake of introducing certain mathematical idealizations of what can be realized. It does not appear to be in harmony with the operational principle in physics which states that one should formulate a physical theory by using observable quantities and realizable states. This motivates us to explore a new formalism based on Klauder's continuous representations for **Q** and **P** which can have only fuzzy values and fuzzy base states $(q|$ and $(p|$ with finite lengths. Let us term such a formalism "fuzzy quantum mechanics."

For simplicity, we assumed the momentum operator **P** to be the same as usual in the previous paper,[1] so that we could concentrate on the fuzziness of the coordinate operator **Q**. In the present paper we first discuss general relations for both fuzzy operators **Q** and **P**. In this general case, Klauder's continuous representations for **Q** and **P** involve a small radical length R and a very large cosmological length S. We consider general physical implications in cosmological and high-energy phenomena based on quantum-mechanical framework. We also discuss the possibility that the usual four-dimensional symmetry of flat space-time becomes exact only in the limit $R \to 0$ and $S \to \infty$ (in which **Q** and **P** are no longer fuzzy).

II. FUZZY OPERATORS Q AND P

Let us consider the mathematical properties of fuzzy state vectors which are denoted by the parenthesis notation rather than the usual bracket notation. A fuzzy state, say, the position of a particle, is represented by a state vector [a "thesis" vector $|\alpha)$ or a "paren" vector $(\alpha|$] in Klauder's continuous representation.[1] As usual, the state thesis $|\alpha)$ of a physical system is postulated to contain "complete information" about the state.

There are particular theses of importance, namely, the base theses of **Q** and **P**. According to Klauder's continuous representations, we have the relation[1]

$$(\mathbf{q}'|\mathbf{q}) = D^2(\boldsymbol{\partial}')\delta^3(\mathbf{q}'-\mathbf{q}), \quad \boldsymbol{\partial}' = -i\hbar\partial/\partial\mathbf{q}' , \tag{1}$$

$$D(\boldsymbol{\partial}') = 1/(2R^2\boldsymbol{\partial}'^2/\hbar^2+1) \tag{2}$$

for the base state $|\mathbf{q})$ which is assumed to be the wave packet with the minimum width $\Delta q_{\min} \approx R$ that can be physically realized in nature. The absence of the position eigenstates in fuzzy quantum mechanics implies that, in principle, one cannot measure the position of a particle with absolute certainty. In this sense, we have made a further departure from the determinism in classical mechanics than the conventional quantum mechanics.

Suppose a state $|\boldsymbol{\alpha})$ is formed by the superposition of the base state $|\mathbf{p})$,

$$|\boldsymbol{\alpha}) = \int F(\mathbf{p})|\mathbf{p})d^3p, \quad \int \equiv \int_{-\infty}^{\infty} , \tag{3}$$

where $|\mathbf{p})$ is the base momentum state with the minimum width $\Delta\mathbf{p}_{\min} \approx \hbar/S$. The length scale S is interpreted as the effective size of the physical universe.[1] [See Eq. (14) below.] The physically realizable wave packet corresponding to $|\boldsymbol{\alpha})$ has a width Δq restricted by

$$S \gtrsim \Delta q \gtrsim R . \tag{4}$$

This relation is considered as a fundamental assumption for fuzzy quantum mechanics based on Klauder's continuous representations.

Thus physically realizable wave packets correspond to a certain sector (or subset) of vectors in a Hilbert space. In this physical sector, state vectors have a finite length and a finite width given by (4). This physical sector does not involve any eigenvector of **Q** or **P** and has nondenumerably infinite dimensionals. We note that this physical sector of vector space does not form a closed linear manifold (or a subspace) in the usual mathematical sense. Nevertheless, one can have a restriction on the superposition of waves, so that realizable states of **Q** and **P** always correspond to vectors in the physical sector and do not contradict previous experiments. [See the discussions after Eq. (44) below.]

Since the mathematics of Hilbert space is so elegant and beautiful, one might deplore the fact that physical states in fuzzy quantum mechanics correspond to only a subset of vectors associated with **Q** and **P** in Hilbert space. But this should not be considered as a drawback of the present formalism. The situation here resembles the fact that observed particles with certain masses and spin in nature correspond to only a subset of particles allowed in the irreducible representations of the fundamental four-dimensional group, i.e., the Poincaré group. (Particles with a continuous spin do not seem to be realized in nature.) Similarly, the physical states in Yang-Mills field theories and the unified electroweak theory are only a subset of all states in the theory.[5]

