Tunneling of squeezed states in asymmetrical double-well potentials

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Tunneling of initially squeezed wave packets in asymmetrical double-well potentials is studied; the fractal nature of the trajectory, previously found in the case of symmetric potentials, is reobtained.

In recent works it was shown that a squeezed initial wave packet in a one-dimensional symmetrical double-well potential gives rise to irregular tunneling behavior. The tunneling trajectory, i.e., the time dependence of the expectation value of the coordinate, was shown to have a chaotic character, and the result was that a fractal dimension could be assigned to the trajectory.[1,2]

The purpose of the present work is to extend such an analysis to moderately asymmetric double-well potentials. The importance of this kind of potential has to be seen also in connection with macroscopic quantum-tunneling phenomena observable, for instance, in Josephson junctions, superconducting quantum interference devices, etc.[3]

Following the approach adopted in Refs. 1 and 2, the time evolution of an initial wave packet $\Phi(x_i,0)$ can be written

$$\Phi(x,t)=\int_{-\infty}^{\infty} K(x,t;x_i,0)\Phi(x_i,0)dx_i \ , \tag{1}$$

where the kernel K can be expressed as an eigenfunction expansion[4]

$$K(x,t;x_i,0)=\sum_n \psi_n(x)\psi_n^*(x_i)\exp(-iE_n t/\hbar) \ , \tag{2}$$

since ψ_n and E_n are the eigenfunctions and the eigenvalues of the problem, respectively. The initial wave packet, for a particle of unit mass, is taken to be a squeezed state described as the Gaussian wave packet

$$\Phi(x_i,0)=\left[\frac{\omega_0 e^{2R}}{\pi\hbar}\right]^{1/4}\exp\left[-\frac{y^2e^{2R}}{2}\right] , \tag{3}$$

where R is a squeezing parameter, $y=(\omega_0/\hbar)^{1/2}(x\pm a)$ is a local coordinate centered at the potential minima, and ω_0 is the vibrational angular frequency in each separate well supposed, for simplicity, perfectly parabolic (see Fig. 1),

$$V(x)=\tfrac{1}{2}\omega_0^2(x\pm a)^2+V(\mp a) \ ,$$

where $V(-a)=0$ and $V(a)=-\sigma$.

Once $\Phi(x,t)$ is determined, the dynamical evolution of the system can be described by the expectation value of the coordinate

$$\langle x(t)\rangle=\int_{-\infty}^{\infty} |\Phi(x,t)|^2 x\,dx \ . \tag{4}$$

The eigenfunctions of Eq. (2) are expressed as linear combinations of wave functions of harmonic oscillators centered at $x=\pm a$, namely ψ_a and ψ_{-a}, respectively,

$$\begin{aligned}\psi_{4n}&=(\sin\varphi_n)\psi_{-a}+(\cos\varphi_n)\psi_a \ ,\\ \psi_{4n+1}&=(\cos\varphi_n)\psi_{-a}-(\sin\varphi_n)\psi_a \ ,\end{aligned} \tag{5}$$

where

$$\psi_{\pm a}=H_{2n}(y)\exp(-y^2/2) \ .$$

By using the assumption of moderate asymmetry $(0<\sigma\ll\hbar\omega_0)$ the mixing coefficients in Eq. (5) can be determined using a truncated matrix formulation by diagonalizing the energy matrix[5,6]

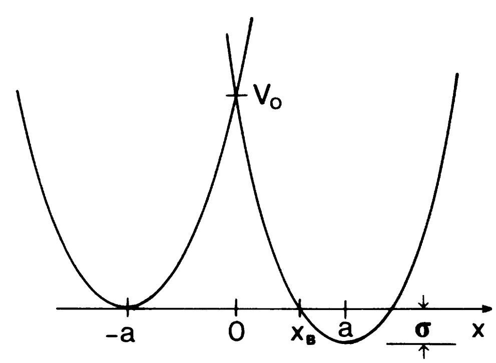

FIG. 1. Asymmetric double-well potential; x_B is the bounce coordinate when the zero is referred to $-a$.

