Exact density-matrix calculations for simple open systems

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Explicit density-matrix solutions are obtained for several simple open quantum systems without the usual trace being taken over the thermal bath. This is achieved using a representation of the density matrix in terms of the dynamic operators of the system. The results are compared with those from conventional reduced calculations. Different types of irreversible behavior are found, and the implications of this distinction for models of quantum measurement are considered.

I. INTRODUCTION

Some recent work in the quantum theory of measurement[1–6] has developed models in which the environment of an open system plays a key role in achieving state reduction. The underlying ideas is, of course, not new, but it is only in the last few years that explicit solutions for the states of some sample systems have been obtained, exhibiting very clearly diagonalization of the reduced density matrix in the basis given by the eigenstates of the measured operator (Zurek's "pointer basis"[7,8]). Such measurement schemes, in which state reduction is modeled as a loss of coherence, of course, do not, and can never, completely solve the so-called "measurement problem;" their dependence on an average being taken over the quantum states of the environment means that they can never be more than a partial (though suggestive) description of the process of measurement. As a response to this, it can be argued that this is a limitation of all physical models of irreversible processes. Thus (goes the argument), while loss-of-coherence measurement models solve nothing completely, they do reduce two problems to one: quantum measurement becomes just a special case of the general problem of irreversibility in quantum systems. For instance, Peres has argued[9] that the unobservability in practice of phase relationships within the environment which carry the lost information means that the irreversibility of a quantum measurement can be explained entirely analogously to "familiar classical irreversibility."

The crucial question here is the relative importance for irreversibility of two quite distinct aspects of open systems (considered together with their environment). Firstly, the fact that they are *large* systems with many degrees of freedom. Secondly, the fact that our knowledge of them is incomplete. Obviously, in practical terms, the second is an almost inevitable consequence of the first, but this does not mean that the distinction is not real. To see the effects of these two properties separately, we need to study open systems without making the usual average over the environment; we must look at the behavior of the unreduced density matrix, including the detailed behavior of the thermal bath.

In this paper I present such complete calculations for several simple open quantum systems: a phase-damped oscillator, an amplitude-damped oscillator, a Raman amplifier, and a photon-number measurement model. Although, as the preceding discussion indicates, these calculations were principally motivated by concerns of measurement theory, they do not use any assumptions or techniques peculiar to it, and may equally well be taken simply as more comprehensive solutions than hitherto presented for some standard quantum models.

II. OPERATOR REPRESENTATION OF DENSITY MATRICES

This paper makes extensive use of density matrices as functions of the operator dynamical variables describing the system. As this approach is not widely used, this section gives some properties and examples of it. As a starting point, one may use the operator representation most commonly found: that for a thermal state in terms of the Hamiltonian[10]

$$\rho_T = \frac{1}{Z}\exp\left[-\frac{H}{kT}\right], \tag{2.1}$$

where

$$Z = \mathrm{Tr}\left[\exp\left[-\frac{H}{kT}\right]\right].$$

A. Single-mode states

For a harmonic oscillator (and all examples in this section will be for the harmonic oscillator),

$$H = \hbar\omega a^\dagger a \tag{2.2}$$

and

$$Z = \frac{1}{1-\exp(-\hbar\omega/kT)} = N+1 ,$$

so that

$$\rho_T = \frac{1}{N+1}\exp\left[-\frac{\hbar\omega}{kT}a^\dagger a\right]. \tag{2.3}$$

Using the operator ordering relation[11]

$$\exp(\kappa a^\dagger a) = :\exp[(e^\kappa - 1)a^\dagger a]: , \tag{2.4}$$

this may be rewritten as

$$\rho_T = \frac{1}{N+1} :\exp\left[-\frac{a^\dagger a}{N+1}\right]: . \tag{2.5}$$

The zero-temperature limit of this gives the vacuum density matrix

$$\rho_{\text{vac}} = |0\rangle\langle 0| = :\exp(-a^\dagger a): . \tag{2.6}$$

From this it follows immediately that for an element in a number state basis the representation is

$$|m\rangle\langle n| = \frac{1}{\sqrt{m!n!}}(a^\dagger)^m :\exp(-a^\dagger a): a^n . \tag{2.7}$$

The representation for a coherent element similarly follows by the application of displacement operators

$$\begin{aligned}|\alpha\rangle\langle\beta| &= D(\alpha)\rho_{\text{vac}}D^\dagger(\beta)\\ &= e^{-|\alpha|^2/2}e^{a^\dagger\alpha}:e^{-a^\dagger a}:e^{\beta^* a}e^{-|\beta|^2/2}\\ &= \langle\beta|\alpha\rangle :\exp[-(a^\dagger-\beta^*)(a-\alpha)]: . \end{aligned} \tag{2.8}$$

In fact, we can go further than this and obtain the representation for the combination of a coherent element with thermal noise:

$$\begin{aligned} D(\alpha)\rho_T D^\dagger(\beta) &= e^{-|\alpha|^2/2-|\beta|^2/2}e^{a^\dagger\alpha}e^{-\alpha^* a}\frac{\exp\left[-\frac{\hbar\omega}{kT}a^\dagger a\right]}{N+1}e^{-a^\dagger\beta}e^{\beta^* a}\\ &= \frac{\langle\beta|\alpha\rangle}{N+1} :\exp\left[-(a^\dagger-\beta^*)(a-\alpha)+\frac{N}{N+1}(a^\dagger-\alpha^*)(a-\beta)\right]: . \end{aligned} \tag{2.9}$$

We see that this reduces to (2.8) for $N=0$, while for $\alpha=\beta$ (coherent state with thermal noise),

$$D(\alpha)\rho_T D^\dagger(\alpha) = \frac{1}{N+1} :\exp\left[-\frac{(a^\dagger-\alpha^*)(a-\alpha)}{N+1}\right]: , \tag{2.10}$$

as we would expect from (2.5).

As a final example for the single mode, the representation of a squeezed state can be produced using squeeze generators.[12–15] An ideal squeezed state is given by

$$\begin{aligned}|\alpha, re^{i\phi}\rangle &= D(\alpha)S(re^{i\phi})|0\rangle\\ &= D(\alpha)\exp\{\tfrac{1}{2}r[e^{i\phi}(a^\dagger)^2 - e^{-i\phi}a^2]\}|0\rangle .\end{aligned} \tag{2.11}$$

Using (A9) with $\sigma_+ = \frac{1}{2}e^{i\phi}(a^\dagger)^2$ and $\sigma_- = -\frac{1}{2}e^{-i\phi}a^2$, this becomes

$$|\alpha, re^{i\phi}\rangle = (\cosh r)^{-1/2}\exp[\tfrac{1}{2}e^{i\phi}(a^\dagger - a^*)^2\tanh r]|\alpha\rangle , \tag{2.12}$$

so that a general density matrix element is

$$|\alpha, re^{i\phi}\rangle\langle\beta, se^{i\psi}| = \frac{\langle\beta|\alpha\rangle}{(\cosh r\cosh s)^{1/2}} :\exp[-(a^\dagger-\beta^*)(a-\alpha)+\tfrac{1}{2}e^{i\phi}(a^\dagger-\alpha^*)^2\tanh r + \tfrac{1}{2}e^{-i\psi}(a-\beta)^2\tanh s]: . \tag{2.13}$$

In similar fashion, using (A9), (A7), and (A8), the general element for a squeezed thermal state is

$$\begin{aligned} &D(\alpha)S(re^{i\phi})\rho_T S^\dagger(se^{i\psi})D^\dagger(\beta)\\ &\quad= M_{rs}^{-1}\langle\beta|\alpha\rangle :\exp\{-(a^\dagger-\beta^*)(a-\alpha)+M_{rs}^{-2}N(N+1)(a^\dagger-\alpha^*)(a-\beta)\\ &\qquad+\tfrac{1}{2}M_{rs}^{-2}[(N+1)^2e^{i\phi}\sinh r\cosh s - N^2e^{i\psi}\cosh r\sinh s](a^\dagger-\alpha^*)^2\\ &\qquad+\tfrac{1}{2}M_{rs}^{-2}[(N+1)^2e^{-i\psi}\sinh s\cosh r - N^2e^{-i\phi}\cosh s\sinh r](a-\beta)^2\}: , \end{aligned} \tag{2.14}$$

where

$$M_{rs}=[(N+1)^2\cosh r\,\cosh s-e^{i(\psi-\phi)}N^2\sinh r\,\sinh s]^{1/2}\ .$$

It may be noted that these normally ordered operator representations of the density matrix are identical in form to the Q function (the antinormal quasiprobability distribution) for the same states. In particular, by taking projections over the coherent states of the on-diagonal cases of (2.13) and (2.14) we can recover the results of Stoler[16] and Vourdas,[17] respectively.

