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Quantization and phase-space methods for slowly varying optical fields in a dispersive nonlinear medium

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We consider the quantization of slowly varying optical fields in a dispersive nonlinear medium and the application of phase-space methods to the resulting quantum field equations. A pragmatic approach to the quantization of the electromagnetic field is adopted whereby we apply canonical quantization to the Hamiltonian expressed in terms of the slowly varying electric field envelope, all approximations (quasimonochromatic and paraxial) having been made at the classical level. This approach allows us to include material dispersion, diffraction, and nonlinearity. Using phase-space methods we then develop a *c*-number functional Fokker-Planck equation from which the quantum statistical properties of propagating optical fields can be deduced.

I. INTRODUCTION

The aim of this paper is to develop systematic methods for the calculation of quantum statistical properties of propagating optical fields.[1] We adopt a pragmatic approach to the quantization of classical field theories with respect to the approximations typically made when specific examples are investigated; these are often necessary so that even numerical progress can be made.

In classical nonlinear optics, beam propagation methods are well developed to deal with envelope field equations. These are sufficiently accurate to describe a wide variety of nonlinear, diffractive, and dispersive phenomena.[2–4] The envelope equations are derived by making use of paraxial (restriction on spatial frequency content) and quasimonochromatic (restriction on temporal frequency content) approximations. Lax *et al.*[5] have shown that under the paraxial approximation even the freely propagating field only obeys Maxwell's equation $\nabla\cdot\mathbf{E}=0$ in a perturbative sense consistent with the paraxial approximation, i.e., the field is not explicitly transverse under the paraxial approximation. We should also note that the use of such envelope equations is not generally valid for certain boundary value problems such as reflection and transmission at the interface between linear or nonlinear media.

The procedure used in developing a quantum theory of a free or interacting electromagnetic field (in the nonrelativistic domain) is to use the Coulomb gauge and demand that fields be transverse.[6] Canonical quantization then results in the introduction of the transverse δ function into the equal time commutation relations. Heisenberg equations for the field operators may then be derived. To address the problem of making approximations it is necessary to define norms for the operators and their spatial and temporal derivatives, and this requires knowledge of the states of the system, or equivalently the density operator. Since such a treatment has not been attempted to our knowledge, and would be difficult, to formulate, we propose a simpler approach: to make approximations at the classical level and then quantize. This is motivated by physical considerations such as the fact that classically paraxial fields are not explicitly transverse. The quasimonochromatic approximation (QMA) was discussed in this way by Graham and Haken[7] when developing a quantum propagation theory for thermal and two-level media, though they did not consider the paraxial approximation. Essentially this procedure allows us to derive envelope equations at the classical level and then quantize in canonical fashion. Operator equations of similar form to the classical equations of motion are then obtained, with some justification. We note that such a procedure has already been used implicitly by many authors, most recently by Garrison *et al.*[8] When the paraxial approximation is made it is not necessary to demand the quantized fields be transverse. Indeed Graham and Haken[7] showed that when the QMA is made postquantization, it is necessary to alter the full transverse field commutation relations. Our approach has the useful feature that transverse effects are included in the quantum theory at a realistic level of approximation. Furthermore, our free space propagator is the same as that derived by Yuen and Shapiro.[9]

In this paper we concentrate on situations where the medium may be treated as a classical nonlinear dielectric, and we include the effects of dispersion. Although dissipation is not considered explicitly, it may be added to the formulation straightforwardly. Indeed the effect of squeezed inputs may also be included.[10]

To calculate moments and correlations from a quantum theory it is often convenient to use phase-space methods. This involves converting the operator master equation into a *c*-number equation for a quasiprobability distribution over the phase space. Popular among these are the Glauber-Sudarshan,[11] Wigner, Q,[12] and positive P distribution of Drummond and Gardiner.[13] For the latter distribution, equivalent (in the distribution sense) Ito-type stochastic differential equations (SDE's) exist, and these have proved useful for the analysis of nonclassical states of the field in a variety of optical systems. Numerical simulations also then become possible by averaging over an ensemble of stochastic realizations.

For the field theories of interest here we extend the Drummond-Gardiner P distributions to spatially distributed systems. Thus we derive functional Fokker-Planck equations and associated stochastic partial differential equations (SPDE's). The techniques may be applied to a whole range of propagation problems. We note that Drummond, Carter, and co-workers[14] have recently derived SPDE's using a discretization procedure which circumvented the definition of functional P distributions. They applied their results to discuss the squeezing of solitary waves in optical fibers. Yurke and co-workers have investigated a related problem, and the squeezing of a train of mode locked pulses in parametric amplification using "phenomenological" operator equations.[15]

The remainder of this paper is organized as follows. In Secs. II and III we develop our classical theory and quantization procedure with reference to a nontrivial example including material dispersion, diffraction, and nonlinearity. In Sec. IV functional phase-space distributions are defined, and functional Fokker-Planck and SPDE's are derived and justified. Section V gives our summary and conclusions.

II. CLASSICAL THEORY

In this section we develop the classical theory of slowly varying optical fields in a dispersive nonlinear medium. Initially we consider only the linear problem and carefully outline the various approximations which are made in obtaining envelope equations.