$$\begin{array}{c} \psi_a \\ \psi_{-a} \end{array} \begin{vmatrix} -\sigma/2 & \gamma_n \\ \gamma_n & \sigma/2 \end{vmatrix} , \tag{6}$$

where γ_n are coupling coefficients due to tunneling through the barrier.[7] They enter the energy-shift expression since the eigenvalues are given by

$$\begin{aligned} E_{4n} &= E_n^{(0)} - (\sigma/2) - \delta_n , \\ E_{4n+1} &= E_n^{(0)} + (\sigma/2) + \delta_n , \end{aligned} \tag{7}$$

where $\delta_n = (\frac{1}{2})[(4\gamma_n^2 + \sigma^2)^{1/2} - \sigma]$ is the energy shift of each level and, for each energy matrix, the energy zero is referred to as

$$E_n^{(0)} = [2n + (\tfrac{1}{2})]\hbar\omega_0 - \sigma/2 .$$

By using this method it is easy to verify that $\tan\varphi_n = \delta_n/\gamma_n$ and $\tan(2\varphi_n) = 2\gamma_n/\sigma$ so that

$$\cos(2\varphi_n) = \frac{\sigma}{(\sigma^2 + 4\gamma_n^2)^{1/2}} = \frac{\sigma}{2\delta_n + \sigma} . \tag{8}$$

For $\sigma = 0$ the wave functions of Eq. (5) reduce to an even and an odd linear combination of harmonic-oscillator wave functions, respectively.

By substituting Eqs. (5) into Eq. (2), by substituting Eqs. (2) and (3) into Eq. (1), and by integrating (neglecting overlap terms), we obtain, assuming that the motion starts at the upper minimum,

$$\begin{aligned} \Phi(x,t) = \sum_n \tilde{A}_{2n}[\, & (\sin\varphi_n)\psi_{4n}\exp(-iE_{4n}t/\hbar) \\ & + (\cos\varphi_n)\psi_{4n+1}\exp(-iE_{4n+1}t/\hbar)] , \end{aligned} \tag{9}$$

where

$$\tilde{A}_{2n} = \frac{\sqrt{(2n)!}}{n!} \left[\frac{1}{\cosh R} \right]^{1/2} \left[-\frac{\tanh R}{2} \right]^n .$$

The expectation value of the position as a function of time, Eq. (4), results in

$$\frac{\langle x(t) \rangle}{\langle x(0) \rangle} = \sum_n (\tilde{A}_{2n})^2 \left[\cos^2(2\varphi_n) + \sin^2(2\varphi_n) \cos\left[\frac{\sigma + 2\delta_n}{\hbar} t \right] \right] , \tag{10}$$

which, for $\sigma = 0$, exactly reproduces the corresponding expression of Refs. 1 and 2, since $2\varphi_n = \pi/2$ and $2\delta_n = \hbar\Delta\omega_{4n}$ is the tunneling splitting relative to the nth level.

The energy shift δ_n can be evaluated according to the relation[8,9]

$$\delta_n = \frac{\exp(-2S_0/\hbar)}{(2\pi n')^2(\sigma + \delta_n)} , \tag{11}$$

where S_0 is the classical action, and the density of states n' is given by

$$n' = \frac{1}{\hbar\bar{\omega}_{2n}} .$$

The frequencies $\bar{\omega}_{2n}$, for small asymmetry, can be taken according to Refs. 1 and 2. By using this method we obtain

$$\frac{2\delta_n}{\hbar} = \left[\left[\frac{\sigma}{\hbar} \right]^2 + \Delta\omega_{4n}^2 \right]^{1/2} - \frac{\sigma}{\hbar} , \tag{12}$$

where $\Delta\omega_{4n} = \beta^{2n}/(2n)!$ and β is related to the energy barrier V_0 by the relation $\beta \simeq 4V_0/\hbar\omega_0$.

The trajectories of Eq. (10) have been evaluated for several values of the asymmetry parameter σ and by using the same values for the squeezing and potential parameters as those in Ref. 2. The cases $\sigma = 0.5$ and 2 are reported in Figs. 2 and 3, respectively; as can be seen, the trajectory amplitude in the asymmetric case is strongly reduced. The quantity σ is expressed in units of the ground-state tunneling splitting which, with the present choice of potential parameters, is of the order 10^{-10} times the vibrational energy in one of the two wells. A further increase in σ completely quenches the $n = 0$ contribution to the tunneling.

Note that these results hold either for a motion starting from the upper minimum, at $x = -a$, or from the lower at $x = a$.