The properties of the base theses $|p)$, the base parens $(p\,|$, and the fuzzy operators **P** and **Q** acting on them can be summarized as follows:

$$(\mathbf{p}\,|\,\mathbf{Q}=\partial_p(\mathbf{p}\,|\,,\quad \partial_p=i\hbar\partial/\partial\mathbf{p}\ , \tag{5}$$

$$\mathbf{Q}\,|\,\mathbf{p})=-\partial_p\,|\,\mathbf{p})\ , \tag{6}$$

$$P_k\,|\,\mathbf{p})=[p_k-i\hbar\partial\ln E(\mathbf{Q})/\partial Q_k]\,|\,\mathbf{p})\ , \tag{7}$$

$$(\mathbf{p}\,|\,P_k=(\mathbf{p}\,|\,[p_k+i\hbar\partial\ln E(\mathbf{Q})/\partial Q_k]\ , \tag{8}$$

$$\begin{aligned}(\mathbf{p}'\,|\,\mathbf{p})&=\int E^2(\mathbf{q})\exp[-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{q}/\hbar]d^3q/(2\pi\hbar)^3\\&=E^2(\partial_p')\delta^3(\mathbf{p}'-\mathbf{p})=E(\partial_p')E(\partial_p)\delta^3(\mathbf{p}'-\mathbf{p})\ ,\end{aligned} \tag{9}$$

$$\begin{aligned}\int d^3p\,\,|\,\mathbf{p})E^{-2}(\partial_p)(\mathbf{p}\,|&=\int d^3p\,[E^{-1}(\partial_p)\,|\,\mathbf{p})]\\&\quad\times[E^{-1}(\partial_p)(\mathbf{p}\,|\,]=1\ ,\end{aligned} \tag{10}$$

where $E(\partial_p)$ and $E^{-2}(\partial_p)$ are understood as integral operators

$$\begin{aligned}F(\partial_p)Y(\mathbf{p})=(2\pi\hbar)^{-3}&\int d^3q\,\exp(-i\mathbf{q}\cdot\mathbf{p}/\hbar)F(\mathbf{q})\\&\times\int d^3p'\exp(i\mathbf{p}'\cdot\mathbf{q}/\hbar)Y(\mathbf{p}')\ ,\end{aligned} \tag{11}$$

$$Y(\mathbf{p})=\,|\,\mathbf{p})\quad\text{or}\ \varphi(\mathbf{p})\ .$$

These relations characterize the fuzzy p representation in the theory. To guide our discussions, we follow the previous paper and assume

$$E(\mathbf{q})=\exp(-\,|\,\mathbf{q}\,|\,/2S)\equiv E(\mathbf{q}/S)\ . \tag{12}$$

That is, the whole physical space may be pictured as a "big fuzzy point."[1] The results in this paper are not sensitive to the spacific form of $E(\mathbf{q})$ in Klauder's continuous representations, provided that

$$E(\mathbf{q})\approx\begin{cases}1 & \text{for }\ |\,\mathbf{q}\,|\ll S\\ 0 & \text{for }\ |\,\mathbf{q}\,|\gg S\ .\end{cases}$$

It appears that the explicit form of $E(\mathbf{q})$ cannot be determined by existing principles of physics or by known experiments. We hope that it can be determined in the future.

With the help of (12), Eq. (9) can be written as

$$\begin{aligned}(\mathbf{p}'\,|\,\mathbf{p})&=(\hbar/\pi S)[(\mathbf{p}'-\mathbf{p})^2+\hbar^2/S^2]^{-2}\\&\to\delta^3(\mathbf{p}'-\mathbf{p})\quad\text{as }S\to\infty\ .\end{aligned} \tag{13}$$

In contrast with $\langle\mathbf{p}\,|\,\mathbf{p}\rangle=\delta^3(0)=\infty$, we have

$$(\mathbf{p}\,|\,\mathbf{p})=S^3/(\hbar^3\pi^2)\neq\infty\ ,$$

which can be used to normalize $(\mathbf{p}\,|\,\mathbf{p}')$. We have seen from (13) and (4) that the length S characterizes the size of the physical three-dimensional space. According to fuzzy quantum mechanics, the physical universe, in which physical phenomena take place, has an effective finite volume V_S because of the presence of $E(\mathbf{q})$ in Klauder's continuous representations. If $E(q)$ is given by (12), we have

$$V_S=\int d^3q\,E^2(\mathbf{q})=8\pi S^3\ , \tag{14}$$

where S is presumably of the order of 10^{10} light years or larger.

Equations (5) and (6) for the base momentum vectors look familiar. But the results (7) and (8) show that they are not eigenvectors of the momentum P and that the observed value of **P** has a fuzziness due to $E(\mathbf{q})$ or the finite size of the physical universe. Such a fuzziness is inherent in nature and cannot be reduced by any artificial method. One may consider (9) as a "fuzzy orthogonal relation" because it can be written as

$$\begin{aligned}E^{-2}(\partial_p')(\mathbf{p}'\,|\,\mathbf{p})&=E^{-1}(\partial_p')E^{-1}(\partial_p)(\mathbf{p}'\,|\,\mathbf{p})\\&=(\mathbf{p}'\,|\,E^{-2}(\mathbf{Q})\,|\,\mathbf{p})=\delta^3(\mathbf{p}'-\mathbf{p})\ .\end{aligned}$$

The fuzzy closure relation (10) enables us to express an arbitrary paren or thesis in terms of the base parens or theses, e.g.,

$$|\,\alpha)=\int d^3q\,\,|\,\mathbf{q})D^{-2}(\partial)(\mathbf{q}\,|\,\alpha)\ .$$