A. Maxwell equations

Our staring point is the Maxwell equations for propagation in a homogeneous, lossless and isotropic dielectric[16]

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},\quad \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}, \tag{2.1}$$

$$\nabla\cdot\mathbf{D}=\nabla\cdot\mathbf{H}=0, \tag{2.2}$$

where $\mathbf{B}=\mu_0\mathbf{H}$, the medium being assumed nonmagnetic, and

$$\mathbf{D}(t)=\epsilon_0\left[\mathbf{E}(t)+\int_0^\infty d\tau\, R(\tau)\mathbf{E}(t-\tau)\right]. \tag{2.3}$$

Here $R(\tau)$ is the linear polarization response function, and we have ignored its tensor character for the sake of clarity in presentation.[16,17] For the moment we omit the explicit spatial dependence of the field. In the usual manner we obtain the wave equations

$$\nabla^2\mathbf{E}=\mu_0\frac{\partial^2\mathbf{D}}{\partial t^2}, \tag{2.4}$$

$$\nabla^2\mathbf{B}=-\mu_0\nabla\times\frac{\partial\mathbf{D}}{\partial t}, \tag{2.5}$$

where we have used Eqs. (2.2) which imply that $\nabla\cdot\mathbf{E}(t)=0$ also.

B. Quasimonochromatic approximation

So far the classical theory is exact. We now discuss these equations in the QMA by considering fields with mean carrier frequency ω, and write them as

$$\mathbf{E}(t)=\mathbf{E}^+(t)e^{-i\omega t}+\text{c.c.}, \tag{2.6a}$$

$$\mathbf{B}(t)=\mathbf{B}^+(t)e^{-i\omega t}+\text{c.c.}, \tag{2.6b}$$

$$\mathbf{D}(t)=\mathbf{D}^+(t)e^{-i\omega t}+\text{c.c.}, \tag{2.6c}$$

where the + superscript denotes a positive frequency component (vector) envelope. Substituting Eqs. (2.6) into (2.3)–(2.5) yields

$$\nabla^2\mathbf{E}^+=\mu_0\left[\frac{\partial^2\mathbf{D}^+}{\partial t^2}-2i\omega\frac{\partial\mathbf{D}^+}{\partial t}-\omega^2\mathbf{D}^+\right], \tag{2.7}$$

$$\nabla^2\mathbf{B}^+=-\mu_0\nabla\times\left[\frac{\partial\mathbf{D}^+}{\partial t}-i\omega\mathbf{D}^+\right], \tag{2.8}$$

$$\mathbf{D}^+(t)=\epsilon_0\left[\mathbf{E}^+(t)+\int_0^\infty d\tau\, R(\tau)\mathbf{E}^+(t-\tau)e^{i\omega\tau}\right]. \tag{2.9}$$

In the QMA the field envelopes are assumed to be slowly varying with respect to the $\exp(i\omega\tau)$ factor. We therefore perform a Taylor series expansion of $\mathbf{E}^+(t-\tau)$ in powers of τ. If we do this and define[16]

$$\epsilon(\omega)=\epsilon_0\left[1+\int_0^\infty d\tau\, R(\tau)e^{i\omega\tau}\right] \tag{2.10}$$

and retain only terms up to order τ^2, or alternatively truncate beyond $\partial^2\mathbf{E}^+/\partial t^2$, then $\mathbf{D}^+$ can be written as

$$\mathbf{D}^+(t)=\epsilon(\omega)\mathbf{E}^+(t)+i\left[\frac{\partial\epsilon}{\partial\omega}\right]\frac{\partial\mathbf{E}^+(t)}{\partial t} -\frac{1}{2}\left[\frac{\partial^2\epsilon}{\partial\omega^2}\right]\frac{\partial^2\mathbf{E}^+(t)}{\partial t^2}. \tag{2.11}$$

The expression (2.10) for the permittivity $\epsilon(\omega)$ ensures that its real and imaginary parts are related by a Kramers-Kronig transformation.[16] However, in the remainder of this paper we shall neglect the imaginary part of the permittivity which is responsible for absorption on the basis that we are considering a nonresonant interaction. By substituting Eq. (2.11) into (2.7)–(2.8) and using Eqs. (2.1) with (2.6), neglecting third- and higher-order derivatives, we obtain after some manipulation

$$\nabla^2\mathbf{E}^++k_0^2\mathbf{E}^++i\left[\frac{\partial k_0^2}{\partial\omega}\right]\frac{\partial\mathbf{E}^+}{\partial t}-\frac{1}{2}\left[\frac{\partial^2k_0^2}{\partial\omega^2}\right]\frac{\partial^2\mathbf{E}^+}{\partial t^2}=0, \tag{2.12}$$

with an identical equation for $\mathbf{B}^+$, and we have defined

$$k_0^2=\frac{\epsilon(\omega)\omega^2}{c^2\epsilon_0}. \tag{2.13}$$

In Eq. (2.12) all fields are defined at time t.

For our purposes it is adventageous to consider ω as a function of k_0, $\omega=\omega(k_0)$. Then using the chain rule for partial derivatives Eq. (2.12) can finally be written in the QMA as

$$\nabla^2\mathbf{E}^+ + k_0^2\mathbf{E}^+ + \frac{2ik_0}{\omega'}\frac{\partial \mathbf{E}^+}{\partial t} + \frac{1}{(\omega')^2}\left[\frac{k_0\omega''}{\omega'} - 1\right]\frac{\partial^2\mathbf{E}^+}{\partial t^2} = 0 , \quad (2.14)$$

where $\omega' = \partial\omega/\partial k_0$ is the group velocity, and the last term in the equation describes material dispersion, $\omega'' = \partial^2\omega/\partial k_0^2$ being the group velocity dispersion.[18]

The QMA imposes constraints on the temporal frequency content of the fields, which for $\mathbf{E}^+$ can be written

$$|\omega^2\mathbf{E}^+| \gg \left|\omega\frac{\partial\mathbf{E}^+}{\partial t}\right| \gg \left|\frac{\partial^2\mathbf{E}^+}{\partial t^2}\right| . \quad (2.15)$$

Since no restrictions have been made concerning the spatial field structure the condition $\nabla\cdot\mathbf{E}^+ = 0$ should still be enforced.