B. Continuous modes

All the single-mode representations of Sec. II A apply in an obvious fashion to multiple independent modes. In particular, in the case of a continuous distribution of modes (for a field or heat bath) the vacuum state is

$$\rho_{\text{vac}}=:\exp\left[-\int d\omega\, b^{\dagger}(\omega)b(\omega)\right]:\ . \tag{2.15}$$

Here $b(\omega)$ has dimensions $[T]^{1/2}$, with the commutator being

$$[b(\omega),b^{\dagger}(\omega)]_{-}=\delta(\omega-\omega')\ . \tag{2.16}$$

Similarly, for the coherent state,

$$\begin{aligned}|[\alpha(\omega)]\rangle\langle[\alpha(\omega)]|&=\prod_{\omega}D_{\omega}(\alpha(\omega))\rho_{\text{vac}}\prod_{\omega}D_{\omega}^{\dagger}(\alpha(\omega))\\&=\exp\left[\int d\omega[b^{\dagger}(\omega)\alpha(\omega)-\alpha^{*}(\omega)b(\omega)]\right]:\exp\left[-\int d\omega\, b^{\dagger}(\omega)b(\omega)\right]:\\&\quad\times\exp\left[\int d\omega[b(\omega)\alpha^{*}(\omega)-b^{\dagger}(\omega)\alpha(\omega)]\right]\\&=:\exp\left[-\int d\omega[b^{\dagger}(\omega)-\alpha^{*}(\omega)][b(\omega)-\alpha(\omega)]\right]:\ .\end{aligned} \tag{2.17}$$

The product over ω in the right-hand side of the first line is a continuous product, interpreted as in the following line. A continuous product of a more awkward type arises when one considers the thermal state, which is not normalizable:

$$\rho_T=\frac{1}{Z_T}\exp\left[-\int d\omega\frac{\hbar\omega}{kT}b^{\dagger}(\omega)b(\omega)\right]=\frac{1}{Z_T}:\exp\left[-\int d\omega\frac{b^{\dagger}(\omega)b(\omega)}{N(\omega)+1}\right]:\ . \tag{2.18}$$

Here the partition function Z_T is a product containing a factor of $1+N(\omega)$ for each of the continuously infinite number of modes. However, this causes no problems in practice. The combination of thermal and coherent elements gives

$$\begin{aligned}\prod_{\omega}D_{\omega}(\alpha(\omega))\rho_T\prod_{\omega}D_{\omega}^{\dagger}(\beta(\omega))=\frac{1}{Z_T}\langle[\beta(\omega)]|[\alpha(\omega)]\rangle:\Bigg[\exp\Bigg[&-\int d\omega[b^{\dagger}(\omega)-\beta^{*}(\omega)][b(\omega)-\alpha(\omega)]\\&-\int d\omega\frac{N(\omega)}{N(\omega)+1}[b^{\dagger}(\omega)-\alpha^{*}(\omega)][b(\omega)-\beta(\omega)]\Bigg]\Bigg]:\ ,\end{aligned} \tag{2.19}$$

where

$$\langle[\beta(\omega)]|[\alpha(\omega)]\rangle=\exp\left[-\int d\omega[\beta^{*}(\omega)\alpha(\omega)-\tfrac{1}{2}|\beta(\omega)|^2-\tfrac{1}{2}|\alpha(\omega)|^2]\right]\ .$$

Multimode squeezed states could similarly be obtained using the appropriate generators.[18]

C. Time-dependent behavior

Although the density matrix is a quantum-mechanical operator, and can be represented as a function of the dynamical variables of the system, it of course obeys not Heisenberg's equation of motion but von Neumann's, differing by an overall sign. The density matrix (in the Schrödinger picture) may therefore be treated as an ordinary operator (in the Heisenberg picture) evolving backwards in time. This means that we can obtain its time evolution by solving the equations for the dynamical operators, and substituting these solutions into the operator representation, changing the sign of the time.

As a trivial example, take a free harmonic oscillator initially in a coherent state:

$$\rho(0)=:e^{-(a^{\dagger}-\alpha^{*})(a-\alpha)}:\ . \tag{2.20}$$

The Heisenberg equation for $a(t)$ gives

$$a(t)=e^{-i\omega t}a(0)=e^{-i\omega t}a\ , \tag{2.21}$$

so we can immediately write

$$\rho(t) = :\exp\{-[a^\dagger(-t)-\alpha^*][a(-t)-\alpha]\}: \\ = :\exp[-(a^\dagger-\alpha^* e^{+i\omega t})(a-\alpha e^{-i\omega t})]: . \quad (2.22)$$

Care may be needed when the evolution mixes annihilation and creation operators; the normal ordering applies *before* substituting in the solutions. Special attention to detail is also required when dealing with a driven system, in which the total Hamiltonian is time dependent.

III. PHASE DAMPING

Phase damping is a suitable first choice for unreduced calculation for two reasons: firstly, it is mathematically straightforward; secondly it is a system for which a conventional reduced calculation is most clearly inadequate—a Markovian master equation can be obtained only in the limit of high bath temperature.

The model Hamiltonian is conventional, with the isolated (system) mode being coupled to the continuous (bath) modes via its number operator $N = a^\dagger a$.

$$H = \hbar\omega_0 N + \tfrac{1}{2}\int d\omega\{p^2(\omega)+[\omega q(\omega)-\kappa(\omega)N]^2\} . \quad (3.1)$$

Obviously, slightly different models are possible; for instance, N could couple to $p(\omega)$ instead of $q(\omega)$. The exceptional simplicity of the calculations for this process results entirely from the fact that the coupling operator is a constant of the motion (i.e., $N(t)=N(0)$ since $-i\hbar^{-1}[N,H]_-=0$); in the terminology of measurement theory, it is a quantum nondemolition variable of the system. The choice of any other such constant coupling (e.g., a quadrature phase operator[19]) would give results formally very similar to those below.

As a first step, the Heisenberg equations of motion for the bath operators are easily solved. The equations are

$$\dot q(\omega,t) = p(\omega,t) , \\ \dot p(\omega,t) = -\omega^2 q(\omega,t) + \omega\kappa(\omega)N . \quad (3.2)$$

In terms of the annihilation operators defined by

$$b(\omega) = \frac{1}{(2\hbar\omega)^{1/2}}[\omega q(\omega)+ip(\omega)] , \quad (3.3)$$

we have the solution

$$b(\omega,t) = e^{-i\omega t}b_0(\omega) + \frac{\kappa(\omega)}{(2\hbar\omega)^{1/2}}N(1-e^{-i\omega t}) , \quad (3.4)$$

where I use $b_0(\omega)=b(\omega,0)$.

Next consider the equation of motion for an arbitrary system operator Y:

$$\dot Y = -\frac{i}{\hbar}[Y,H]_- = -i\omega_0[Y,N]_- + \frac{i}{2\hbar}\int d\omega\,\kappa(\omega)\left[[Y,N]_-, \frac{1}{(2\hbar\omega)^{1/2}}[b(\omega)+b^\dagger(\omega)]-\kappa(\omega)N\right]_+ , \quad (3.5)$$

where $[A,B]_+$ denotes the anticommutator of A and B. Since bath and system operators commute at equal times, this can be rearranged as

$$\dot Y = -i\omega_0[Y,N]_- + i\int d\omega\,\kappa(\omega)\left[\frac{\omega}{2\hbar}\right]^{1/2}\left[e^{i\omega t}b_0^\dagger(\omega)+\frac{\kappa(\omega)}{(2\hbar\omega)^{1/2}}N(1-e^{i\omega t})\right][Y,N]_- \\ + i\int d\omega\,\kappa(\omega)\left[\frac{\omega}{2\hbar}\right]^{1/2}[Y,N]_-\left[e^{-i\omega t}b_0(\omega)+\frac{\kappa(\omega)}{(2\hbar\omega)^{1/2}}N(1-e^{-i\omega t})\right] - \frac{i}{2\hbar}\int d\omega\,\kappa^2(\omega)[[Y,N]_-,N]_+ \\ = -i\omega_0[Y,N]_- + i\int d\omega\,\kappa(\omega)\left[\frac{\omega}{2\hbar}\right]^{1/2}\{b_0^\dagger(\omega)e^{i\omega t}[Y,N]_- + [Y,N]_- e^{-i\omega t}b_0(\omega)\} \\ - \frac{i}{2\hbar}\int d\omega\,\kappa^2(\omega)\{[[Y,N]_-,N]_+\cos(\omega t) - i[[Y,N]_-,N]\sin(\omega t)\} , \quad (3.6)$$

where the solution (3.4) for the bath operators has been used. At this point we may select an appropriate basis in which to represent the system density matrix. As N is constant, the number states are the obvious choice. Specializing Y to a matrix element in this basis gives

$$\frac{d}{dt}|m\rangle\langle n| = -i(n-m)\omega_0|m\rangle\langle n| + i(n-m)\int d\omega\,\kappa(\omega)\left[\frac{\omega}{2\hbar}\right]^{1/2}[b_0^\dagger(\omega)e^{i\omega t}|m\rangle\langle n| + |m\rangle\langle n|e^{-i\omega t}b_0(\omega)] \\ - \frac{i}{2\hbar}\int d\omega\,\kappa^2(\omega)[(n^2-m^2)\cos(\omega t) - i(n-m)^2\sin(\omega t)]|m\rangle\langle n| , \quad (3.7)$$