Similarly, the fuzzy position operator **Q**, the base theses $|\,\mathbf{q})$, and the base parens $(\mathbf{q}\,|$ are assumed to have properties corresponding to Eqs. (5)–(10):[1]

$$(\mathbf{q}\,|\,\mathbf{P}=-i\hbar\partial/\partial\mathbf{q}(\mathbf{q}\,|\,\equiv\partial(\mathbf{q}\,|\ , \tag{15}$$

$$\mathbf{P}\,|\,\mathbf{q})=-\partial\,|\,\mathbf{q})\ , \tag{16}$$

$$\mathbf{Q}\,|\,\mathbf{q})=[\mathbf{q}+i\hbar\partial\ln D(\mathbf{P})/\partial\mathbf{P}]\,|\,\mathbf{q})\ , \tag{17}$$

$$(\mathbf{q}\,|\,\mathbf{Q}=(\mathbf{q}\,|\,[\mathbf{q}-i\hbar\partial\ln D(\mathbf{P})/\partial\mathbf{P}]\ , \tag{18}$$

$$\begin{aligned}(\mathbf{q}'\,|\,\mathbf{q})&=(2\pi\hbar)^{-3}\int D^2(\mathbf{p})\exp[i(\mathbf{q}'-\mathbf{q})\cdot\mathbf{p}/\hbar]d^3p\\&=D^2(\partial')\delta^3(\mathbf{q}'-\mathbf{q})\ ,\end{aligned} \tag{19}$$

$$\begin{aligned}\int d^3q\,\,|\,\mathbf{q})D^{-2}(\partial)(\mathbf{q}\,|&=\int d^3q\,[D^{-1}(\partial)\,|\,\mathbf{q})][D^{-1}(\partial)(\mathbf{q}\,|\,]\\&=1\ ,\end{aligned} \tag{20}$$

where $D^2(\partial)$ and $D^{-2}(\partial)$ are integral operators:

$$f(\partial)\varphi(\mathbf{q})=\frac{1}{(2\pi\hbar)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{q}/\hbar}f(\mathbf{p})\times\int d^3q' e^{-i\mathbf{q}'\cdot\mathbf{p}}\varphi(\mathbf{q}') . \quad (21)$$

The function $D^2(\mathbf{p})$ in (19) is given by[1]

$$D^2(\mathbf{p})=1/(2R^2\mathbf{p}^2/\hbar^2+1)^2 , \quad (22)$$

so that $(\mathbf{q}'\mid\mathbf{q})$ can be pictured as a fuzzy point with a bell shape and a width approximately equal to R. From (19) and (22) we obtain

$$(\mathbf{q}'\mid\mathbf{q})=(16\pi 2^{1/2}R^3)^{-1}\exp(-\mid\mathbf{q}'-\mathbf{q}\mid/2^{1/2}R) \to\delta^3(\mathbf{q}'-\mathbf{q}) \text{ as } R\to\infty , \quad (23)$$

which shows that the base thesis $\mid\mathbf{q})$ is normalizable.

The sector of fuzzy thesis states under consideration can be viewed as being spanned either by the set $\{\mid\mathbf{q})\}$ or by the set $\{\mid\mathbf{p})\}$. It can be shown that the relations (5)–(10) expressed in the set $\{\mid\mathbf{p})\}$ and the relations (15)–(20) expressed in the set $\{\mid\mathbf{q})\}$ are connected by a new transformation function $(\mathbf{q}\mid\mathbf{p})$,

$$(\mathbf{q}\mid\mathbf{p})=(2\pi\hbar)^{-3/2}E(\partial_p)D(\partial)\exp(i\mathbf{p}\cdot\mathbf{q}/\hbar)$$
$$=(2\pi\hbar)^{-3/2}E(\partial_p)D(\mathbf{p})\exp(i\mathbf{p}\cdot\mathbf{q}/\hbar) , \quad (24)$$

where $E(\partial_p)$ and $D(\partial)$ are integral operators given in (11) and (21). For example, the relation (17) can be obtained from (6), (10), and (24):

$$\mathbf{Q}\mid\mathbf{q})=\int\mathbf{Q}\mid\mathbf{p})E^{-2}(\partial_p)(\mathbf{p}\mid\mathbf{q})d^3p$$
$$=[\mathbf{q}+i\hbar\partial\ln D(\mathbf{P})/\partial\mathbf{P}]\mid\mathbf{q}), \quad \mathbf{P}\equiv\partial . \quad (25)$$

We note that the new transformation function (24) corresponds to the plane wave in ordinary quantum mechanics. In the present formalism, the expression (24) is interpreted as the physically realizable "plane wave" for a free particle. In other words, the usual plane wave $\exp(i\mathbf{p}\cdot\mathbf{q}/\hbar)$ is an idealized function which cannot be physically realized (see Appendix A).