C. Paraxial approximation

We now consider linearly polarized, traveling-wave solutions which are propagating mainly in the z direction, and write the field envelopes as

$$\mathbf{E}^+(\mathbf{r},t) = \mathbf{x}E(\mathbf{r},t)e^{ik_0z} , \quad (2.16)$$

$$\mathbf{B}^+(\mathbf{r},t) = yB(\mathbf{r},t)e^{ik_0z} , \quad (2.17)$$

where $\mathbf{x}$ and $\mathbf{y}$ are unit vectors transverse to the direction of propagation; thus $\mathbf{E}^+$ and $\mathbf{B}^+$ are orthogonal to each other and to the direction of propagation. These field representations are consistent with Eqs. (2.1), (2.4), and (2.5), but not with Eqs. (2.2). Only for the case of plane waves (E and B constants) are the conditions $\nabla\cdot\mathbf{E} = \nabla\cdot\mathbf{H} = 0$ satisfied. However, Lax *et al.*[5] have shown that within the paraxial approximation the fields are transverse to lowest order in a perturbation expansion. The paraxial approximation applies subject to constraints on the spatial frequency content of the fields. In particular, Lax *et al.*[5] show that if the characteristic width w_0 of the optical field is much less than the corresponding Rayleigh range $k_0w_0^2$, then the fields may be treated as transverse to zeroth order in the perturbation parameter

$$f = \frac{1}{k_0w_0} \ll 1 . \quad (2.18)$$

The constraints on the spatial frequency content of the fields may be written

$$|k_0^2\mathbf{E}| \gg \left|k_0\frac{\partial\mathbf{E}}{\partial z}\right| \gg \left|\frac{\partial^2\mathbf{E}}{\partial z^2}\right| \quad (2.19)$$

along with identical equations for $\mathbf{B}$. We shall assume throughout that these conditions are satisfied. Then substituting Eq. (2.16) into (2.14) yields the following equation for the scalar electric field envelope $\mathbf{E}$

$$\left[\nabla_T^2 + \frac{\partial^2}{\partial z^2} + 2ik_0\left[\frac{\partial}{\partial z} + \frac{1}{\omega'}\frac{\partial}{\partial t}\right] + \frac{1}{(\omega')^2}\left[\frac{k_0\omega''}{\omega'} - 1\right]\frac{\partial^2}{\partial t^2}\right]\mathbf{E}(\mathbf{r},t) = 0 , \quad (2.20)$$

where ∇_T^2 is the transverse Laplacian describing beam diffraction $[\mathbf{r} \equiv (\mathbf{r}_T, z)]$.

As it stands, Eq. (2.20) is not suitable for applying nonrelativistic canonical quantization due to the presence of the second-order time derivative. However, it is possible to transform this equation so that this term is replaced by a second-order spatial (z) derivative. We outline the procedure here: (1) Transform Eq. (2.20) to a moving coordinate frame defined by $t' = t$, $\zeta = z - \omega't$, and drop the term $\partial^2E/\partial t'^2 - 2\omega'\partial^2E/\partial\zeta\partial t'$ in comparison to $(\omega')^2\partial^2E/\partial\zeta^2$ since the group velocity $\omega' \gg 1$. (2) Transform back to the original coordinates via $t = t'$, $z = \zeta + \omega't'$. The resulting equation for E is

$$\left[\nabla_T^2 + 2ik_0\left[\frac{\partial}{\partial z} + \frac{1}{\omega'}\frac{\partial}{\partial t}\right] + \frac{k_0\omega''}{\omega'}\frac{\partial^2}{\partial z^2}\right]E(\mathbf{r},t) = 0 . \quad (2.21)$$

The basis of the derivation of Eq. (2.21) is that the quantity

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{(\omega')^2}\frac{\partial^2}{\partial t^2}\right]E(\mathbf{r},t)$$

is of order 0. Thus for a monochromatic field $\partial^2E/\partial t^2 = 0$, and $\partial^2E/\partial z^2$ should correspondingly be set equal to 0 also. Equations (2.20) and (2.21) therefore reduce to the same equation for a monochromatic field. Equation (2.21) is our basic equation describing propagation in a linear dielectric in the QMA and paraxial approximation.

D. Nonlinearity

To include a nonlinearity in our model a nonlinear displacement vector term $\mathbf{D}^{\mathrm{NL}}$ must be added to Eq. (2.3). For an isotropic Kerr medium with instantaneous response this term may be written[17]

$$\mathbf{D}^{\mathrm{NL}} = \epsilon_0\chi^{(3)}(\mathbf{E}\cdot\mathbf{E})\mathbf{E} , \quad (2.22)$$

where the third-order susceptibility $\chi^{(3)}$ is real and we have ignored its tensor character for simplicity. By following though the analysis of the previous subsections and neglecting all the time derivatives of $\mathbf{D}^{\mathrm{NL}}$ we obtain the following nonlinear envelope equation for E:

$$\left[\nabla_T^2 + 2ik_0\left[\frac{\partial}{\partial z} + \frac{1}{\omega'}\frac{\partial}{\partial t}\right] + \frac{k_0\omega''}{\omega'}\frac{\partial^2}{\partial z^2}\right]E(\mathbf{r},t) = -3\left[\frac{\omega}{c}\right]^2\chi^{(3)}|E|^2E . \quad (2.23)$$

This result is our basic equation describing propagation of the scalar electric field envelope in a dispersive Kerr-type nonlinear medium.