which can be directly exponentiated to give

$$
\begin{aligned}
(\,|\,m\,\rangle\langle\,n\,|\,)_t &= e^{-i(n-m)\omega_0 t}\exp\left[-\frac{i}{2\hbar}\int d\omega\,\kappa^2(\omega)\int_0^t d\tau[(n^2-m^2)\cos(\omega\tau)-i(n-m)^2\sin(\omega\tau)]\right]\\
&\quad\times\exp\left[i(n-m)\int d\omega\,\kappa(\omega)\int_0^t d\tau\left[\frac{\omega}{2\hbar}\right]^{1/2} b_0^\dagger(\omega)e^{i\omega\tau}\right](\,|\,m\,\rangle\langle\,n\,|\,)_0\\
&\quad\times\exp\left[i(n-m)\int d\omega\,\kappa(\omega)\int_0^t d\tau\left[\frac{\omega}{2\hbar}\right]^{1/2} b_0(\omega)e^{-i\omega\tau}\right]\\
&= e^{-i(n-m)\omega_0 t}\exp\left[-i\int d\omega\frac{\kappa^2(\omega)}{2\hbar\omega}\{(n^2-m^2)\sin(\omega t)+i(n-m)^2[\cos(\omega t)-1]\}\right]\\
&\quad\times\exp\left[(n-m)\int d\omega\frac{\kappa(\omega)}{(2\hbar\omega)^{1/2}}b_0^\dagger(\omega)(e^{i\omega t}-1)\right](\,|\,m\,\rangle\langle\,n\,|\,)_0\\
&\quad\times\exp\left[-(n-m)\int d\omega\frac{\kappa(\omega)}{(2\hbar\omega)^{1/2}}b_0(\omega)(e^{-i\omega t}-1)\right].
\end{aligned}
\tag{3.8}
$$

We now have all the information we need to write down the solution for the total density matrix for the process. We start with a density matrix of the form

$$
\rho(0)=\rho_{\text{sys}}\otimes\rho_T=\prod_{m,n} c_{mn}(\,|\,m\,\rangle\langle\,n\,|\,)_0\frac{1}{Z_T}:\exp\left[-\int d\omega\frac{b_0^\dagger(\omega)b_0(\omega)}{N(\omega)+1}\right]:.
\tag{3.9}
$$

A typical term in the solution is

$$
\begin{aligned}
\rho_{mn}(t) &= \frac{1}{Z_T}:(\,|\,m\,\rangle\langle\,n\,|\,)_{-t}\exp\left[-\int d\omega\frac{b^\dagger(\omega,-t)b(\omega,-t)}{N(\omega)+1}\right]:\\
&= \frac{1}{Z_T}e^{i(n-m)\omega_0 t}\exp\left[i(n^2-m^2)\int d\omega\frac{\kappa^2(\omega)}{2\hbar\omega}\sin(\omega t)\right]\\
&\quad\times\exp\left[(n-m)^2\int d\omega\frac{\kappa^2(\omega)}{2\hbar\omega}[\cos(\omega t)-1]\right]:(\,|\,m\,\rangle\langle\,n\,|\,)_0\\
&\quad\times\exp\left[-\int d\omega\left[b_0^\dagger(\omega)-\frac{n\kappa(\omega)}{(2\hbar\omega)^{1/2}}(1-e^{i\omega t})\right]\left[b_0(\omega)-\frac{m\kappa(\omega)}{(2\hbar\omega)^{1/2}}(1-e^{-i\omega t})\right]\right.\\
&\qquad\left.+\int d\omega\frac{N(\omega)}{N(\omega)+1}\left[b_0^\dagger(\omega)-\frac{m\kappa(\omega)}{(2\hbar\omega)^{1/2}}(1-e^{i\omega t})\right]\left[b_0(\omega)-\frac{n\kappa(\omega)}{(2\hbar\omega)^{1/2}}(1-e^{-i\omega t})\right]\right]:.
\end{aligned}
\tag{3.10}
$$

From (2.19) we can recognize the bath part of this as off-diagonal coherent excitation of the original thermal state, allowing us to write

$$
\begin{aligned}
\rho(t)=\sum_{m,n}c_{mn}\rho_{mn}(t)&=\sum_{m,n}c_{mn}\exp\left[i(n-m)\omega_0 t+i(n^2-m^2)\int d\omega\frac{\kappa^2(\omega)}{2\hbar\omega}\sin(\omega t)\right]\\
&\quad\times\exp\left[m\int d\omega[b_0^\dagger(\omega)\beta(\omega)-\beta^*(\omega)b_0(\omega)]\right](\rho_T\otimes\,|\,m\,\rangle\langle\,n\,|\,)_0\\
&\quad\times\exp\left[n\int d\omega[\beta^*(\omega)b_0(\omega)-b_0^\dagger(\omega)\beta(\omega)]\right],
\end{aligned}
\tag{3.11}
$$

where

$$
\beta(\omega)=\frac{\kappa(\omega)}{(2\hbar\omega)^{1/2}}(1-e^{-i\omega t}).
$$

Thus the effect of the coupling between bath and system is twofold: a phase shift given by the factor involving an integral over $\sin(\omega t)$, which for $\kappa(\omega)$ slowly varying will rapidly reach a steady value; and more importantly, the coherent excitation of a range of bath modes in correlation with the state of the system. From this point of view there is no sign of phase damping, or loss of coherence. These two (related) effects may be made to appear by tracing over the bath modes, to give

$$\langle \rho(t) \rangle = \sum_{m,n} c_{mn} \exp\left[i(n-m)\omega_0 t + i(n^2-m^2)\int d\omega \frac{\kappa^2(\omega)}{2\hbar\omega}\sin(\omega t)\right] \times \exp\left[(n-m)^2 \int d\omega \frac{\kappa^2(\omega)}{\hbar\omega}[N(\omega)+\tfrac{1}{2}][\cos(\omega t)-1]\right](|m\rangle\langle n|)_0 . \quad (3.12)$$

It is the extra factor involving $(n-m)^2$, arising from ignorance of the bath rather than from the intrinsic nature of the physical process, which gives the decay of the coefficients of off-diagonal elements, i.e., the loss of coherence which can be described as phase damping.

Equation (3.12) may be compared with the approximate result obtained from a Markovian master equation in Ref. 2. The interaction picture master equation is

$$\langle \dot{\rho} \rangle = \lambda[a^\dagger a \rho a^\dagger a - (a^\dagger a)^2 \rho - \rho (a^\dagger a)^2] , \quad (3.13)$$

giving

$$\langle \rho(t) \rangle = \sum_{m,n} c_{mn} e^{-\lambda(n-m)^2 t}(|m\rangle\langle n|)_0 . \quad (3.14)$$

Similar behavior may be seen in (3.12) in the high-temperature limit $N(\omega)+\frac{1}{2}\approx kT/\hbar\omega$, giving [for $\kappa^2(\omega)$ constant]

$$\int_0^\infty d\omega \frac{\kappa^2(\omega)}{\hbar\omega}[N(\omega)+\tfrac{1}{2}][\cos(\omega t)-1] \approx -\frac{\pi\kappa^2 kT}{2\hbar^2} t ; \quad (3.15)$$

that is, $\lambda = \pi\kappa^2 kT/2\hbar^2$. For times short compared with the thermal correlation time ($t < \hbar/kT$), (3.12) yields highly non-Markovian phase damping. The n^2-m^2 anharmonicity is also relatively more important at short times.

IV. AMPLITUDE DAMPING

I use here a standard model for a harmonic oscillator linearly coupled to a heat bath of oscillators in the rotating-wave approximation (RWA). The Hamiltonian is

$$H = \hbar\omega_0 a^\dagger a + \hbar \int_{-\infty}^{\infty} d\omega\, \omega b^\dagger(\omega) b(\omega) + i\int_{-\infty}^{\infty} d\omega\, \hbar\kappa(\omega)[b^\dagger(\omega) a - a^\dagger b(\omega)] . \quad (4.1)$$

The extension of the bath frequency range down to $-\infty$ is an unphysical idealization, but appropriate within the RWA. As a further idealization, the coupling constant $\kappa(\omega)$ will be taken to be a constant. The equations of motion are

$$\dot{a} = -i\omega_0 a - \int d\omega\, \kappa(\omega) b(\omega) , \quad (4.2)$$

$$\dot{b}(\omega) = -i\omega b(\omega) + \kappa(\omega) a . \quad (4.3)$$

The formal solution of (4.3) is

$$b(\omega,t) = e^{-i\omega t} b_0(\omega) + \kappa \int_0^t d\tau\, e^{-i\omega(t-\tau)} a(\tau) , \quad (4.4)$$

which substituted into (4.2) gives

$$\dot{a} = [-i\omega_0 - \pi\kappa^2 \mathrm{sgn}(t)]a - \kappa \int d\omega\, e^{-i\omega t} b_0(\omega) , \quad (4.5)$$

the solution of which is

$$a(t) = e^{-i\omega_0 t - \pi\kappa^2|t|} a_0 - \kappa \int d\omega \frac{e^{-i\omega t} - e^{-i\omega_0 t - \pi\kappa^2 |t|}}{\mathrm{sgn}(t)\pi\kappa^2 - i(\omega-\omega_0)} b_0(\omega) . \quad (4.6)$$

