PHYSICAL REVIEW A VOLUME 38, NUMBER 4 AUGUST 15, 1988

Discrete-state phasor neural networks

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(Received 18 May 1988)

An associative memory network with local variables assuming one of q equidistant positions on the unit circle (q-state phasors) is introduced, and its recall behavior is solved exactly for any q when the interactions are sparse and asymmetric. Such models can describe natural or artificial networks of (neuro-)biological, chemical, or electronic limit-cycle oscillators with q-fold instead of circular symmetry, or similar optical computing devices using a phase-encoded data representation.

Solvable neural-network models can be helpful for understanding some of the basic aspects of biological information processing, and can aid the emerging technology of "neurocomputing" by suggesting or analyzing new design strategies. Until recently, most of the theoretical attention was concentrated on models having either binary variables or scalars constrained to the unit interval by sigmoidal nonlinearities.[1–3] Very recently, a model with continuous variables ranging over the unit circle (phasors) was introduced,[4] and its performance as a phasor associative memory was solved exactly,[5] assuming the (vector-valued) interactions to be sparse and asymmetric. In this paper, I propose and solve a class of related models in which the phasors are discrete q-state variables on the circle. The fundamental difference with Kanter's[6] recent q-state Potts neural network lies in the cyclic group structure of the value-space of q-state phasors, which allows for simpler synapses and yields different dynamical behavior. The motivations for studying general phasor models derive from natural as well as technological backgrounds.[4,5] Interacting limit-cycle oscillators occur in a wide variety of settings,[7] and some parts of the brain (e.g., olfactory bulb[8] and motor-pattern generators in simple animals) seem to use local feedback neural circuits as building blocks in networks. Problems for which phasor networks would be especially appropriate involve the processing of signals with a circular value space, such as edge orientations or optic-flow directions in images, or phase patterns over detector arrays. Finally, it seems technologically attractive to encode information in the phases of beams in optical neurocomputers, enabling the relatively slow optical switching used so far in amplitude-based machines to be replaced by much faster saturating optical amplification. First attempts at the design and construction of phasor devices are now being made.[9,10]

The models analyzed here are meant to be associative memories, i.e., given a noise-corrupted pattern as initial state, they should relax towards the nearest pattern occurring in a set of learned patterns. Exact results and useful approximations are derived for the accuracy and speed of recall in sparse, asymmetrically connected q-state phasor nets. From a statistical physics point of view, such phasor models are clockface or planar Potts (not standard Potts)[11] models with rather unusual sparse, asymmetric, and vector-valued interactions. I shall use complex number notation for all two vectors with $|x|$ denoting the modulus, and $\bar{x}$ the complex conjugate of x.

The network has $N \to \infty$ phasors s_i, with $|s_i| = 1$ and $s_i \in \{\sigma^n\}$ $(n = 0, \ldots, q-1)$, the set of qth roots of unity, i.e., $\sigma^n = \exp(\sqrt{-1}\, n 2\pi/q)$. Interactions occur via couplings $c_{ij} = K_{ij} C_{ij}$. The $K_{ij} \in \{0,1\}$ represents the sparseness and asymmetry of the connection matrix by taking them as N^2 independent samples from the distribution

$$p(K_{ij}) = \frac{Z}{N}\delta(K_{ij}-1) + \left(1-\frac{Z}{N}\right)\delta(K_{ij}) . \qquad (1)$$

The exact results apply to very sparse connectivities, $Z \sim (\ln N)^z$ as $N \to \infty$, for some fixed $0 < z < \infty$. Note that this causes almost sure asymmetry. The numbers Z_i of inputs $K_{ij} = 1$ per cell have mean Z and variance Z, hence become deterministic as N (and thus Z) go to infinity. Sparse, asymmetric connectivity is biologically more realistic, as well as easier to implement in Si chips, but may be unnecessary in optical networks. In this paper, I am more motivated by the theoretical advantages of sparseness, introduced and applied successfully to various binary automata and neural networks by Derrida and coworkers[12,13] and Hilhorst and Nijmijer.[14] The analysis becomes much simpler because no buildup of correlations occurs during the evolution of the model. A cell's state at time t depends only on that of roughly Z^t cells at time 0. These ancestors are all distinct as long as $Z^t \ll N^{1/2}$, which is guaranteed for any finite t by the chosen scaling of Z. To complete the specification of the interactions, the C_{ij} are defined via a complex-valued generalization[4,5] of Hebb's rule[15]