A relation between E and B can be obtained by putting Eqs. (2.16)–(2.17) into (2.6) and then into Eqs. (2.1):

$$B(\mathbf{r},t)=\left[\frac{\epsilon(\omega)}{c^2\epsilon_0}\right]^{1/2}E(\mathbf{r},t)\ . \tag{2.24}$$

Here we have used Eq. (2.11) and have neglected all nonlinear and derivative terms. Under the same conditions the cycle-averaged energy of the optical field is easily calculated as[16]

$$\begin{aligned}U(t)&=\tfrac{1}{2}\int d^3\mathbf{r}\left\langle\frac{d}{d\omega}[\omega\epsilon(\omega)]\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}\right\rangle\\&=\frac{2\epsilon(\omega)\omega}{\omega' k_0}\int d^3\mathbf{r}\,|E(\mathbf{r},t)|^2\ .\end{aligned}\tag{2.25}$$

We shall adopt Eqs. (2.23)–(2.25) as the appropriate classical theory of a traveling-wave optical field. Thus all assumptions concerning the quasioptic nature of the theory are being made at the classical level. In particular, in contrast to the exact theory we do not impose the constraint $\nabla\cdot\mathbf{E}=0$, this being consistent with the paraxial approximation.

E. Hamiltonian formulation

By multiplying through Eq. (2.23) by a constant m which has units of mass and rearranging we obtain

$$\begin{aligned}i\kappa\frac{\partial E}{\partial t}&=-\frac{\kappa^2}{2m}\nabla_T^2E-i\kappa\omega'\frac{\partial E}{\partial z}\\&\quad-\frac{\kappa\omega''}{2}\frac{\partial^2E}{\partial z^2}-\frac{3\omega'\kappa}{2k_0}\left[\frac{\omega}{c}\right]^2\chi^{(3)}|E|^2E\ ,\end{aligned}\tag{2.26}$$

where $\kappa=m\omega'/k_0$ is a constant with the dimensions of action. Since m is arbitrary we can choose it such that $\kappa=\hbar$. We further define scaled fields Ψ and $\Psi^\dagger$

$$E=i\left[\frac{\hbar\omega' k_0}{2\epsilon(\omega)}\right]^{1/2}\Psi,\quad E^*=-i\left[\frac{\hbar\omega' k_0}{2\epsilon(\omega)}\right]^{1/2}\Psi^\dagger \tag{2.27}$$

for which the field equations can be written

$$\begin{aligned}i\hbar\frac{\partial\Psi}{\partial t}&=-\frac{\hbar^2}{2m}\nabla_T^2\Psi-i\hbar\omega'\frac{\partial\Psi}{\partial z}\\&\quad-\frac{\hbar\omega''}{2}\frac{\partial^2\Psi}{\partial z^2}-\hbar\sigma\Psi^\dagger\Psi\Psi\ ,\end{aligned}\tag{2.28}$$

$$\begin{aligned}-i\hbar\frac{\partial\Psi^\dagger}{\partial t}&=-\frac{\hbar^2}{2m}\nabla_T^2\Psi^\dagger+i\hbar\omega'\frac{\partial\Psi^\dagger}{\partial z}\\&\quad-\frac{\hbar\omega''}{2}\frac{\partial^2\Psi^\dagger}{\partial z^2}-\hbar\sigma\Psi^\dagger\Psi^\dagger\Psi\ ,\end{aligned}\tag{2.29}$$

and we have defined

$$\sigma=\frac{3\hbar\chi^{(3)}\omega^2(\omega')^2}{4\epsilon(\omega)c^2}\ . \tag{2.30}$$

The field energy is then given by

$$U(t)=\hbar\omega\int d^3\mathbf{r}\Psi^\dagger\Psi\ . \tag{2.31}$$

Equation (2.28) has the form of a nonlinear Schrödinger equation for the field Ψ. It is important to note, however, that although we have chosen to introduce Planck's constant $\hbar$ into the formalism at this stage the theory is still classical: Eqs. (2.28) and (2.31) are equivalent to (2.23) and (2.25) which do not involve $\hbar$. Our scaling has been adopted for later convenience when we apply canonical quantization (Sec. III).

To write the field equations in Hamiltonian form it is necessary to introduce a new field Π related to $\Psi^\dagger$:

$$\Pi=i\hbar\Psi^\dagger\ . \tag{2.32}$$

The Hamiltonian can then be written as

$$H(t)=\int d^3\mathbf{r}\mathcal{H}(\mathbf{r},t) \tag{2.33}$$

where the Hamiltonian density is given by

$$\begin{aligned}\mathcal{H}(\mathbf{r},t)&=-\frac{i\hbar}{2m}\nabla_T\Pi\cdot\nabla_T\Psi-\frac{\omega'}{2}\left[\Pi\frac{\partial\Psi}{\partial z}-\Psi\frac{\partial\Pi}{\partial z}\right]\\&\quad-\frac{i\omega''}{2}\left[\frac{\partial\Pi}{\partial z}\right]\left[\frac{\partial\Psi}{\partial z}\right]+\frac{\sigma}{2\hbar}\Pi\Pi\Psi\Psi\ .\end{aligned}\tag{2.34}$$

One can easily verify that the Hamilton equations

$$\frac{\partial\Psi}{\partial t}=\frac{\delta H}{\delta\Pi},\quad\frac{\partial\Pi}{\partial t}=-\frac{\delta H}{\delta\Psi} \tag{2.35}$$

are identical to the field equations (2.28) and (2.29), where δ means a functional derivative.[19] Thus Π is identified as the momentum canonically conjugate to Ψ.

It is important to note that the energy $U(t)$ and the Hamiltonian $H(t)$ are distinct objects. $U(t)$ given by Eq. (2.25), or (2.31), is an expression for the field energy, and is that generally used in classical nonlinear electromagnetics. In contrast, the Hamiltonian $H(t)$ does not in general equal $U(t)$ and it contains in it the dynamics of the electromagnetic system, including the nonlinearity. Hillery and Mlodinow,[20] and Abram[21] have critically examined the model Hamiltonian approach to field quantization, and pointed out the conceptual problems which arise when one attempts to develop an exact quantized theory starting from the full Maxwell equations. Even after such a procedure has been carried out successfully it is necessary to make the quasioptic approximations to address the problems of interest here. In this paper we avoid such a route by making quasioptic approximations at the classical level. Our justification is the belief that the two methods must result in essentially the same final quantum theories, although further work is required to fully clarify this.