The full solution for $b(\omega,t)$ is not really needed here, but for completeness (4.6) can be substituted into (4.4) to give

$$b(\omega,t) = e^{-i\omega t} b_0(\omega) + \kappa \frac{e^{-i\omega t} - e^{-i\omega_0 t - \pi\kappa^2|t|}}{\mathrm{sgn}(t)\pi\kappa^2 - i(\omega-\omega_0)} a_0 - \kappa^2 \int d\omega' \left[\frac{e^{i\omega' t} - e^{i\omega t}}{i(\omega-\omega')} + \frac{e^{-i\omega_0 t - \pi\kappa^2|t|} - e^{-i\omega t}}{\mathrm{sgn}(t)\pi\kappa^2 - i(\omega-\omega_0)}\right] \frac{b_0(\omega')}{\mathrm{sgn}(t)\pi\kappa^2 - i(\omega'-\omega_0)} . \quad (4.7)$$

The solution (4.6) can be rewritten in a very simple form as follows. Define a combined bath mode operator B_t by

$$B_t = \int d\omega\, g(\omega) b(\omega) , \quad (4.8)$$

where

$$g(\omega) = \frac{\kappa}{(1-e^{-2\pi\kappa^2|t|})^{1/2}} \frac{e^{-i(\omega-\omega_0)t} - e^{-\pi\kappa^2|t|}}{\mathrm{sgn}(t)\pi\kappa^2 - i(\omega-\omega_0)} .$$

This is normalized so that

$$\int d\omega\, |g(\omega)|^2 = 1 , \quad (4.9)$$

and hence

$$[B_t, B_t^\dagger]_- = 1 .$$

For convenience, further define the overall loss to be

$$\Gamma = e^{-\pi\kappa^2|t|} \; .$$

Then (4.6) becomes

$$a(t) = e^{-i\omega_0 t}[\Gamma a_0 - (1-\Gamma^2)^{1/2}B_t] \; . \tag{4.10}$$

In effect, for any chosen time t, one can write the multimode combination of (4.6) as the two-mode combination (4.10). Instead of describing the bath by $N(=\infty)$ equally spaced modes, we are doing so with one isolated mode, and $N-1$ remaining modes with some new spacing.

As before, we allow an arbitrary initial state of the system, and assume an independent thermal state for the bath. This time, the appropriate basis states for expanding the system density matrix are the coherent states:

$$\begin{aligned}\rho(0) &= \int d^2\alpha \int d^2\beta \, P(\alpha,\beta^*) \frac{(|\alpha\rangle\langle\beta|)_{\text{sys}}}{\langle\beta|\alpha\rangle} \otimes \rho_T \\ &= \int d^2\alpha \int d^2\beta \, P(\alpha,\beta^*) \frac{1}{Z_T} :\exp\left[-(a^\dagger - \beta^*)(a-\alpha) - \frac{1}{N+1}\int d\omega \, b^\dagger(\omega) b(\omega) \right]: \, .\end{aligned} \tag{4.11}$$

As the evolution does not mix creation and annihilation operators, there are no ordering problems. To be able to write down the evolved density matrix $\rho(t)$ we simply need

$$\begin{aligned}&[a^\dagger(-t)-\beta^*][a(-t)-\alpha] + \frac{1}{N+1}\int d\omega \, b^\dagger(\omega,-t) b(\omega,-t) \\ &= \frac{N}{N+1}[\Gamma a_0^\dagger - (1-\Gamma^2)^{1/2}B_{-t}^\dagger][\Gamma a_0 - (1-\Gamma^2)^{1/2}B_{-t}] - e^{i\omega_0 t}\beta^*[\Gamma a_0 - (1-\Gamma^2)^{1/2}B_{-t}] \\ &\quad - e^{-i\omega_0 t}[\Gamma a_0^\dagger - (1-\Gamma^2)^{1/2}B_{-t}^\dagger]\alpha + \beta^*\alpha + \frac{1}{N+1}\left[\int d\omega \, b_0^\dagger(\omega) b_0(\omega) + a_0^\dagger a_0 \right] \\ &= \frac{(a_0^\dagger - e^{i\omega_0 t}\Gamma\beta^*)(a_0 - e^{-i\omega_0 t}\Gamma\alpha)}{N(1-\Gamma^2)+1} + \frac{1}{N+1}\left[\int d\omega \, b_0^\dagger(\omega) b_0(\omega) - B_{-t}^\dagger B_{-t} \right] \\ &\quad + \frac{1+N(1-\Gamma^2)}{1+N}\left[B_{-t}^\dagger - \frac{N\Gamma(1-\Gamma^2)^{1/2}a_0^\dagger}{1+N(1-\Gamma^2)} + \frac{(N+1)(1-\Gamma^2)^{1/2}e^{i\omega_0 t}\beta^*}{1-N(1-\Gamma^2)} \right] \\ &\quad \times \left[B_{-t} - \frac{N\Gamma(1-\Gamma^2)^{1/2}a_0}{1+N(1-\Gamma^2)} + \frac{(N+1)(1-\Gamma)^{1/2}e^{-i\omega_0 t}\alpha}{1+N(1-\Gamma^2)} \right] ,\end{aligned} \tag{4.12}$$

where the first step uses the fact that $\int d\omega \, b^\dagger(\omega) b(\omega) + a^\dagger a$ is a constant of the motion. So

$$\begin{aligned}\rho(t) = \int d^2\alpha \int d^2\beta \, P(\alpha,\beta^*) \frac{1}{Z_T} :\exp\Bigg[&- \frac{(a_0^\dagger - e^{i\omega_0 t}\Gamma\beta^*)(a_0 - e^{-i\omega_0 t}\Gamma\alpha)}{N(1-\Gamma^2)+1} \\ &+ \frac{1+N(1-\Gamma^2)}{1+N}\left[B_{-t}^\dagger - \frac{(1-\Gamma^2)^{1/2}}{1+N(1-\Gamma^2)}[N\Gamma a_0^\dagger - (N+1)e^{i\omega_0 t}\beta^*] \right] \\ &\times \left[B_{-t} - \frac{(1-\Gamma^2)^{1/2}}{1+N(1-\Gamma^2)}[N\Gamma a_0 - (N+1)e^{-i\omega_0 t}\alpha] \right] \\ &+ \frac{1}{N+1}\left[\int d\omega \, b_0^\dagger(\omega) b_0(\omega) - B_{-t}^\dagger B_{-t} \right] \Bigg]: \, .\end{aligned} \tag{4.13}$$

That is, the system mode a has amplitudes decaying as $\Gamma = e^{-\pi\kappa^2|t|}$, with the number of thermal quanta rising as $N(1-\Gamma^2)$; the isolated bath mode B_t has amplitude rising as $1-\Gamma^2$, and its thermal fluctuations are transiently correlated with the system, with the number of thermal quanta decaying as Γ^2 [this can be confirmed by rearranging (4.12) so that the correlation terms are grouped with a instead of with B_t]; and the other bath modes remain unaffected. Tracing over the bath now gives

$$\begin{aligned}\mathrm{Tr}_B[\rho(t)] &= \int d^2\alpha \int d^2\beta \frac{P(\alpha,\beta^*)}{N(1-\Gamma^2)+1} :\exp\left[-\frac{(a_0^\dagger - e^{i\omega_0 t}\Gamma\beta^*)(a_0 - e^{-i\omega_0 t}\Gamma\alpha)}{N(1-\Gamma^2)+1} \right]: \\ &= \int d^2\alpha \int d^2\beta \frac{P(\alpha,\beta^*)}{N(t)+1} \langle\beta|\alpha\rangle^{-\exp(-2\pi\kappa^2|t|)/[1+2N(t)]} D(\bar{\alpha}) :\exp\left[-\frac{a_0^\dagger a_0}{N(t)+1} \right]: D^\dagger(\bar{\beta}) \, ,\end{aligned} \tag{4.14}$$

where

$$N(t)=N(1-e^{-2\pi\kappa^2|t|}) ,$$

$$\bar{\alpha}=e^{-i\omega_0 t-\pi\kappa^2|t|}\frac{[N(t)+1]\alpha+N(t)\beta}{2N(t)+1} , \tag{4.15}$$

$$\bar{\beta}=e^{-i\omega_0 t-\pi\kappa^2|t|}\frac{[N(t)+1]\beta+N(t)\alpha}{2N(t)+1} ,$$

in agreement with the result obtained by the fully reduced calculation of Appendix B.