In this connection, we also note that the position operator $\mathbf{Q}$ in Eqs. (5) and (6) of the fuzzy p representation behaves like ordinary the Hermitian operator, $Q_k=i\hbar\partial/\partial P_k$, and its fuzziness is not obvious. However, with the help of the new transformation function (24), we have the result (25) which clearly shows the momentum-dependent fuzziness of $\mathbf{Q}$. Also, from the expression (25) for $\mathbf{Q}\mid\mathbf{q})$, it is not clear whether $(\mathbf{q}\mid\mathbf{Q}\mid\mathbf{q}')$ is a real number. Nevertheless, we can show that, for an arbitrary $\mid\alpha)$ and $(\beta\mid$, the quantity $(\alpha\mid\mathbf{Q}\mid\beta)$ is a real number,

$$(\alpha\mid\mathbf{Q}\mid\beta)=(\beta\mid\mathbf{Q}\mid\alpha)^*=\int\varphi_\alpha^*(\mathbf{q})\mathbf{q}\varphi_\beta(\mathbf{q})E^2(\mathbf{q})d^3q , \quad (26)$$

$$E(\mathbf{q})\varphi_\alpha^*(\mathbf{q})=D^{-1}(\partial)(\alpha\mid\mathbf{q}) . \quad (27)$$

III. FUZZY MATRIX REPRESENTATIONS AND FUZZY UNCERTAINTY RELATIONS

Suppose we have an equation for operators T, U, and V:

$$T=UV .$$

Using the fuzzy closure relation (20), it can be written as

$$(\mathbf{q}\mid T\mid\mathbf{q}'')=(\mathbf{q}\mid UV\mid\mathbf{q}'')$$
$$=\int(\mathbf{q}\mid U\mid\mathbf{q}')D^{-2}(\partial')(\mathbf{q}'\mid V\mid\mathbf{q}'')d^3q' . \quad (28)$$

Thus the operators T, U, and V in this formalism can be represented by matrices $(\mathbf{q}\mid T\mid\mathbf{q}'')$, $(\mathbf{q}\mid U\mid\mathbf{q}')$, and $(\mathbf{q}'\mid V\mid\mathbf{q}'')$, which may be termed "fuzzy matrices" because they cannot be diagonalized. We stress that these fuzzy matrices follows a modified rule of multiplication due to the presence of $D^{-2}(\partial')$, as shown in (28). Some other examples of fuzzy matrices and their new multiplication rule are as follows:

$$(\mathbf{q}\mid U\mid\psi)=\int(\mathbf{q}\mid U\mid\mathbf{q}')D^{-2}(\partial')(\mathbf{q}'\mid\psi)d^3q' \quad (29a)$$
$$=\int(\mathbf{q}\mid\mathbf{q}')D^{-2}(\partial')(\mathbf{q}'\mid U\mid\psi)d^3q' , \quad (29b)$$
$$(\phi\mid\psi)=\int(\phi\mid\mathbf{q})D^{-2}(\partial)(\mathbf{q}\mid\psi)d^3q \quad (30a)$$
$$=\int(\phi\mid\mathbf{p})E^{-2}(\partial_p)(\mathbf{p}\mid\psi)d^3p , \quad (30b)$$

where we have used Eqs. (10) and (20). We see that $(\mathbf{q}'\mid\psi)$ can be looked upon as a continuous fuzzy column matrix and $(\phi\mid\mathbf{q})$ a fuzzy row matrix. The complex number $(\phi\mid\psi)$ in (30) can be expressed as the multiplication of the two matrices $(\phi\mid\mathbf{q})$ and $(\mathbf{q}\mid\psi)$ in the fuzzy q representation. From (29b), we see that $(\mathbf{q}\mid\mathbf{q}')$ is a "unit fuzzy matrix." We note that fuzzy matrices obey all rules of the usual matrices except the multiplication rule. Evidently, fuzzy matrices reduce to usual continuous matrices in the limit $R\to 0$ and $S\to\infty$. Of course, the matrix representations of discrete observables such as the angular momentum $\mathbf{Q}\times\mathbf{P}$, etc. and their states have the usual properties.

As usual, $\mathbf{Q}$ and $\mathbf{P}$ are assumed to satisfy the commutation relation

$$P_jQ_k-Q_kP_j=-i\hbar\delta_{jk} , \quad (31)$$

as one can see from (5) and (8). Therefore one has the uncertainty relation

$$\Delta P_k\Delta Q_k\geq\hbar/2, \quad k=1,2,3 \quad (32)$$

except that neither ΔP_k nor ΔQ_k can be arbitrarily small.