For our purposes it is convenient to consider a system of volume V, to which periodic boundary conditions are applied. The volume may be taken to infinity later. In this case one can prove using Noether's theorem[22] that the field energy U and the Hamiltonian H are constants of the motion, and both have energy units.

III. QUANTUM THEORY

A. Commutation relations

To quantize the classical field theory given in the previous section we replace the classical fields Ψ and Π by field operators which we postulate to obey the equal-time commutation relations[19]

$$[\Psi(\mathbf{r},t),\Psi(\mathbf{r}',t)]=0 \; , \tag{3.1a}$$

$$[\Pi(\mathbf{r},t),\Pi(\mathbf{r}',t)]=0 \; , \tag{3.1b}$$

$$[\Psi(\mathbf{r},t),\Pi(\mathbf{r}',t)]=i\hbar\delta(\mathbf{r}-\mathbf{r}') \; . \tag{3.1c}$$

Note that since we do not require $\nabla\cdot\mathbf{E}=0$ we use a standard δ function instead of the transverse δ function required in the exact quantized theory. By using the definition of Π in Eq. (2.32) we obtain from (3.1c)

$$[\Psi(\mathbf{r},t),\Psi^\dagger(\mathbf{r}',t)]=\delta(\mathbf{r}-\mathbf{r}') \; . \tag{3.2}$$

The Hamiltonian and energy operators of the quantized theory are

$$H(t)=\int d^3\mathbf{r}\,\mathcal{H}(\mathbf{r},t) \; , \tag{3.3}$$

$$U(t)=\hbar\omega\int d^3\mathbf{r}\,\Psi^\dagger\Psi \; , \tag{3.4}$$

where the Hamiltonian density is written in terms of Ψ and $\Psi^\dagger$ as

$$\mathcal{H}(\mathbf{r},t)=\frac{\hbar^2}{2m}\nabla_T\Psi^\dagger\cdot\nabla_T\Psi-\frac{i\hbar\omega'}{2}\left[\Psi^\dagger\frac{\partial\Psi}{\partial z}-\Psi\frac{\partial\Psi^\dagger}{\partial z}\right]+\frac{\hbar\omega''}{2}\left[\frac{\partial\Psi^\dagger}{\partial z}\right]\left[\frac{\partial\Psi}{\partial z}\right]-\frac{\sigma\hbar}{2}\Psi^\dagger\Psi^\dagger\Psi\Psi \; . \tag{3.5}$$

Here we have used normal ordering of the operators in the nonlinear term of Eq. (3.5) and in the energy operator (3.4).[20] The operator ordering of the remaining terms of the Hamiltonian density (3.5) can be shown to be irrelevant.

B. Heisenberg equations

In the Heisenberg picture, for a general operator O we have the equation of motion[19]

$$i\hbar\frac{\partial O(\mathbf{r},t)}{\partial t}=[O(\mathbf{r},t),H(t)]=\int d^3\mathbf{r}'[O(\mathbf{r},t),\mathcal{H}(\mathbf{r}',t)] \; . \tag{3.6}$$

By substituting $O=\Psi,\Psi^\dagger$ and using the commutation relations (3.1) and (3.2) we obtain the field operator equations

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla_T^2\Psi-i\hbar\omega'\frac{\partial\Psi}{\partial z}-\frac{\hbar\omega''}{2}\frac{\partial^2\Psi}{\partial z^2}-\hbar\sigma\Psi^\dagger\Psi\Psi \; , \tag{3.7}$$

$$-i\hbar\frac{\partial\Psi^\dagger}{\partial t}=-\frac{\hbar^2}{2m}\nabla_T^2\Psi^\dagger+i\hbar\omega'\frac{\partial\Psi^\dagger}{\partial z}-\frac{\hbar\omega''}{2}\frac{\partial^2\Psi^\dagger}{\partial z^2}-\hbar\sigma\Psi^\dagger\Psi^\dagger\Psi \; . \tag{3.8}$$

Thus the classical equations are reproduced in operator form in the quantum theory. This implies that in the absence of nonlinearity ($\sigma=0$) the Green's function of the classical scalar problem is the propagator for the quantum theory. This result is well known and has been used by Hopf and Meystre,[1] and Yuen and Shapiro[9] when dispersion is also absent ($\omega''=0$). It can be easily shown that even in the quantum theory $U(t)$ and $H(t)$ are constants of the motion. Furthermore, the field equations (3.7) and (3.8) preserve the commutation relations (3.1) and (3.2), which is a requirement of a consistent quantum theory.

The operator solutions may be expressed in terms of boson operators via a plane-wave expansion

$$\Psi(\mathbf{r},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}a_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} \; , \tag{3.9}$$

$$\Psi^\dagger(\mathbf{r},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}a_{\mathbf{k}}^\dagger(t)e^{-i\mathbf{k}\cdot\mathbf{r}} \; , \tag{3.10}$$

where the boson operators $a_{\mathbf{k}},a_{\mathbf{k}}^\dagger$ satisfy the usual nonzero commutation relation

$$[a_{\mathbf{k}}(t),a_{\mathbf{k}'}^\dagger(t)]=\delta_{\mathbf{k},\mathbf{k}'} \tag{3.11}$$

consistent with (3.2). The quantization volume V on which periodic boundary conditions are applied may later be taken to infinity. The wave vector $\mathbf{k}$ is defined relative to the carrier wavevector $\mathbf{k}_0=k_0\mathbf{z}$, i.e., $\mathbf{k}=\mathbf{k}_T+\Delta k_z\mathbf{z}$, with $\Delta k_z=k_z-k_0$ and $\mathbf{k}_0+\mathbf{k}=\mathbf{k}_T+\mathbf{z}k_z$.