In the reduced density matrix (4.14) we can see not only the overall decay of amplitude but also preferential decay of the off-diagonal elements. This preferential decay has two aspects. Firstly, the coefficients of off-diagonal elements are reduced compared to those of diagonal elements by a factor which starts as 1 and ends as $\langle\beta|\alpha\rangle$. This is similar to the decay of off-diagonal elements in phase damping in Sec. III. Secondly, for finite temperatures, any off-diagonal component of amplitude is reduced compared to the diagonal component. The reduction is given explicitly in (B13). In this case then, unlike the phase damping of Sec. III, the damping of the system mode arises out of the motion of the system, but exhibiting the loss of coherence still requires ignorance of the state of the bath.

V. RAMAN AMPLIFICATION

Raman amplification is superficially similar to the damping process of Sec. IV—indeed the reduced equations have exactly the same form (see Appendix B). However, in this case there is additional complexity resulting from the process mixing annihilation and creation operators. The physical difference from simple damping is that the coupling between local system and bath is not direct but is achieved via a high-frequency driving field (usually with the pump frequency ω_p roughly twice the system frequency ω_0). This means that in the rotating-wave description the important interaction terms are those which couple system and bath operators with the same frequency sign rather than the opposite one. The same effect can be achieved with an ordinary damping coupling to a local system modeled as an inverted (i.e., negative frequency) oscillator.

The Hamiltonian is

$$H=\hbar\omega_0 a^\dagger a+\hbar\int_{-\infty}^{\infty}d\omega\,\omega b^\dagger(\omega)b(\omega)$$
$$+i\int d\omega\,\hbar\kappa(\omega)[e^{-i\omega_p t}b^\dagger(\omega)a^\dagger-e^{i\omega_p t}ab(\omega)] , \tag{5.1}$$

where the explicit time dependence of the interaction term indicates that this is a driven process: the effective coupling constant κ is in fact the product of the real coupling constant with the amplitude of the driving field, which is assumed to be strong enough to be treated classically. Clearly, for the rotating-wave Hamiltonian (5.1) to be valid, $\omega_p-\omega_0$ must be positive and large compared to the gain rate [which turns out to be $\pi\kappa^2(\omega)$]. The solutions of the equations of motion can be found in just the same fashion as for amplitude damping, giving

$$a(t)=e^{-i\omega_0 t+\pi\kappa^2|t|}$$
$$-\kappa\int d\omega\frac{e^{i(\omega_p-\omega)t}-e^{-i\omega_0 t+\pi\kappa^2|t|}}{\mathrm{sgn}(t)\pi\kappa^2-i(\omega+\omega_0-\omega_p)}b_0^\dagger(\omega) . \tag{5.2}$$

Again, similarly to the damped case, we can write this as

$$a(t)=e^{-i\omega_0 t}[Ga_0-(G^2-1)^{1/2}B_t^\dagger] , \tag{5.3}$$

where the combined bath mode is given by

$$B_t=\int d\omega\,g(\omega)b(\omega) ,$$

with

$$g(\omega)=\frac{\kappa}{(e^{2\pi\kappa^2|t|}-1)^{1/2}}\frac{e^{-i(\omega_p-\omega_0-\omega)t}-e^{\pi\kappa^2|t|}}{\mathrm{sgn}(t)\pi\kappa^2-i(\omega+\omega_0-\omega_p)}$$

so that

$$[B_t,B_t^\dagger]_-=\int d\omega\,|g(\omega)|^2=1 , \tag{5.4}$$

and the overall gain is given by

$$G=e^{\pi\kappa^2|t|} .$$

At this point we can usefully begin to disentangle creation and annihilation operators, by noting that (5.3) is of the same form as the solution for a two-mode parametric interaction. That is, we can write

$$a(t)=e^{-i\omega_0 t}e^{-r(a_0B_t-a_0^\dagger B_t^\dagger)}a_0e^{r(a_0B_t-a_0^\dagger B_t^\dagger)} , \tag{5.5}$$

where r corresponds to (coupling multiplied by time) in the two-mode case. Here $r=\cosh^{-1}G$. The analogy should not be pressed too far, since important differences between the two are hidden by the time-dependent definition of the mode B_t. Since $a^\dagger a-\int d\omega\,b^\dagger(\omega)b(\omega)$ is a constant of the motion, it follows from (5.5) that

$$\int d\omega\,b^\dagger(\omega,t)b(\omega,t)$$
$$=e^{r(a_0^\dagger B_t^\dagger-a_0B_t)}\int d\omega\,b_0^\dagger(\omega)b_0(\omega)e^{-r(a_0^\dagger B_t^\dagger-a_0B_t)} . \tag{5.6}$$

This can be established as follows:

$$\begin{aligned}\int d\omega\, b^\dagger(\omega,t)b(\omega,t) &= \int d\omega\, b^\dagger(\omega,t)b(\omega,t) - a^\dagger(t)a(t) + a^\dagger(t)a(t)\\ &= \int d\omega\, b_0^\dagger(\omega)b_0(\omega) - a_0^\dagger a_0 + [Ga_0^\dagger - (G^2-1)^{1/2}B_t][Ga_0 - (G^2-1)^{1/2}B_t^\dagger]\\ &= \int d\omega\, b_0^\dagger(\omega)b_0(\omega) - B_t^\dagger B_t + [GB_t^\dagger - (G^2-1)^{1/2}a_0][GB_t - (G^2-1)^{1/2}a_0^\dagger]\\ &= \int d\omega\, b_0^\dagger(\omega)b_0(\omega) - B_t^\dagger B_t + e^{r(a_0^\dagger B_t^\dagger - a_0 B_t)} B_t^\dagger B_t e^{r(a_0 B_t - a_0^\dagger B_t^\dagger)} ,\end{aligned} \tag{5.7}$$

and (5.6) follows since

$$\left[(a_0 B_t - a_0^\dagger B_t^\dagger),\, \left[\int d\omega\, b_0^\dagger(\omega) b_0(\omega) - B_t^\dagger B_t \right] \right]_- = 0 .$$

Note that it is *not* true that the transformation used in (5.5) and (5.6) gives the correct time dependence of the bath modes individually. Using (5.6) it is, however, possible to write

$$\int d\omega\, b^\dagger(\omega,t)b(\omega,t) = \int d\omega \bar{b}^\dagger(\omega,t)\bar{b}(\omega,t) , \tag{5.8}$$

where

$$\begin{aligned}\bar{b}(\omega,t) &= e^{-i\omega t} e^{r(a_0^\dagger B_t^\dagger - a_0 B_t)} b_0(\omega) e^{r(a_0 B_t - a_0^\dagger B_t^\dagger)}\\ &= e^{-i\omega t}[b_0(\omega) + (G-1)g^*(\omega)B_t\\ &\quad - (G^2-1)^{1/2} g^*(\omega) a_0^\dagger] .\end{aligned}$$

For an initial density matrix representing a vacuum state for the system and a thermal state for the bath,

$$\rho_0(0) = \frac{1}{Z_T} :\exp(-a^\dagger a): \exp\left[-\frac{\hbar\omega}{kT} \int d\omega\, b^\dagger(\omega) b(\omega) \right] , \tag{5.9}$$

the density-matrix solution is then

$$\begin{aligned}\rho_0(t) &= e^{r(a_0^\dagger B_{-t}^\dagger - a_0 B_{-t})} \rho_0(0) e^{r(a_0 B_{-t} - a_0^\dagger B_{-t}^\dagger)}\\ &= \frac{1}{G^2 Z_T} :\exp\left[-\frac{a_0^\dagger a_0}{1+(G^2-1)(N+1)} - \frac{1}{N+1} \left[\int d\omega\, b_0^\dagger(\omega) b_0(\omega) - B_{-t}^\dagger B_{-t} \right]\right.\\ &\quad \left. - \frac{1+(1-G^{-2})N}{1+N} \left[B_{-t}^\dagger - \frac{G(G^2-1)^{1/2}(1+N)a_0}{1+(G^2-1)(1+N)} \right] \left[B_{-t} - \frac{G(G^2-1)^{1/2}(1+N)a_0^\dagger}{1+(G^2-1)(1+N)} \right] \right]: ,\end{aligned} \tag{5.10}$$

using successively (A14) with $\sigma_+ = a_0^\dagger B_{-t}^\dagger$, $\sigma_- = a_0 B_{-t}$, and $\sigma_z = a_0^\dagger a_0 + B_{-t}^\dagger B_{-t} + 1$; the vacuum state of the system $[a_0 \rho_0(0) = \rho_0(0) a_0^\dagger = 0]$; the ordering relation (2.4); and the definition of r.