Let us consider the spatial extension $\Delta Q^k_{\min}$ of the fuzzy base state $\mid\mathbf{q})$,

$$\Delta Q^k_{\min}=(\langle Q_k^2\rangle_{\min}-\langle Q_k\rangle^2_{\min})^{1/2}, \quad k=1,2,3 \quad (33)$$

where

$$\langle F(\mathbf{Q})\rangle=(\mathbf{q}\mid F(\mathbf{Q})\mid\mathbf{q})/(\mathbf{q}\mid\mathbf{q}) , \quad (34)$$

$$(\mathbf{q}\mid Q_k^2\mid\mathbf{q})=\int\mid D^{-1}(\partial')(\mathbf{q}'\mid\mathbf{q})\mid^2 q_k'^2d^3q'$$
$$=\int\left[\frac{\exp(-\mid\mathbf{q}'-\mathbf{q}\mid/2^{1/2}R)}{8\pi R^2\mid\mathbf{q}-\mathbf{q}'\mid}\right]^2 q_k'^2d^3q'$$
$$=\frac{1}{48\pi R2^{1/2}} , \quad k=1,2,3 . \quad (35)$$

From Eqs. (34), (35), and (23), we find

$$\Delta Q^k_{\min} = R/3^{1/2}, \quad k=1,2,3 \ . \tag{36}$$

It follows from (36) and the minimum uncertainty relation $\Delta P_k \Delta Q_k = \hbar/2$ that

$$\Delta P^k_{\max} = 3^{1/2}\hbar/(2R) \ , \tag{37}$$

which implies that it is impossible to have arbitrarily large momentum (or short wavelength). Otherwise, if one has an indefinitely short wavelength, then one can determine the position of a particle with unlimited accuracy. As a matter of fact, a particle with a momentum p is suppressed by an inherent probability proportional to $D^2(p)$, as shown in (19). The suppression is significant only when $p \gtrsim \hbar/R$. In high-energy laboratories, we now have particles with momenta $p \sim 10^3$ GeV/c and we have not yet seen any anomaly related to the inherent suppression of momenta. This enables us to estimate that $\hbar/R$ should be roughly 10^4 GeV/c or larger.

Similarly, we can calculate the minimum uncertainty $\Delta P^k_{\min}$,

$$\Delta P^k_{\min} = (\langle P_k^2 \rangle_{\min} - \langle P_k \rangle^2_{\min})^{1/2}, \quad k=1,2,3$$

where

$$\begin{aligned}(\mathbf{p} \mid (P^k)^2 \mid \mathbf{p}) &= \int d^3p' p_k'^2 \mid E^{-1}(\partial_p')(\mathbf{p}' \mid \mathbf{p}) \mid^2 \\ &= \int d^3p' p_k'^2 \left[\frac{\hbar/(2\pi^2 S)}{[(\mathbf{p}-\mathbf{p}')^2 + \hbar^2/4S^2]^2} \right]^2 \\ &= \frac{S}{12\pi^2\hbar} \ .\end{aligned}$$

From these relations and (13) with $\mathbf{p}' = \mathbf{p}$, we obtain

$$\Delta P^k_{\min} = \hbar/(2 \times 3^{1/2} \times S) \ . \tag{38}$$

This result together with the minimum uncertainty relation $\Delta P_k \Delta Q_k = \hbar/2$ lead to a maximum uncertainty for ΔQ^k,

$$\Delta Q^k_{\max} = 3^{1/2} S, \quad k=1,2,3 \ . \tag{39}$$

Otherwise, there will be a contradiction because if the space is infinite, i.e., $\Delta Q^k_{\max} = \infty$, then one can have the usual plane wave with a definite momentum (with $\Delta P^k = 0$). Therefore, the result (38) is consistent with the effectively finite size of the physical universe expressed in (14).

IV. EFFECTIVE DENSITY OF STATES AND SUPERPOSITION OF WAVES

For the discussion in this section, we take the viewpoint that the four-dimensional symmetry of flat space-time is exact only in the limit $S \to \infty$ and $R \to 0$. Evidently, if the physical universe is effectively finite, the Lorentz transformation and the four-dimensional symmetry are only approximately true. Unfortunately, this approximate nature probably can never be detected by measuring new effects due to S in, say, (39), because S is extremely large. Nevertheless, let us consider the modified density of state of a free particle solely on the basis of the fuzziness of $\mathbf{P}$ and $\mathbf{Q}$. Such a modification is intimately related to the probabilistic nature of the dynamical variables. We believe that this modification will not be changed in the future when one takes the four-dimensional symmetry of spacetime into consideration (see Appendix B).

The suppression of the momentum $\mathbf{p}$ by an "inherent probability" $D^2(\mathbf{p})$ in (19) is difficult to be implemented in a theory to calculate probability amplitudes of scattering or decay processes in detail, unless the theory involves creation and annihilation operators. However, we still can see its partial effect in a physical process through an effective density of states.