$$C_{ij} = \sum_{k=1}^{P} s_i^{(k)} \bar{s}_j^{(k)} , \qquad (2)$$

where $s_i^{(k)}$ is phasor i in the kth of P learned patterns. The $s_i^{(k)}$ will be assumed independent and uniformly distributed over the set $\{\sigma^n\}$, $n = 0, \ldots, q-1$. It should be possible to realize such interactions by means of holograms in optical networks, and via (multiple) real-valued synapses in networks of (biological) oscillators, where the complex-valued couplings represent the phase shifts due to finite propagation delays or hidden variables.[5]

The dynamics of the phasors depends only on their local fields

$$h_i = n_i + \frac{1}{Z}\sum_j c_{ij} s_j , \qquad (3)$$

where Z is the average number of inputs $K_{ij}=1$ per cell. The n_i are complex-valued local white Gaussian noise, i.e.,

$$\langle n_i(t)\bar{n}_j(t')\rangle = b\delta(i-j)\delta(t-t') .$$

Stochastic dynamics is defined via the Gaussian noise, not via a more usual finite-T spin dynamics. Assumption of detailed balance would be unrealistic because of the nonsymmetrical interactions. Noise due to nonequilibrium physical phenomena will be present in any realization, and will usually be Gaussian. In any event, generalization to other noise distributions would not involve great difficulties.

The evolution of the model will be analyzed under a range of discrete-time dynamics differing in their degree of parallelism, but all having the same rule for updating individual phasors:

$$s_i(t+\delta t)=\sigma^n \text{ if } |\mathrm{Arg}[h_i(t)/\sigma^n]| < \pi/q , \qquad (4)$$

where $\mathrm{Arg}(x)$, the phase angle of x, is taken to range over $(-\pi,\pi)$. Thus, each updated phasor assumes the value σ^n closest to the direction of its local field. Borderline cases can be resolved randomly without effect since they occur with vanishing probability. The degree of parallelism of the updating is allowed to vary from random purely sequential (type A) to fully parallel (type B). The natural time scale for type-A dynamics is obtained by taking the time interval δt as $1/N$; the proper choice for type-B dynamics is $\delta t=1$. Extension of the results to intermediate degrees of parallelism is easily done by taking $\delta t = x/N$, and updating a random subset (fraction $0<x/N<1$) of the phasors per time step.

Similarity of a networkstate to the patterns is expressed by overlaps M_k

$$M_k=\frac{1}{N}\sum_i s_i\bar{s}_i^{(k)} . \qquad (5a)$$

Although the evolution of the overlaps will prove to be fundamental in the analysis, it is useful for many practical applications to define another measure E_k, the fraction of phasors "in error" with respect to a pattern k.

$$E_k=\frac{1}{N}\mathrm{Card}\{s_i \,|\, s_i \neq s_i^{(k)},\ i=1,2,\ldots,N\} , \qquad (5b)$$

where Card is the cardinality. Note that the M_k rotate and E_k vary under rigid phase rotations transforming all s_i (or $s_i^{(k)}$) to Ss_i (or $Ss_i^{(k)}$) with a fixed $S \forall i$ and $|S|=1$ (i.e., picking a new global reference phase), whereas the c_{ij} and the dynamics are invariant under such pure gauge transforms. It is convenient to work in the standard gauge in which s_i is transformed to $s_i\bar{M}_1/|M_1|$; this rotates M_1 to the positive real axis, and removes from E_1 all unintended sensitivity to the global phase. If necessary the same can be done to the M_k and E_k $(k>1)$, independently for each k, by rigidly rotating the $s_i^{(k)}$.