Some comments on the representations (3.9) and (3.10) are in order. If the range of summation is unrestricted the equal time commutation relations (3.2) and (3.11) are immediately satisfied through Fourier transformation. However, since we assume the field to be paraxial and quasimonochromatic in nature, it is necessary to restrict attention to states of the field which are paraxial ($k_0^2\gg k_T^2$), and occupy some bandwidth small compared with the carrier frequency ω. This excludes for example, evanescent wave states which decay exponentially in the z direction.[3] Effectively the domain on which the operators act is restricted to some finite range of $\mathbf{k}$ vectors, without altering the validity of the physical theory. Restricting the range does however alter the commutation relation (3.2), which becomes proportional to the range of $\mathbf{k}$. This notion has been used widely in the literature,[23] usually in relation to the detection of a finite bandwidth of field states. It is worth stressing that altering (3.2) is consistent only because attention is restricted to a finite range of field states.

IV. PHASE-SPACE METHODS

In this section we develop c-number phase space methods which enable the quantum statistical properties of propagating optical fields to be investigated. Our analysis represents a field-theoretic extension of the P distribution methods of Drummond and Gardiner.[13]

A. Review

Given the operator field equations, we are faced with the task of calculating observable correlations and moments of the field. Graham and Haken[7] used phase-space methods based on functional Fokker-Planck and Langevin equations to discuss the quantum statistics of propagating fields in thermal and two-level media. More recently the Drummond-Gardiner P distribution[13] has been widely used to discuss the quantum statistics of few-mode optical systems. The P distributions are associated with a phase space integration measure $d\mu$, and the equation of motion for the P distribution specifies the dynamics. Often the equation of motion may be written in Fokker-Planck form, for example in the case of a single-field mode

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\alpha^{\mu}}A^{\mu}P+\frac{1}{2}\frac{\partial^2}{\partial\alpha^{\mu}\partial\alpha^{\nu}}D^{\mu\nu}P \ , \tag{4.1}$$

where A and D are the drift vector and diffusion matrix, respectively, and $\alpha,\alpha^{\dagger}$ are the complex random variables associated with the mode annihilation and creation operators. Indices μ and ν indicate dagger and non-dagger variables, and the summation convention is employed. In the case of the positive P distribution, which always exists for a physical density operator, the diffusion matrix is explicitly positive semidefinite ($\mathbf{D}=\mathbf{B}\mathbf{B}^T$) when expressed in real variables, and $\alpha,\alpha^{\dagger}$ are then independent random variables which are complex conjugate in the mean. This establishes an equivalence (in the distribution sense[24]) between (4.1) and the set of Ito SDE's

$$\frac{d\alpha^{\mu}}{dt}=A^{\mu}+B^{\mu\nu}\xi^{\nu}(t) \ , \tag{4.2}$$

where the zero mean, white-noise sources $\xi^{\mu}(t)$ obey

$$\langle\xi(t)\xi(t')\rangle=\langle\xi^{\dagger}(t)\xi^{\dagger}(t')\rangle=\delta(t-t') \ ,$$
$$\langle\xi(t)\xi^{\dagger}(t')\rangle=0 \ . \tag{4.3}$$

Often it is more efficient to work with the SDE's directly, since approximation methods are well developed, and this has commonly been the route taken in quantum optical applications.[25] Moreover for nonlinear problems where analytical progress is limited, numerical simulations of (4.2) are possible.[26]

Extending these ideas to spatially distributed systems, it is at once clear that the equation of motion for a suitably defined P distribution will be functional in nature. One may expect that a functional Fokker-Planck equation for the functional positive P distribution (with a suitably defined functional measure) will have a positive semidefinite diffusion matrix and be equivalent to a set of Ito SPDE's. For the example of guided wave propagation in an optical fiber, Drummond, Carter, and coworkers[14] have derived such SPDE's, by taking the continuum limit of multimode Ito SDE's. This circumvents the problem of defining functional P distributions as the continuum limit is taken at a later stage in the formulation. However, from our viewpoint, with the coherent interactions expressed in terms of a Hamiltonian density, it is most natural to derive equations for functional P distributions, and from these deduce the SPDE's. As we shall show, this method has the advantage that the effects of diffraction in addition to dispersion, may be naturally incorporated into the formulation, at a realistic level of approximation.

The quantum statistical properties of propagating fields in nonlinear media are likely to be very complex, however, when quantum fluctuations act as a small perturbation, linearization of the SPDE's may enable some degree of analytical progress to be made. We have, however, succeeded in finding some exact moments for the field in the case of self-phase modulation of optical pulses, and these will be published elsewhere.[27] In cases where a linearization procedure is not used numerical simulation of the SPDE's may be the most efficient means of investigation.

B. Heuristic deviation of a stochastic nonlinear Schrödinger equation

To introduce our methods and also serve as a useful and nontrivial example we use the quantized Hamiltonian density derived in Sec. III, which describes optical propagation in a lossless, dispersive nonlinear medium to define and derive a functional equation for the functional complex P distribution, and from this infer associated SPDE's. This method is heuristic in the same sense as in the case of the usual complex P distribution (when it exists), as only the positive P distribution implies the existence of associated SDE's. However, the naive and faster derivation based on the complex P or Glauber-Sudarshan P distributions, leads to the same set of SDE's suitably interpreted.[13] In Sec. IV C we present a general justification of the heuristically derived SPDE's based on the functional positive P distribution and its associated measure.