Mathematically, the coordinates $\mathbf{q}$ and the momenta $\mathbf{p}$ can have values from $-\infty$ to $+\infty$, as shown in Eqs. (9)–(11). But the region in which matters or waves exist is effectively finite, so that physically the three-dimensional space appears to be non-Euclidean (see Appendix A). It appears to be some sort of quantum-mechanical analog of the Riemannian space because one may picture that the space has the volume element $E^2(\mathbf{q})d^3q$. The space is physically finite with the volume given by (14) and has no boundary. Similarly, the momentum space is also physically finite and has no boundary. From Eq. (19), we see that the volume element d^3p in the neighborhood of $\mathbf{p}$ is suppressed by the factor $D^2(\mathbf{p})$. Thus the number of one particle states in the range $\mathbf{p}$ and $\mathbf{p}+d\mathbf{p}$ is given by the following effective density of states:

$$\left[\int E^2(\mathbf{q})d^3q \right] D^2(\mathbf{p})d^3p/(2\pi\hbar)^3 = V_S D^2(\mathbf{p})d^3p/(2\pi\hbar)^3 \tag{40}$$

for a spin-zero particle. This is an important physical result of the new restriction (4) to the Heisenberg uncertainty relation. It is also consistent with the modified inner product in (30a) and (30b),

$$(\alpha \mid \alpha) = \int \mid \psi_\alpha(\mathbf{q}) \mid^2 E^2(\mathbf{q})d^3q = \int \mid \psi_\alpha(\mathbf{p}) \mid^2 D^2(\mathbf{p})d^3p \ , \tag{41}$$

where

$$E(\mathbf{q})\psi_\alpha(\mathbf{q}) = D^{-1}(\partial)(\mathbf{q} \mid \alpha) \ , \tag{42}$$

$$D(\mathbf{p})\psi_\alpha(\mathbf{p}) = E^{-1}(\partial_p)(p \mid \alpha) \ . \tag{43}$$

It follows from (43) and (24) that the amplitudes $E(\mathbf{q})\psi_\alpha(\mathbf{q})$ and $D(\mathbf{p})\psi_\alpha(\mathbf{p})$ are connected by the usual Fourier transform,

$$E(\mathbf{q})\psi_\alpha(\mathbf{q}) = \int d^3p \, (2\pi\hbar)^{-3/2} D(\mathbf{p})\psi_\alpha(\mathbf{p}) \exp(i\mathbf{p}\cdot\mathbf{q}/\hbar) \ . \tag{44}$$

Based on the inner product (41), we interprete that, for a particle in the state $\mid \alpha)$, its wave packets in the coordinate and the momentum spaces are described by the probability densities $\mid E(\mathbf{q})\psi_\alpha(\mathbf{q}) \mid^2$ and $\mid D(\mathbf{p})\psi_\alpha(\mathbf{p}) \mid^2$, respectively. The properties (41) and (44) ensure that the new restrictions (4) can be consistently imposed on the superposition of waves in (3) [where the coefficient $F(\mathbf{p})$ is related to $(\mathbf{p} \mid \alpha)$ through Eq. (10)], so that the physically realizable wave packets always correspond to vectors in the "fuzzy vector sector." We stress that such a restriction on the superposition of waves modifies only physics at very small momentum $p \sim \hbar/S$ and at very large momentum $p \sim \hbar/R$ and, hence, it does not contradict previous experiments.

Since the effective density of states is given by (40), the differential decay rate $dw_{\rm FQM}$ (FQM indicating fuzzy quantum mechanics) for the decay process, $1 \to 2+3+4 \cdots +n$, is given by

$$dw_{\rm FQM} \approx dw_{\rm QM} D^2(\mathbf{p}_2) D^2(\mathbf{p}_3) \cdots D^2(\mathbf{p}_n) , \tag{45}$$

where $dw_{\rm QM}$ is the decay rate for the same process calculated in ordinary quantum mechanics and the sign $\approx$ indicates that the suppression of the momentum by the inherent probability $D^2(p)$ in the intermediate steps of the decay process has not been taken into account.

Similarly, the differential cross section $d\sigma_{\rm FQM}$ for the process, $1+2 \to 3+4+ \cdots +n$, is related to $d\sigma_{\rm QM}$ in ordinary quantum mechanics by the relation

$$d\sigma_{\rm FQM} \approx d\sigma_{\rm QM} D^2(\mathbf{p}_3) D^2(\mathbf{p}_4) \cdots D^2(\mathbf{p}_n) . \tag{46}$$

These modifications become important only at very large momentum $p \gtrsim \hbar/R \sim 10^4$ GeV/c.

We believe that the fundamental fuzziness of dynamical variables **P** and **Q** and their base states in Klauder's continuous representations is independent of the four-dimensional symmetry of space-time. Based on the result (45), we conclude that, apart from the usual relativistic time dilatation, the lifetime of an unstable particle decay in flight (with the momentum p) will be further delated by a "radical dilatation" when $p \gtrsim \hbar/R$. Otherwise, the concept of quantum-mechanical probability is probably not really fundamental and could be modified in the future. For the lifetime $\tau(\Lambda)$ of the decay process $\Lambda \to p + \pi$, suppose $p_p \sim p_\pi \sim 10^3$ GeV/c and $\hbar/R \sim 10^4$ GeV/c; we estimate that

$$[\tau_{\rm FQM}(\Lambda) - \tau_{\rm QM}(\Lambda)]/\tau_{\rm QM}(\Lambda) \sim 0.04 , \tag{47}$$

where we have used (45) and (22). This can be tested by the superconducting supercollider in the future.