The memory-recall behavior of the model is analyzed using an ensemble average over the Gaussian dynamical noise n_i, the random connections K_{ij}, the patterns $\{s_i^{(k)}\}$, and all initial conditions having some fixed finite overlap with only one pattern, say $M_1(t)>0$ and $M_{>1}(t)=0$ at $t=0$. I shall drop the k and t from $M_k(t)$ and $E_k(t)$ when $k=1$ and $t>0$ are meant. Let ∂_i denote the set of inputcells of cell i, $\partial_i := \{j \,|\, K_{ij}=1\}$, and let Z_i be their size, $Z_i=\mathrm{Card}(\partial_i)$. From (2) and (3), the local fields become

$$h_i=n_i+\frac{1}{Z}\sum_{\partial_i} s_j\left(\sum_{k=1}^{P} s_i^{(k)}\bar{s}_j^{(k)}\right)=n_i+h_i^{(1)}+h_i^* ,$$

where

$$h_i^{(1)}=\frac{1}{Z}s_i^{(1)}\left(\sum_{\partial_i}\bar{s}_j^{(1)}s_j\right) \qquad (6)$$

and

$$h_i^*=\frac{1}{Z}\sum_{k=2}^{P} s_i^{(k)}\left(\sum_{\partial_i}\bar{s}_j^{(k)}s_j\right) .$$

In the standard gauge, the $x_i^{(1)}=h_i^{(1)}\bar{s}_i^{(1)}$ are sums of Z_i biased vectors with mean $\langle x_i^{(1)}\rangle = M(t)$ and variance $\langle |x_i^{(1)}-\langle x_i^{(1)}\rangle|^2\rangle \le 1/Z$, at any time t. Thus, $h_i^{(1)}$ become deterministic as N and $Z\sim(\ln N)^z$ go to infinity. From here on, I will first discuss $q>2$ models, and then the special case $q=2$. The h_i^* are sums of $(P-1)Z_i$ random vectors σ^n, so their distribution as $Z\to\infty$ becomes a complex Gaussian with circular symmetry around 0 and variance $(P-1)/Z$. Recall that the complex noise n_i has $\langle n_i\rangle=0$ and $\langle |n_i|^2\rangle=b$.

Clearly, with a low loading $(P/Z\to 0)$ and no noise $(b=0)$, recall is perfect $[M(1)\to 1]$ in just a single parallel updating step. The more interesting cases are extensively loaded and noisy nets with $P/Z=a>0$ and $b>0$. Then the distribution of $x_i=h_i\bar{s}_i^{(1)}$ is Gaussian with variance $d=a+b$, and circular symmetry about $M(t)$. Since the $s_i(t+\delta t)$ depend only on the phase of $h_i(t)$, the evolution of the model can be written in terms of a dimensionless "reduced" overlap $m=M/\sqrt{d}$. As $N\to\infty$, the distribution of x_i in terms of Cartesian coordinates $(v,w) = [\mathrm{Re}(x),\mathrm{Im}(x)]/\sqrt{d/2}$ becomes

$$p(v,w)=\frac{1}{2\pi}\exp\{-[(v-\sqrt{2}m)^2+w^2]/2\} . \qquad (7)$$

The error measure E equals the probability mass falling outside the sector satisfying $v>0$, $|w|<v\tan(\pi/q)$. This will be studied in detail below. To analyze the exact evolution of $m(t)$, it is more appropriate to transform Eq. (7) to polar coordinates, integrate out the radial dependence, and thus obtain[5] the distribution $p(u)$ of the phase errors $u_i=\mathrm{Arg}(x_i)$

$$p(u)=\frac{1}{2\pi}(\exp[-m^2]+\sqrt{\pi}m\cos(u)\exp[-m^2\sin^2(u)]\{1+\mathrm{erf}[m\cos(u)]\}) , \qquad (8)$$

where $\mathrm{erf}(x)$ is the standard error function.[16] A useful approximation for $p(u)$ in the $m \ll 1$ regime is

$$p(u)=\frac{1}{2\pi}\{1+\sqrt{\pi}m\cos(u)[1-m^2\sin^2(u)]+m^2\cos(2u)\}+O(m^4) . \qquad (9)$$