We introduce the multimode coherent states $|\{\alpha\}\rangle$, defined by

$$a_{\mathbf{k}}|\{\alpha\}\rangle=\alpha_{\mathbf{k}}|\{\alpha\}\rangle \quad \text{for all } \mathbf{k} \ , \tag{4.4}$$

where $\mathbf{k}$ is defined relative to $\mathbf{k}_0$, the plane-wave wave-vector component. Thus the field operator Ψ given by (3.14) satisfies

$$\Psi(\mathbf{r})|\{\alpha\}\rangle=\psi(\mathbf{r})|\{\alpha\}\rangle \ , \tag{4.5}$$

where

$$\psi(\mathbf{r})\equiv\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\alpha_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} \ . \tag{4.6}$$

Following Drummond and Gardiner[13] we expand the density operator for the system $\rho(t)$ in terms of a functional P distribution $P(\{\boldsymbol{\alpha}\})$ and associated measure $\mathcal{D}\mu(\{\boldsymbol{\alpha}\})$, using the off-diagonal coherent state projector $\Lambda(\{\boldsymbol{\alpha}\})$ (with $\{\boldsymbol{\alpha}\}\equiv\{\alpha,\alpha^{\dagger}\}$)

$$\rho(t)=\int P(\{\boldsymbol{\alpha}\})\Lambda(\{\boldsymbol{\alpha}\})\mathcal{D}\mu(\{\boldsymbol{\alpha}\}) \ , \tag{4.7}$$

where

$$\Lambda(\{\boldsymbol{\alpha}\})\equiv\frac{|\{\alpha\}\rangle\langle\{\alpha^{\dagger}\}^*|}{\langle\{\alpha^{\dagger}\}^*|\{\alpha\}\rangle} \ . \tag{4.8}$$

In terms of a Hamiltonian density $\mathcal{H}(\mathbf{r})$ the Liouville–von Neumann equation reads

$$i\hbar\frac{\partial\rho}{\partial t}=\int d^3\mathbf{r}[\mathcal{H}(\mathbf{r}),\rho(t)] \ , \tag{4.9}$$

and substituting the generalized P expansion (4.7) for $\rho(t)$ we have

$$i\hbar\int\mathcal{D}\mu\Lambda\frac{\partial P}{\partial t}=\int\mathcal{D}\mu\int d^3\mathbf{r}\,P[\mathcal{H}(\mathbf{r}),\Lambda] \ . \tag{4.10}$$

Now with the Hamiltonian density (3.5) expressed in terms of the field operators Ψ and $\Psi^\dagger$, an equation of motion for a P distribution may be derived as follows. (1) Evaluate the commutator on the right-hand side of (4.10) using the mode expansions for Ψ and $\Psi^\dagger$ and the identities[13]

$$a_{\mathbf{k}}\Lambda=\alpha_{\mathbf{k}}\Lambda\ , \tag{4.11a}$$

$$a^\dagger_{\mathbf{k}}\Lambda=\left[\alpha^\dagger_{\mathbf{k}}+\frac{\partial}{\partial\alpha_{\mathbf{k}}}\right]\Lambda\ , \tag{4.11b}$$

$$\Lambda a^\dagger_{\mathbf{k}}=\alpha^\dagger_{\mathbf{k}}\Lambda\ , \tag{4.11c}$$

$$\Lambda a_{\mathbf{k}}=\left[\alpha_{\mathbf{k}}+\frac{\partial}{\partial\alpha^\dagger_{\mathbf{k}}}\right]\Lambda\ . \tag{4.11d}$$

(2) For the case of the functional complex P distribution define the measure

$$\mathcal{D}\mu_c\equiv\prod_{\mathbf{k}}d\alpha_{\mathbf{k}}d\alpha^\dagger_{\mathbf{k}}\ . \tag{4.12}$$

Now integrate by parts over independent contours for each of the independent variables. Then an equation of motion is found by equating the functional integrands on both sides. (3) Where necessary, integrate by parts over configuration space so that the equation of motion can be expressed entirely in terms of the random fields $\psi(\mathbf{r})$ defined by Eq. (4.6), $\psi^\dagger(\mathbf{r})$ which is found from (4.6) by the replacements $\psi\to\psi^\dagger$, $a_{\mathbf{k}}\to a^\dagger_{\mathbf{k}}$, and $i\to -i$, and the functional derivatives defined by

$$\frac{\delta}{\delta\psi(\mathbf{r})}\equiv\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{\partial}{\partial\alpha_{\mathbf{k}}} \tag{4.13}$$

and

$$\frac{\delta}{\delta\psi^\dagger(\mathbf{r})}\equiv\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\partial}{\partial\alpha^\dagger_{\mathbf{k}}}\ . \tag{4.14}$$

The order of the configuration and phase-space integrations may be reversed. For the case of the positive P it is more convenient to perform the configuration space integrations before the functional integrals (see next section).

For the Hamiltonian density (3.5), this leads to the functional Fokker-Planck equation

$$\frac{\partial P[\psi;t]}{\partial t}=\int d^3\mathbf{r}\left[-\frac{\delta}{\delta\psi(\mathbf{r})}\left[-\omega'\frac{\partial\psi(\mathbf{r})}{\partial z}+\frac{i\hbar}{2m}\nabla_T^2\psi(\mathbf{r})+\frac{1}{2}i\omega''\frac{\partial^2\psi(\mathbf{r})}{\partial z^2}+i\sigma\psi^\dagger(\mathbf{r})\psi^2(\mathbf{r})\right]+\frac{1}{2}\frac{\delta^2}{\delta\psi^2(\mathbf{r})}[i\sigma\psi^2(\mathbf{r})]+\text{c.c.}\right]P[\psi;t]\ , \tag{4.15}$$

where c.c. indicates similar terms with $\psi\to\psi^\dagger$, $i\to -i$, and $\delta/\delta\psi,\delta^2/\delta\psi^2\to\delta/\delta\psi^\dagger,\delta^2/\delta\psi^{\dagger 2}$, respectively. We may naively convert this into equivalent stochastic partial differential equations by means of the correspondence

$$\frac{\partial P[\psi;t]}{\partial t}=\int d^3\mathbf{r}\left[-\frac{\delta}{\delta\psi^\mu}[A^\mu]+\frac{1}{2}\frac{\delta^2}{\delta\psi^\mu\delta\psi^\nu}[B^{\mu\sigma}B^{\nu\sigma}]\right]P[\psi;t]$$
$$\to\frac{\partial\psi^\mu}{\partial t}=A^\mu+B^{\mu\nu}\xi^\nu(\mathbf{r},t)\ , \tag{4.16}$$

where, as before, superscripts indicate dagger or nondagger indices, the summation convention for repeated indices is employed, and ξ is a vector of white-noise fields defined by the nonzero correlations

$$\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle=\langle\xi^\dagger(\mathbf{r},t)\xi^\dagger(\mathbf{r}',t')\rangle=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\ . \tag{4.17}$$