For a coherently displaced initial system element, i.e.,

$$\rho_{\alpha\beta}(0) = \frac{\langle \beta | \alpha \rangle}{Z_T} :\exp[-(a^\dagger - \beta^*)(a-\alpha)]: \exp\left[-\frac{\hbar\omega_0}{kT} \int d\omega\, b^\dagger(\omega) b(\omega) \right] , \tag{5.11}$$

we can use a modification of the transformation (5.5), since

$$a(t) - \alpha = e^{-i\omega_0 t}[Ga - (G^2-1)^{1/2} B_t^\dagger - e^{i\omega_0 t}\alpha] = e^{-i\omega_0 t} U_t (a_0 - G e^{i\omega_0 t}\alpha) U_t^{-1} , \tag{5.12}$$

where

$$U_t = \exp(r\{(a_0^\dagger - Ge^{-i\omega_0 t}\beta^*)[B_t^\dagger - (G^2-1)^{1/2} e^{i\omega_0 t}\alpha] - (a_0 - Ge^{i\omega_0 t}\alpha)[B_t - (G^2-1)^{1/2} e^{-i\omega_0 t}\beta^*]\}) ,$$

and also

$$e^{r(a_0^\dagger B_t^\dagger - a_0 B_t)} B_t e^{r(a_0 B_t - a_0^\dagger B_t^\dagger)} = GB_t - (G^1-1)^{1/2} a_0^\dagger = U_t [B_t - (G^2-1)^{1/2} e^{-i\omega_0 t}\beta^*] U_t^{-1} . \tag{5.13}$$

So the calculation can proceed in exactly the same fashion as for the vacuum case, with the replacements

$$\begin{aligned} &a_0 \Longrightarrow a_0 - Ge^{i\omega_0 t}\alpha, \quad a_0^\dagger \Longrightarrow a_0^\dagger - Ge^{-i\omega_0 t}\beta^* ,\\ &B_t \Longrightarrow B_t - (G^2-1)^{1/2} e^{-i\omega_0 t}\beta^*, \quad B_t^\dagger \Longrightarrow B_t^\dagger - (G^2-1)^{1/2} e^{i\omega_0 t}\alpha .\end{aligned} \tag{5.14}$$

This gives as the complete density-matrix solution

$$\rho(t)=\int d^2\alpha\int d^2\beta\, P(\alpha,\beta^*)\frac{1}{G^2 Z_T}:\exp\left[-\frac{(a_0^\dagger-Ge^{i\omega_0 t}\beta^*)(a_0-Ge^{-i\omega_0 t}\alpha)}{1+(G^2-1)(N+1)}\right.$$
$$-\frac{1+(1-G^{-2})N}{1+N}\left[B_{-t}^\dagger-\frac{G}{1+(G^2-1)(N+1)}\times[(G^2-1)^{1/2}(1+N)a_0+Ne^{-i\omega_0 t}\alpha]\right]$$
$$\times\left[B_{-t}-\frac{G}{1+(G^2-1)(N+1)}[(G^2-1)^{1/2}(1+N)a_0^\dagger+Ne^{i\omega_0 t}\beta^*]\right]$$
$$\left.-\frac{1}{N+1}\left[\int d\omega\, b_0^\dagger(\omega)b_0(\omega)-B_{-t}^\dagger B_{-t}\right]\right]:\,. \tag{5.15}$$

Tracing over the bath gives for the reduced density matrix

$$\mathrm{Tr}_B[\rho(t)]=\int d^2\alpha\int d^2\beta\frac{P(\alpha,\beta^*)}{1+(G^2-1)(N+1)}:\exp\left[-\frac{(a_0^\dagger-e^{i\omega_0 t}G\beta^*)(a_0-e^{-i\omega_0 t}G\alpha)}{1+(G^2-1)(N+1)}\right]:$$
$$=\int d^2\alpha\int d^2\beta\, P(\alpha,\beta^*)\frac{\langle\beta\,|\,\alpha\rangle^{-\exp(2\pi\kappa^2|t|)/[1+2N(t)]}}{N(t)+1}D(\bar\alpha):\exp\left[-\frac{a_0^\dagger a_0}{N(t)+1}\right]:D^\dagger(\bar\beta)\,,$$

where

$$N(t)=(1+N)(e^{2\pi\kappa^2|t|}-1),\quad \bar\alpha=e^{-i\omega_0 t+\pi\kappa^2|t|}\frac{[N(t)+1]\alpha+N(t)\beta}{2N(t)+1}\,,$$

$$\bar\beta=e^{-i\omega_0 t+\pi\kappa^2|t|}\frac{[N(t)+1]\beta+N(t)\alpha}{2N(t)+1}\,,$$

again agreeing with the result of Appendix B.

As for amplitude damping, we can see in (5.16) decay of both the coefficients and the off-diagonal amplitude of off-diagonal matrix elements. The latter effect is, however, much more important here, since the diagonal amplitudes are now being amplified (as G), while the off-diagonal ones are still being damped (as G^{-1}). Thus in this case all elements of the density matrix eventually become diagonal. Temperature considerations are less important than in the damping case, as any amplifying process must in any case introduce extra noise.[20] (The extra noise introduced by the Raman amplifier is in fact the least noise allowed by the uncertainty principle.) This diagonalization is of course *not* present in the unreduced density matrix (5.15), although the amplification is, being a direct consequence of the substitution (5.14).

VI. A MEASUREMENT MODEL

As a final example of unreduced calculation, I will consider the measurement model of Walls, Collett, and Milburn.[4] This is a three-part model, including the quantum *system* being measured, the measurement apparatus or *meter,* coupled to the system; and the *environment,* modeled as a heat bath coupled to the meter. The specific choice of Hamiltonian is

$$H=\hbar\Omega N+\hbar\omega_0 a^\dagger a-i\hbar N(\epsilon^* a e^{i\omega_0 t}-a^\dagger\epsilon e^{-i\omega_0 t})$$
$$+\int d\omega\,\hbar\omega b^\dagger(\omega)b(\omega)+i\int d\omega\,\hbar\kappa[b^\dagger(\omega)a-a^\dagger b(\omega)]\,, \tag{6.1}$$

where N is the system number operator and a is the meter annihilation operator. The reasons for these choices are given in detail in Ref. 4, the principle considerations being the following.

(i) The coupling to the system is back-action evading, giving a more nearly ideal measurement.

(ii) Amplification (a necessary component of any genuine measurement scheme[20]) is built in by the driven system-meter coupling.

(iii) The amplitude coupling of meter and environment means that after the bath is traced out the meter is diagonalized in a basis of coherent states, which have a well-defined classical limit.

The equations of motion are

$$\frac{d}{dt}|m\rangle\langle n| = [-i\Omega + (a^\dagger \epsilon e^{-i\omega_0 t} - \epsilon^* a e^{i\omega_0 t})] \times [|m\rangle\langle n|, N]_- ,$$
$$\dot{a} = -i\omega_0 a + e^{-i\omega_0 t}\epsilon N - \int d\omega\, \kappa b(\omega) , \qquad (6.2)$$
$$\dot{b}(\omega) = -i\omega b(\omega) + \kappa a .$$

As in the phase-damping model of Sec. III, the system number operator is a constant of the motion

$$N(t) = N(0) . \qquad (6.3)$$

Solving (6.2) in a fashion similar to preceding sections gives

$$a(t) = e^{-i\omega_0 t}\left[\Gamma a_0 + \frac{\epsilon}{\mathrm{sgn}(t)\pi\kappa^2}N(1-\Gamma) - (1-\Gamma^2)^{1/2}B_t\right] ,$$

$$|m\rangle\langle n| = e^{-i\Omega(n-m)t}\exp\left[-(n-m)\epsilon\int_0^t d\tau\, a^\dagger(\tau)e^{-i\omega_0\tau}\right](|m\rangle\langle n|)_0 \exp\left[-(n-m)\epsilon^*\int_0^t d\tau\, a(\tau)e^{-i\omega_0\tau}\right] , \qquad (6.4)$$

where Γ and B_t are as defined in Sec. IV.

To avoid unnecessary complexity, I shall specialize to the zero-temperature case, with the meter initially in a vacuum state:

$$\rho(0) = \sum_{m,n} c_{mn} :\exp\left[-a^\dagger a - \int d\omega\, b^\dagger(\omega)b(\omega)\right]|m\rangle\langle n|: . \qquad (6.5)$$

This choice of initial state exhibits the measurement process most clearly, although some interesting effects might well be seen with the meter and bath prepared in, for instance, a squeezed state.[21]

It would be possible to proceed from here as in earlier sections, just substituting the solutions (6.4) into the initial density matrix. In this case the additional mode makes such an approach considerably more complicated. With the simple initial state (6.5), it is, however, quite straightforward to solve the evolution of the density matrix directly. We have

$$\frac{d}{dt}\rho_{mn} = \frac{d}{dt} :(|m\rangle\langle n|)_{-t}\exp\left[-a^\dagger(-t)a(-t) - \int d\omega\, b^\dagger(\omega,-t)b(\omega,-t)\right]:$$
$$= i\Omega[\rho_{mn}, N]_- - [\rho_{mn}, N[a^\dagger(-t)\epsilon e^{i\omega_0 t} - \epsilon^* a(-t)e^{-i\omega_0 t}]]_-$$
$$= i\Omega(n-m)\rho_{mn} + [\rho_{mn} n\epsilon^* a(-t)e^{-i\omega_0 t} + m\epsilon a^\dagger(-t)e^{i\omega_0 t}\rho_m n] . \qquad (6.6)$$