The evolution of $m(t)$ can be calculated from (8) by choosing some specific type of dynamics (degrees of parallelism) based on the update rule Eq. (4). For type-B (parallel) dynamics, the states of all $s_i(t+1)$ are thus given in terms of the $s_i(t)$ by substituting $m(t)=M(t)/\sqrt{d}$, given $M(0)>0$. Each new $s_i(t+1)=\sigma^n s_i^{(1)}$ contributes $\mathrm{Re}(\sigma^n)/N$ to $M(t+1)$. The evolution of the model in terms of M, a, and b becomes $M(t+1)=F(a,b,M(t))$, where

$$F(a,b,M(t))=G(m)=\sum_{n=0}^{q-1}\left\{\cos(2n\pi/q)\int_{(2n-1)\pi/q}^{(2n+1)\pi/q}du\,p(u)\right\}. \tag{10}$$

Extension to type-A (serial) dynamics is easy, since this corresponds to

$$\begin{aligned}M(t+\delta t)&=(1-1/N)M(t)+(1/N)M(t+1)\\&=M(t)+\delta t[F(a,b,M(t))-M(t)].\end{aligned}$$

Thus, as $N\rightarrow\infty$ with $\delta t=1/N$,

$$\frac{d}{dt}M(t)=F(a,b,M(t))-M(t). \tag{11}$$

Extension to dynamics with intermediate parallelism should be obvious. In any case, the attractive fixed points $M^*(d)=G(M^*/\sqrt{d})$ represent the $t\rightarrow\infty$ recall accuracy of the phasor models with loading a and noise b.

Equations (10) and (11) are the exact solution of the recall process, and the integrals and sums can be evaluated numerically without serious difficulty. However, a bit more analysis will extract useful closed-form results and approximations of the solution in the low-m and high-m regimes. Substituting Eq. (9), the third-order $m\ll1$ expansion of $p(u)$, into Eq. (10) one finds that after some trigonometric algebra that the trivial fixed point $M=0$ becomes unstable for d below the critical point

$$d_c=q^2\sin^2(\pi/q)/(4\pi)\ \text{for any}\ q>2. \tag{12}$$

For all $q>3$, a stable recall solution $M^*>0$ branches off as

$$M^*\cong A_q(d_c-d)^{1/2}\ \text{with}\ A_{>4}=\sqrt{2},\ \text{but}\ A_4=\sqrt{3}. \tag{13}$$

The characteristic relaxation time T of $M(t)$, which is of order 1 for $d\ll1$, shows critical slowing down $T\sim|d_c-d|^{-1}$ for d near d_c and $q>3$. The case $q=3$ is peculiar; here, an unstable fixed point $M'>0$ branches off towards higher d, behaving as $M'\cong4\pi(d-d_c)/(3\sqrt{3})$. The fixed-point line now bulges out above $d_c\cong0.537$, leading to a first-order transition above d_c. Numerical evaluation of Eq. (10) shows that the attractive upper branch still has $M^*\cong0.7027$ at d_c, and extends to $d\cong0.613$ where $M^*\cong0.39$.

The high-m regime is more interesting for practical applications since this is where the recall process converges well to the proper pattern. Crossover to the high-m regime occurs when the bulk of $x_i=h_i\bar{s}^{(1)}$ vectors fall within the $n=0$ sector ($|u_i|<\pi/q$). This begins as $m>q$. The error-density E can be determined using two simple bounds $E^-<E<E^+$. I take E^+ as twice the probability mass in the halfplane $w>v\tan(\pi/q)$. This overestimates E by double counting the mass in the $v<0$ mirror image of the $n=0$ sector, which suggests subtracting (once) the mass in the halfplane $v<0$ from E^+ to get a lower bound E^-. The bounds then become

$$\begin{aligned}E^+&=\sqrt{2/\pi}\int_{\sqrt{2}m\sin(\pi/q)}^{\infty}dy\exp(-y^2/2)\\&=\mathrm{erfc}[m\sin(\pi/q)],\end{aligned} \tag{14a}$$

and

$$\begin{aligned}E^-&=E^+-\frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}m}^{\infty}dy\exp(-y^2/2)\\&=E^+-\mathrm{erfc}[m]/2,\end{aligned} \tag{14b}$$