Note added in proof

The difficulty with nonsquare integrable coordinate (or momentum) representations by itself may not be sufficiently strong motivation for introducing a new physical formulation of quantum mechanics (i.e., fuzzy quantum mechanics). However, the new physical formulation is also motivated by the relation between this difficulty and the problem of locality and, hence, the ultraviolet divergence in quantum field theories. This could be significant because the radical length R, which characterizes the inherent fuzziness at short distances, is related to Dirac's conjecture of a fundamental length λ and to Feynman's width for a modified δ function for interactions. In 1949, Dirac said the following: "Present-day atomic theories involve the assumption of localizability, which is sufficient but is very likely too stringent .... A less drastic assumption may be adequate, e.g., that there is a fundamental length λ such that the Poisson bracket of two dynamical variables must vanish if they are localized at two points whose separation is space-like and greater than λ, but need not vanish if it is less than λ."[7] A similar view was expressed by Feynman: "There were several suggestions for interesting modifications of electrodynamics. We discuss lots of them, but I shall report on only one. It was to replace this delta function in the interaction by another function f (which has the width of order a^2) .... Interactions will now occur when $T^2 - R_s^2$ is of order a^2 roughly, where T is the time difference and R_s is the separation of the charges."[8]

Furthermore, it is interesting to note that the concept of fuzzy coordinate variables is in harmony with Schwinger's conclusion that "a convergent theory (QED) cannot be formulated consistently within the framework of present space-time concepts. To limit the magnitude of interaction while retaining the customary coordinate description is contradictory, since no mechanism is provided for precisely localized measurements."[9]

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APPENDIX A: MODIFIED PLANE WAVES AND WAVE EQUATIONS FOR A FINITE UNIVERSE

Let us consider the global picture of the finite and fuzzy universe with an effective size approximately equal to S. For simplicity of discussions, we set $R = 0$. The position amplitude $E(\mathbf{q})\psi_\alpha(\mathbf{q})$ in (41) involves two parts: One part $E(\mathbf{q})$ is related to the macroscopic universe and another part $\psi_\alpha(q)$ describes the microscopic system under consideration.

As usual, the time evolution of the microscopic amplitude $\psi_\alpha = \psi_\alpha(\mathbf{q},t)$ is postulated to satisfy the Schrödinger equation with the usual Hermitian Hamiltonian H_u:

$$i\hbar \partial\psi_\alpha/\partial t = H_u(\partial, \mathbf{q})\psi_\alpha , \tag{A1}$$

$$H_u(\partial, \mathbf{q}) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{q}^2} + V(\mathbf{q}) . \tag{A2}$$

Since $\psi_\alpha(\mathbf{q},t)$ is related to $(\mathbf{q},t \mid \alpha)$ by (42), one may rewrite (A1) in the form

$$i\hbar \frac{\partial}{\partial t} \mid \alpha) = E(\mathbf{Q}) H_u(\mathbf{P},\mathbf{Q}) E^{-1}(Q) \mid \alpha) , \tag{A3}$$

where we have used the relation $(\mathbf{q} \mid D^{-1}(\mathbf{P}) E(\mathbf{Q}) = \langle \mathbf{q} \mid E(\mathbf{Q}) = \langle \mathbf{q} \mid E(\mathbf{q})$.

When $V(\mathbf{q}) = 0$, we have the usual free particle solution $\psi_\alpha = \exp(-iEt/\hbar + i\mathbf{p}\cdot\mathbf{q}/\hbar)$, $|\mathbf{p}| = (2mE)^{1/2} \equiv \alpha$. Thus we have a modified plane wave given by $E(\mathbf{q})\psi_\alpha = \exp(-iEt/\hbar + i\mathbf{p}\cdot\mathbf{q}/\hbar - |\mathbf{q}|/2S)$ and the probability for finding this free particle between $\mathbf{q}$ and $\mathbf{q} + d\mathbf{q}$ is $E^2(\mathbf{q}) d^3 q$.

We note that if one takes the effective finiteness of the physical universe seriously, one probably has to consider

$E^2(\mathbf{q})d^3q$ in (14) as an "invariant volume element" and to use curvilinear coordinates for describing physical phenomena. In this sense, fuzzy quantum mechanics based on Klauder's continuous representations suggests that the physical space is non-Euclidean. The Hamiltonian in (A2) will be only an approximation. Such a non-Euclidean property of the physical universe will be significant only when S is small (i.e., at very early universe, provided S is time dependent). In view of the observed expansion of the universe, the cosmological scale S in Klauder's continuous representations (9) and (12) is probably dependent on the age of the universe. It is hoped that, in the future, one may discover a more fundamental equation which takes gravity into consideration and determines the evolution of both the microscopic system $\psi_\alpha(\mathbf{q})$ and the whole universe described by $\psi_{\rm univ}[\mathbf{q}/S(t)]=E(\mathbf{q})$.