Since $[a(t), a(t')]_- = 0$ for any times t, t', we can directly exponentiate (6.6) to give

$$\rho_{mn} = e^{i\Omega(n-m)t}\exp\left[m\epsilon\int_0^t d\tau\, a^\dagger(-\tau)e^{i\omega_0\tau}\right]\rho_{mn}(0)\exp\left[n\epsilon^*\int_0^t d\tau\, a(-\tau)e^{-i\omega_0\tau}\right] . \qquad (6.7)$$

From (6.4),

$$\epsilon\int_0^t d\tau\, a(-\tau)e^{-i\omega_0 t} = \alpha(t)a_0 + \int d\omega\, \beta(\omega,t)b_0(\omega) - \tfrac{1}{2}N\left[|\alpha(t)|^2 + \int d\omega\, |\beta(\omega,t)|^2\right] , \qquad (6.8)$$

where

$$\alpha(t) = \frac{\epsilon}{\pi\kappa^2}(1-e^{-\pi\kappa^2|t|}), \quad \beta(\omega,t) = \frac{\kappa\epsilon}{\pi\kappa^2 + i(\omega-\omega_0)}\left[\frac{1-e^{-\pi\kappa^2|t|}}{\pi\kappa^2} - \frac{e^{i(\omega-\omega_0)t}-1}{i(\omega-\omega_0)}\right] .$$

Putting all this together gives

$$\rho(t) = \sum_{m,n} c_{mn}(|m\rangle\langle n|)_{\mathrm{sys}} \otimes [|m\alpha(t)\rangle\langle n\alpha(t)|]_M \otimes \{|[m\beta(\omega,t)]\rangle\langle[n\beta(\omega,t)]|\}_B . \qquad (6.9)$$

Thus both meter and bath are coherently excited in correlation with the number of photons in the system. Tracing out the bath gives

$$\mathrm{Tr}_B[\rho(t)] = \sum_{m,n} c_{mn} \exp\left\{ \frac{|\epsilon|^2}{(2\pi\kappa^2)^2}(n-m)^2[1-\pi\kappa^2 t - e^{-\pi\kappa^2 t} + \tfrac{1}{2}(1-e^{-\pi\kappa^2 t})^2] \right\}$$
$$\times (|m\rangle\langle n|)_{\mathrm{sys}} \otimes \left\{ \left| \frac{\epsilon n}{2\pi\kappa^2}(1-e^{-\pi\kappa^2 t}) \right\rangle \left\langle \frac{\epsilon m}{2\pi\kappa^2}(1-e^{-i\pi\kappa^2 t}) \right| \right\}_M ,$$

as obtained by reduced calculation in Ref. 4.

VII. INTERPRETATION AND CONCLUSION

The aim of this paper has been to try and shed some light on the behavior of open quantum systems by a method which allows the detailed description of the environment to be retained, at least in the simple cases studied here. All the results here follow exactly from the initial (usually approximate) Hamiltonians. One possible approach to more complex systems would be to calculate the behavior of the bath along the lines of the input/output formalism of Gardiner and Collett.[22] Given an "input" field (specified by the initial state of the bath) the behavior of the local system can be calculated by conventional methods (such as a master equation), and the final state of the bath is then the "output" field, given by boundary conditions between system and bath. To construct a description in terms of an unreduced density matrix one would also need to know the system-bath correlations. For a general nonlinear system this is clearly not possible exactly, but might reasonably be attempted with a Gaussian approximation for the fluctuations. Effectively, one would be assuming an unreduced density matrix in the form of an exponential quadratic, as in Secs. IV and V above, and determined the coefficients by inverting the antinormally ordered covariance matrix, neglecting higher-order cumulants.

Interpretation of the complete density-matrix solutions is not necessarily obvious, especially when correlated quantum noise terms are present. In the case of zero temperature, where no such correlations arise we can venture an identification of two distinct sorts of irreversible behavior, as suggested in the Introduction. Firstly, there is the deterministic loss or gain of amplitude seen in Secs. IV and V, and in the meter mode in Sec. VI. This is of the same nature as classical damping or amplification, and does not depend on any tracing out of the bath, though of course for true irreversibility we will require the limit of a continuous number of bath modes. Secondly, there is a loss of coherence, manifested as a diagonalization of the system density matrix, and also appearing as phase damping in Sec. III. This is evidently a specifically quantum-mechanical effect—the expression of the initial state as a superposition is necessary for the notion of diagonalization to have any significance. It is purely a property of the *reduced* density matrix, and thus, unlike the first sort of irreversible behavior, only appears when all possibility of any detailed knowledge of the bath is denied.

At finite temperature, the picture is less clear. In the case of phase damping, a thermal randomization of phase would be expected even classically, but the only apparent effect of the increase in temperature is that the quantum loss of coherence becomes stronger. Part of the difficulty here, of course, is that the chosen number-state basis has no classical limit or analogue. Where a classical analogue *can* be found, in the amplitude-coupled cases, we can see thermal contributions to both types of irreversible behavior: even before tracing out, thermal excitation of the system mode is evident (though the system-bath correlations leave some ambiguity as to just how much there is), while loss of coherence after tracing out is also increased.

As far as measurement models are concerned, we noted at the beginning of the paper one principal objection to all loss-of-coherence measurement schemes. That is that to work as models of a collapse and hence perhaps of a measurement, they require not merely the inclusion of the environment in the model, but the explicit reduction of the density matrix by tracing out the environment. One might perhaps say that a scheme such as that of Sec. VI does not so much show state reduction arising out of the behavior of a real physical system, as arising out of the calculational technique of tracing over the bath. Some knowledge still present in the model has to be actively suppressed before it can exhibit the empirically familiar features of a quantum measurement.

It is tempting to appeal here to the enormous practical difficulty, amounting to a practical impossibility, of "undoing" the processes which distribute the quantum coherences over the bath modes. But this is to miss the point; were we concerned solely with practical predictions, there would be no reason to question the "textbook" account of quantum measurement, which has, after all, passed strenuous experimental tests. Merely to raise the problem of measurement in the first place is to indicate that one is concerned with matters of theoretical principle and consistency. In any case, this appeal is somewhat circular; to make use of the difficulty of measuring phase relationships in the environment is to assume the availability of some acceptable account of measurement, which is just what the model seeks to provide. Increasing the complexity of the already infinite bath would just further increase the practical difficulties without introducing anything new in principle.

I also noted in the Introduction another possible response to this objection. While accepting that models of this type do not represent the whole answer to the problem of quantum measurement, it may be pointed out that questions of the legitimacy of tracing out the environment are not specific to measurement models. They are, rather, a common feature of all models of irreversible quantum processes. It might seem reasonable then to consider that measurement models of this type, while not solving any problems completely, do reduce two problems to one: measurements may be considered as just a special case of irreversible processes in general.

The unreduced solution for the model considered here, (6.9), allows us now to make some assessment of this defence. Recalling our distinction between the two sorts of irreversible behavior, one which does not depend on the tracing out and the other which does, we can see both at work here. The development of correlations between the system and the meter is an irreversible process of the first kind, like the amplitude damping of Sec. IV. It appears, described by the quantity $\alpha(t)$, even in the unreduced solution. But the system part of each term in the density matrix is constant in this solution. The diagonalization which gives a mixture of number states as the system state only appears when the bath is traced out, like the phase damping of Sec. III. If amplitude damping is taken as the more typical instance of an irreversible process, as I think it must be, then the (reluctant) conclusion must be that the assumptions needed to make loss of coherence a source of diagonalization do go further than those required for normal dissipative irreversibility.

In saying this, I do not wish to deny the value of measurement models of this type. What such a model actually shows is that: if in a sufficiently large and complex system the coherences between distinct subsystems are negligible (allowing us to trace over at least some of those parts of the system which we are not directly interested in), then the measurement postulates (or something very like them) must be approximately true. If this assumption holds, then we would in effect be justified in treating the local system as being genuinely open, in which case the diagonalization of the density matrix follows trivially (as Peres[9] points out). The problem is that while this crucial assumption is intuitively plausible, and seems far less of an arbitrary addition to quantum mechanics than the measurement postulates themselves, it is still by no means clear that it is in any way a consequence of quantum mechanics, or even that the two are compatible.

In summary, standard theories of open systems not only include a heat bath, perhaps with the essentially classical ignorance represented by a thermal initial state, but also, by tracing out over this bath, introduce a much more thoroughgoing ignorance amounting to a denial of any possibility of more detailed knowledge. The former is a necessary part of the theory; the latter can be considered a legitimate calculational technique, greatly reducing the complexity of the problem without affecting the conclusions for the system alone. But if wider conclusions are to be drawn concerning the loss of quantum coherence, as measurement theory would like to, this more complete ignorance also becomes indispensible. However attractive measurement models which exploit this loss of coherence may seem, the fact remains that the specifically quantum-mechanical aspects of irreversibility still require explanation beyond that which is needed for ordinary classical irreversibility.

More broadly, we have here an instance of an ambiguity in the representation of quantum states: the *mixture* of states was originally introduced to allow classical uncertainty to be represented within quantum mechanics; it is also used to describe incomplete quantum systems. Whatever their mathematical similarities, the two are physically distinct, and any careful interpretation of the density matrix must recognize this.