where $\mathrm{erfc}[x]=1-\mathrm{erf}[x]$ is the standard complementary error function. For $x\gg1$, the first term of its asymptotic expansion will usually suffice $\mathrm{erfc}[x]=\exp[-x^2]\times(x\sqrt{\pi})^{-1}[1+O(x^{-2})]$. Note further that E^+ is an accurate $m\gg1$ approximation of E, since the $\sin(\pi/q)<1$ in the argument of E^+ makes the difference E^+-E^- decay much faster than E^+ itself. Estimating the mass in wider sectors one finds, as expected, that almost every phasor with an erroneous value $s_i\neq s_i^{(1)}$ merely falls into one of the neighboring sectors around $s_i^{(1)}\sigma^n$ with $|n|=1$, as soon as m is moderately larger than q. We still have to find the fixed point $m^*=M^*/\sqrt{d}$ in order to compute the fixed point error E^*. Since $|n|=1$ errors dominate, this means solving $M^*\cong1-E^*+E^*\cos(2\pi/q)$. Using the approximation $E^*\cong E^+$ proposed above, one obtains the asymptotic behavior for $d\ll\sin^2(\pi/q)$

$$1-M^*\cong2\sqrt{d/\pi}\sin(\pi/q)\exp[-\sin^2(\pi/q)/d], \tag{15a}$$

and

$$E^*\cong\sqrt{d/\pi}\exp[-\sin^2(\pi/q)/d]/\sin(\pi/q). \tag{15b}$$

Finally, the case $q=2$. The reason for treating it separately is that the distribution of the x_i is no longer circularly symmetric around M since the signal $h_i^{(1)}$ and interference h_i^* vectors are now distributed on the real axis, while the noise n_i is still complex. In view of the dynamics Eq. (4), which is now independent of $\mathrm{Im}(h_i)$, only the projections on the real line play any role. Thus, half of the noise variance b has no effect, but all of the variance a from interfering $k>1$ patterns does. Defining $d'=a+b/2$, the evolution equation for type-B dynamics becomes $M(t+1)=\mathrm{erf}[M(t)/\sqrt{2d'}]$, leading to the fixed-point approximations

$$M^*\cong[2(d_c'-d')]^{1/2}\ \text{as}\ d'\rightarrow d_c'=2/\pi\ \text{from below}, \tag{16}$$

and

$$1-M^*\cong\sqrt{2d'/\pi}\exp[-1/(2d')]\ \text{for}\ d'\ll\tfrac{1}{2}. \tag{17}$$

For $b=0$ Eqs. (16) and (17) equal the $T=0$ results for the fully binary network of Derrida, Gardner, and Zippelius.[12] The relation between E and M is now simply $E=(1-M)/2$.

I conclude with a few remarks about these results. The maximal memory capacity [Eq. (12)] converges for large q to the value $\pi/4$ obtained earlier for the continuous phasor model.[4,5] The fixed-point overlap M^* and error density E^* [Eqs. (15a) and (15b)] are very flat for small loading and noise. Their derivatives of all orders vanish at $d=0$, where they possess an essential singularity only. This is similar to the standard binary networks,[2,12] but quite distinct from the asymptotically linear $1-M$ error found recently in the continuous phasor model.[4,5] This feature of the q-state models is due to the strong nonlinearity that enforces the discreteness of state space. It persists in the limit for $q \rightarrow \infty$, but applies to the low-error regime shrinking with q as $\sqrt{d} < \sin(\pi/q)$. One has to study the $1 \gg d \gg q^{-2}$ regime to notice the linear error emerging in the $q \rightarrow \infty$ limit. The effectively recallable number of bits in the low-E regime of the $q=3$ model is 2.37 times that of the $q=2$ case, but this ratio drops for larger q, since the $\log_2(q)$ gain from the increasing resolution per phasor looses out against the decreasing width of the low-E regime.

As for applications, it is at present somewhat unclear which biological realizations of q-state phasor nets may exist. Most likely candidates are interacting limit-cycle oscillators[7,17] with q-fold cycle symmetry, based on local feedback involving several chemical species or hidden neurons. The latter seem to occur in the olfactory bulb,[8] and as primitive motorpattern generators[18] in small animals. In any event, it is very likely even now that technical implementations of phasor models in optical hardware are feasible; in fact, two different kinds of designs for such machines have been proposed independently[9,10] while this paper was being written. Other architectures (backpropagation nets, etc.) could also be used with phasor encoding. On the theoretical side, further generalizations seem possible. For example, one can let the variables range over other value spaces, or use interactions that are more general functions of the distance in value space between the local variables.

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