Though a small asymmetry rapidly quenches the averaged tunneling trajectory, it has practically no effect on the length and fractal dimension of the trajectory. In Fig. 4 we report the log-log plot of the trajectory length as a function of the inverse sampling interval; by using a faster computer than that in Refs. 1 and 2 (Olivetti M24 plus 87 Basic Inline compiler) it was possible to evaluate

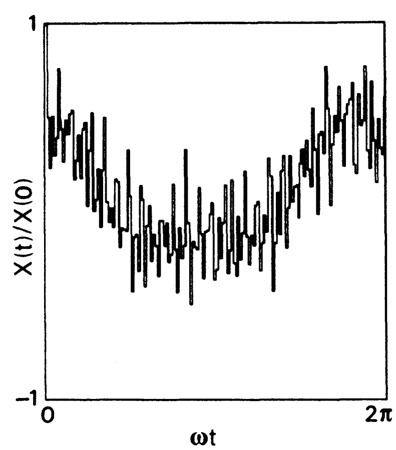

FIG. 2. Expectation value of the normalized position as a function of time for $R = 2$, $\beta = 48$, and $\sigma = 0.5$ in units of the ground-state tunneling splitting. The sampling time is $2\pi/157$.

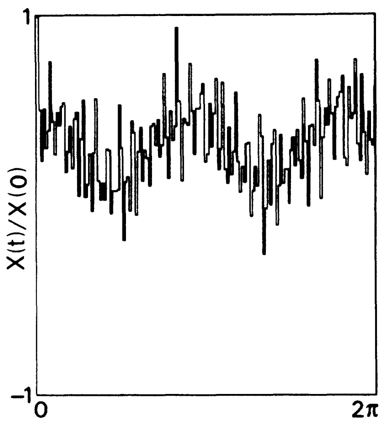

FIG. 3. Same as Fig. 2 but with $\sigma = 2$.

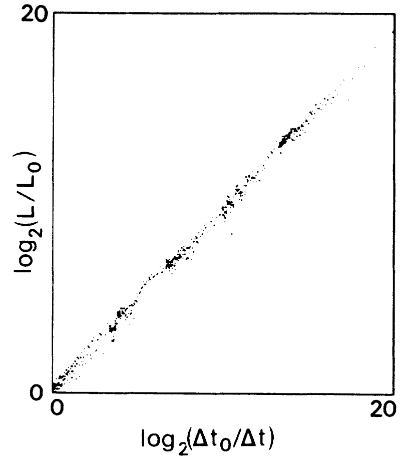

FIG. 4. Normalized length of the trajectory with the same parameters as those in Fig. 3. $\Delta t_0 = \pi/25$; L_0 is the length corresponding to Δt_0. The slope is fitted to 0.9574, giving a fractal dimension $d = 1.9574$.

the length in 500 points. The curve has a well-defined slope over several decades, even though structures are present which are due to the deterministic nature of the trajectory as discussed in Ref. 2. For any value of σ small enough for the above analytical treatment to be valid, no appreciable change is observed either in the slope or in the shape of Fig. 4. This can be understood by considering that for $n > 0$ only very small changes, in the time-dependent arguments of the cosine function in Eq. (10), are caused by the considered values of σ.

The strong reduction of the amplitude of $\langle x(t) \rangle$, even with small values of the asymmetry parameter σ, can be quantified as follows. Let us consider Eq. (10) for $R = 0$, that is, when we have only the $n = 0$ contribution. In this case it is easily seen that the maximum averaged amplitude $\langle x_M \rangle$ results in

$$\frac{\langle x_M \rangle}{\langle x(0) \rangle} = \frac{2(\hbar\Delta\omega_0)^2}{\sigma^2 + (\hbar\Delta\omega_0)^2} , \tag{13}$$

which, for $\sigma = 0$, rightly gives $\langle x_M \rangle / \langle x(0) \rangle = 2$, while for $\sigma = \hbar\Delta\omega_0$ we have $\langle x_M \rangle / \langle x(0) \rangle = 1$ as confirmed by computer calculation. For $\sigma \gg \hbar\Delta\omega_0$, Eq. (13) can be approximated as

$$\frac{\langle x_M \rangle}{\langle x(0) \rangle} = 2 \left[\frac{\hbar\bar{\omega}_0}{\pi\sigma} \right]^2 \exp(-2S_0/\hbar) , \tag{14}$$

being $\Delta\omega_0 = [\bar{\omega}_0 \exp(-S_0/\hbar)]/\pi$ and, according to Refs. 1 and 2,

$$\bar{\omega}_0 \simeq 2a\omega_0 \left[\frac{\pi\omega_0}{\hbar} \right]^{1/2} .$$

This means that the coherent oscillation between the two wells is quenched by the small asymmetry.

The present analysis has been performed without considering dissipative effects, whose inclusion is important especially from the point of view of applications. Dissipation, while strongly increasing the complexity of the analysis,[10] also influences, as mentioned above, the dynamical behavior of the system which, in a finite time, tends to decay to the lower minimum. This fact cannot be predicted in the undamped scheme of the present work, where an important aspect is represented by the fractal character of the tunneling trajectory.

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