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APPENDIX A: SOME OPERATOR RELATIONS

In using operator representations of density matrices, one frequently wishes to reorder exponentiated operators. I summarize here some of the operator relations which have been particularly useful in preparing this paper. The starting point in each case is of course the Baker-Hausdorff expansion

$$e^{B}Ae^{-B}=\sum_{n}\frac{1}{n!}\,{}^{(n)}[B,A]_{-}\ , \tag{A1}$$

where ${}^{(n)}[B,A]_{-}$ denotes the n-fold-nested commutator of B with A. In particular, for the case that

$$[A,B]_{-}=kA\ , \tag{A2}$$

we have

$$e^{\lambda B}Ae^{-\lambda B}=e^{-\lambda k}A\ , \tag{A3}$$

which for $A=a$, $B=a^{\dagger}a$ gives

$$e^{\beta a^{\dagger}a}e^{\alpha a}=e^{\exp(-\beta)\alpha a}e^{\beta a^{\dagger}a}\ , \tag{A4}$$

needed for the derivation of (2.9). For A,B satisfying (A2), we can further show that

$$e^{A+B}=\exp\left[\frac{A}{k}(1-e^{-k})\right]e^{B}=e^{B}\exp\left[\frac{A}{k}(e^{k}-1)\right]. \tag{A5}$$

This can be done in standard fashion by differentiation of the product $e^{\lambda(A+B)}e^{-\lambda B}e^{-\lambda A}$.

Now consider a set of operators σ_{+}, σ_{-}, σ_{z} with spin commutation relations

$$[\sigma_{+},\sigma_{-}]_{-}=\sigma_{z},\quad [\sigma_{z},\sigma_{\pm}]_{-}=\pm 2\sigma_{\pm}\ . \tag{A6}$$

Using (A1) we immediately obtain

$$e^{\kappa\sigma_{z}}\sigma_{\pm}e^{-\kappa\sigma_{z}}=e^{\pm 2\kappa}\sigma_{\pm}\ . \tag{A7}$$

Again using the technique of differentiating and then reintegrating, we may also derive two particularly useful ordering relations:

$$e^{\kappa\sigma_\pm}e^{\lambda\sigma_\mp}=e^{\lambda\sigma_\mp/(1+\lambda\kappa)}(1+\lambda\kappa)^{\pm\sigma_z}e^{\kappa\sigma_\pm/(1+\lambda\kappa)} \tag{A8}$$

and

$$e^{\kappa(\sigma_++\sigma_-)}=e^{\sigma_\pm \tanh\kappa}(\cosh\kappa)^{\mp\sigma_z}e^{\sigma_\mp \tanh\kappa} . \tag{A9}$$

APPENDIX B: REDUCED DAMPING CALCULATIONS

The reduced density matrix $\bar\rho=\mathrm{Tr}_B(\rho)$ can be calculated directly for amplitude damping using the usual master equation for this process:

$$\dot{\bar\rho}=\tfrac{1}{2}\gamma(1+N)(2a\bar\rho a^\dagger-a^\dagger a\bar\rho-\bar\rho a^\dagger a)$$
$$+\tfrac{1}{2}\gamma N(2a^\dagger\bar\rho a-aa^\dagger\bar\rho-\bar\rho aa^\dagger) , \tag{B1}$$

where a rotating frame has been used to remove the free system frequency from the equation. This has been solved by direct integration for the zero-temperature case.[2] Here we make use of the characteristic function, defined by

$$\chi(\lambda)=\mathrm{Tr}(e^{\lambda a^\dagger-\lambda^* a}\bar\rho) . \tag{B2}$$

Using the operator correspondences which follow from (B2), we find

$$\frac{\partial\chi(\lambda)}{\partial t}=-\tfrac{1}{2}\gamma\left[(2N+1)|\lambda|^2+\lambda\frac{\partial}{\partial\lambda}+\lambda^*\frac{\partial}{\partial\lambda^*}\right]\chi(\lambda) . \tag{B3}$$

We can produce a formal solution of this by exponentiation, and then use (A5) with

$$A=-\tfrac{1}{2}\gamma t(2N+1)|\lambda|^2,\quad B=-\tfrac{1}{2}\gamma t\left[\lambda\frac{\lambda}{\partial\lambda}+\lambda^*\frac{\partial}{\partial\lambda^*}\right] ,$$

to give

$$\chi(\lambda,t)=e^{A+B}\chi(\lambda,0)$$
$$=\exp[-(N+\tfrac{1}{2})|\lambda|^2(1-e^{-\gamma t})]$$
$$\times\exp\left[-\tfrac{1}{2}\gamma t\left[\lambda\frac{\partial}{\partial\lambda}+\lambda^*\frac{\partial}{\partial\lambda^*}\right]\right]\chi(\lambda,0) . \tag{B4}$$

We obtain a standard form for $\chi(\lambda,0)$ by expanding $\bar\rho$ over coherent states

$$\bar\rho(0)=\int d^2\alpha\int d^2\beta\,P(\alpha,\beta^*)\frac{|\alpha\rangle\langle\beta|}{\langle\beta|\alpha\rangle} , \tag{B5}$$

so that

$$\chi(\lambda,0)=\int d^2\alpha\int d^2\beta\,P(\alpha,\beta^*)\chi_{\alpha\beta}(\lambda,0) , \tag{B6}$$

where

$$\chi_{\alpha\beta}(\lambda,0)=e^{-|\lambda|^2/2+\lambda\beta^*-\lambda^*\alpha} . \tag{B7}$$

Now using (A3) with $A=\lambda$, $B=-\tfrac{1}{2}\gamma t(\partial/\partial\lambda)$, we have

$$\exp\left[-\tfrac{1}{2}\gamma t\lambda\frac{\partial}{\partial\lambda}\right]e^{\alpha\lambda}=\exp(e^{-\gamma t/2}\alpha\lambda) , \tag{B8}$$

so that

$$\chi_{\alpha\beta}(\lambda,t)=\exp[-(N+\tfrac{1}{2})|\lambda|^2(1-e^{-\gamma t})]$$
$$\times\exp[-e^{-\gamma t}\tfrac{1}{2}|\lambda|^2+e^{-\gamma t/2}(\lambda\beta^*-\lambda^*\alpha)] \tag{B9}$$

The Q function can be obtained as the Fourier transform of the antinormally ordered characteristic function $\chi^A=e^{-|\lambda|^2/2}\chi$:

$$Q_{\alpha\beta}(\delta)=\frac{1}{N(1-e^{-\gamma t})+1}$$
$$\times\exp\left[-\frac{(\delta-e^{-\gamma t/2}\alpha)(\delta^*-e^{-\gamma t/2}\beta^*)}{N(1-e^{-\gamma t})+1}\right] , \tag{B10}$$

so that

$$\rho(t)=\int d^2\alpha\int d^2\beta\frac{P(\alpha,\beta^*)}{N(1-e^{-\gamma t})+1}:\exp\left[-\frac{(a^\dagger-e^{-\gamma t/2}\beta^*)(a-e^{-\gamma t/2}\alpha)}{N(1-e^{-\gamma t})+1}\right]: . \tag{B11}$$

The operator part of this represents a coherent element with thermal noise, as one might expect, but it is not quite in the form of (2.9). To put it into this form we want to find effective displacements $\bar\alpha$ and $\bar\beta$, such that

$$:\exp\left[-\frac{(a^\dagger-e^{-\gamma t/2}\beta^*)(a-e^{-\gamma t/2}\alpha)}{N(1-e^{-\gamma t})+1}\right]:=C:\exp\left[-(a^\dagger-\bar\beta^*)(a-\bar\alpha)+\frac{N(t)}{N(t)+1}(a^\dagger-\bar\alpha^*)(a-\bar\beta)\right]: , \tag{B12}$$

where $N(t)=N(1-e^{-\gamma t})$. Identifying coefficients of a and $a^\dagger$ gives

$$\begin{aligned}\bar{\alpha}+\bar{\beta}&=e^{-\gamma t/2}(\alpha+\beta)\ ,\\ \bar{\alpha}-\bar{\beta}&=e^{-\gamma t/2}\frac{\alpha-\beta}{2N(t)+1}\ .\end{aligned} \tag{B13}$$

Then

$$\begin{aligned}C&=|\langle\beta|\alpha\rangle|^{\{2N(t)/[2N(t)+1]^2\}e^{-\gamma t}}\\ &=\langle\bar{\beta}|\bar{\alpha}\rangle\langle\beta|\alpha\rangle^{-e^{-\gamma t}/[2N(t)+1]}\ ,\end{aligned} \tag{B14}$$

giving a final result identical to (4.14), where $\gamma=2\pi\kappa^2$.

If in (B1) we make the replacements

$$\gamma \Longrightarrow -\epsilon, \quad N \Longrightarrow -(N+1)\ , \tag{B15}$$

we have the equation for a Raman amplifier with gain ϵ. Making the corresponding substitution into the solution (4.14), now with $\epsilon=2\pi\kappa^2$, gives the solution for the Raman amplifier (5.16).

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