APPENDIX B: EFFECTIVE DENSITY OF STATES AND FOUR-DIMENSIONAL SYMMETRY

The density of state (40) at large momenta is modified by a new factor $D^2(\mathbf{p})=1/(2\mathbf{p}^2R^2/\hbar^2+1)^2$. One can replace d^3p by a four-dimensional invariant quantity $d^3p/(\mathbf{p}^2+m^2)^{1/2}$. Nevertheless, one wonders whether the $\mathbf{p}$-dependent function $D(\mathbf{p})$ is compatible with the four-dimensional symmetry which is believed to be important at large momenta. Evidently, the $\mathbf{p}$-dependent function $D(\mathbf{p})$ cannot be invariant within the framework of special relativity. However, if one looks at the four-dimensional symmetry from a different angle,[6] namely, introducing a new scalar evolution variable σ (which can be interpreted as a common time) for all inertial reference frames, then one has a new scalar quantity $G(\mathbf{p})$ which is momentum dependent and can be used to express invariant $\mathbf{p}$-dependent functions. This can be seen as follows.

Suppose we choose an arbitrary inertial frame F in which the speed of light is assumed to be isotropic and constant c. We can set up a synchronized clock system in F (say, the ground) by using the light signals. Suppose another frame F' (say, a train) is moving with a constant velocity $(V,0,0)$ and the observers in F' use the clock system on the ground F to record time, so that all observers in both frames have a common time $\sigma=t=t'$. Within the four-dimensional framework, an event is now specified by

$$x^\mu=(c\sigma,x,y,z) \text{ and } x'^\mu=(b'\sigma,x',y',z') \tag{B1}$$

in F and F', respectively, where b' is no longer a constant. Since F and F' are assumed to be completely equivalent, as required by the principle of relativity for physical laws, these two sets of coordinates can be shown to be connected by a new four-dimensional transformation[6]

$$x'=\gamma(x-\beta c\sigma),\quad y'=y\ ,\quad z'=z,\quad b'\sigma=\gamma(c\sigma-\beta x)\ , \quad \sigma=t=t'=\cdots\ , \tag{B2}$$

where $\cdots$ is common time, $\beta=V/c$, and $\gamma=(1-\beta^2)^{-1/2}$. The physical meaning of the new variable b' is completely specified by the transformation (B2) which preserves the four-dimensional interval $c^2\sigma^2-x^2-y^2-z^2=b'^2\sigma^2-x'^2-y'^2-z'^2$. The transformation of velocities, measured with respect to the common time σ, is given by

$$v'_x=\gamma(v_x-\beta_c),\quad v'_y=v_y\ ,\quad v'_z=v_z,\quad c'=\gamma(c-\beta v_x)\ , \tag{B3}$$

where $c'\equiv d(b'\sigma)/d\sigma$, $v'_x=dx'/d\sigma$, $v_x=dx/d\sigma$, etc. Although the speed of light is no longer a universal constant, we still can postulate an invariant "action function" A for a free particle,

$$A=-\int m\,ds,\quad ds=(dx_0^2-dr^2)^{1/2}=(dx_\mu dx^\mu)^{1/2}\ .$$

The "four-momentum" of the particle is

$$\begin{aligned}p^\mu&=m\,dx^\mu/ds\\&=(p^0,\mathbf{p})\\&=(m/(1-v^2/C^2)^{1/2},(m\mathbf{v}/C)/(1-v^2/C^2)^{1/2})\ ,\end{aligned}$$

where $C=dx^0/d\sigma$. Such covariant four-momenta p^μ have the dimension of a mass and transform as follows:

$$p'_x=\gamma(p_x-\beta p_0),\quad p'_y=p_y\ ,\quad p'_z=p_z,\quad p'_0=\gamma(p_0-\beta p_x)\ . \tag{B4}$$

It follows from (B3) and (B4) that the ratio p'_0/c' is an invariant:

$$p'_0/c'=p_0/c\equiv G(p),\quad p_0=(\mathbf{p}^2+m^2)^{1/2}\ . \tag{B5}$$

The scalar property of $G(\mathbf{p})$ can also be seen in the following form:

$$G(\mathbf{p})=m/(C^2-\mathbf{v}^2)^{1/2}=m/(ds/d\sigma)\ , \tag{B6}$$

where m, ds, and $d\sigma$ are all scalars under the new transformation (B2).

Thus we have seen that the function $D(\mathbf{p})$ necessary for Klauder's continuous representations for $\mathbf{Q}$ and $(\mathbf{q}|$ can be compatible with a four-dimensional symmetry with a common time, provided that we define $D(\mathbf{p})$ in terms of the scalar $G(\mathbf{p})$ associated with a physical particle with a mass m, namely,

$$D(\mathbf{p})=1/[I^2G^2(\mathbf{p})+1]\ , \tag{B7}$$

where I is a new constant which is related to the radical length R in the frame F by the relation $I=Rc^2/\hbar$. The expression (B7) is essentially the same as $D(p)$ in Eq. (40) because $I^2m^2/c^2=(Rmc/\hbar)^2\ll 1$. For a detailed discussion of the four-dimensional transformation (B2) with a common time (i.e., common relativity), we refer to Ref. 6.

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