This gives

$$\left[\frac{\partial}{\partial t}+\omega'\frac{\partial}{\partial z}\right]\psi(\mathbf{r},t)=\left[\frac{i\hbar}{2m}\nabla_T^2+i\sigma\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t)+\frac{i\omega''}{2}\frac{\partial^2}{\partial z^2}+\sqrt{i\sigma}\xi(\mathbf{r},t)\right]\psi(\mathbf{r},t)\ , \tag{4.18}$$

with a similar equation for $\psi^\dagger$ found by the replacements $\psi\to\psi^\dagger$, $i\to -i$, and $\xi\to\xi^\dagger$.

Equation (4.18) and the corresponding equation for $\psi^\dagger$ are nonlinear multiplicative stochastic partial differential equations and form the basis of our quantum statistical theory. Standard stochastic methods appropriate to spatially distributed systems may be used to calculate the linearized quantum fluctuations about any deterministic solution.[13,14] By definition of the P distributions, Eq. (4.18) allows only normally ordered moments of field operators to be calculated. In the deterministic limit that the noise $\xi,\xi^\dagger\to 0$, Eq. (4.18) reduces to one form of the nonlinear Schrödinger equation, with an additional term represented by the transverse Laplacian which accounts for diffraction.

C. Functional positive P distribution

Here we attempt to justify the correspondence between the functional Fokker Planck equation and the set of Ito SPDE's given in Eq. (4.16). Our treatment is merely a field-theoretic extension of the arguments made by Drummond and Gardiner.[13] In their treatment there exists a rigorous mathematical correspondence, in the distribution sense, between the Markovian solution of a Fokker-Planck equation and the Markov process defined by the solution of associated Ito SDE's.[24] Although the solutions of SPDE's have been shown to exhibit the Markovian property,[28] to our knowledge the precise relationship between functional equations and associated SPDE's remains to be shown. Nevertheless we assume that an extension of the usual arguments of Drummond and Gardiner will be sufficient to establish the correspondence in the functional case considered here.

We assume that from an underlying field theory which may include dissipation, a Markovian master equation may be derived, and in terms of a functional P distribution may be expressed in the general form

$$\frac{\partial\rho}{\partial t}=\int \mathcal{D}\mu\Lambda\frac{\partial P}{\partial t}=\int\mathcal{D}\mu\int dz\, P\left[\frac{1}{\sqrt{L}}A^{\mu}e^{-i\eta(\mu)kz}\frac{\partial}{\partial\alpha_k^{\mu}}+\frac{1}{2L}D^{\mu\nu}e^{-i[\eta(\mu)k+\eta(\nu)k']z}\frac{\partial^2}{\partial\alpha_k^{\mu}\partial\alpha_{k'}^{\nu}}\right]\Lambda\ , \tag{4.19}$$

where

$$\eta(\mu)=\begin{cases}1 & \text{if } \mu \text{ is a dagger variable}\\ -1 & \text{if } \mu \text{ is a non-dagger variable,}\end{cases} \tag{4.20}$$

and summation over μ, ν, k and k' is implied. The form of Eq. (3.19) may generally be found by using mode expansions for the field operators and where necessary performing integration by parts over configuration space, which is taken to be one dimensional (z) for simplicity here: L is the quantization length.

Since by construction $\mathbf{D}$ is a symmetric matrix, it may be written in the form $\mathbf{D}=\mathbf{B}\mathbf{B}^T$, where the complex matrix $\mathbf{B}$ is unique up to orthogonal transformations. We write all complex variables in terms of their real and imaginary parts (indicated by subscripts x and y, respectively), that is

$$\alpha_k^{\mu}=\alpha_{k,x}^{\mu}+i\alpha_{k,y}^{\mu}\ , \tag{4.21}$$

$$\mathbf{A}^{\mu}=\mathbf{A}_x^{\mu}+i\,\mathbf{A}_y^{\mu}\ , \tag{4.22}$$

$$\mathbf{B}=\mathbf{B}_x+i\mathbf{B}_y\ , \tag{4.23}$$

and define the (real) functional positive P measure

$$\mathcal{D}\mu_+\equiv\prod_{\mu,k}d\alpha_{k,x}^{\mu}d\alpha_{k,y}^{\mu}\ . \tag{4.24}$$

The projection operator Λ is analytic in the complex variables $\{\alpha,\alpha^{\dagger}\}$, so that

$$\frac{\partial}{\partial\alpha_k^{\mu}}\Lambda=\frac{\partial}{\partial\alpha_{k,x}^{\mu}}\Lambda=-i\frac{\partial}{\partial\alpha_{k,y}^{\mu}}\Lambda\ . \tag{4.25}$$

We now substitute expressions (4.21)–(4.24) into Eq. (4.19), and choose the derivatives using the analyticity property (4.25) in such a way as to make the term in parentheses completely real. We find

$$\frac{\partial\rho}{\partial t}=\int\mathcal{D}\mu_+\Lambda\frac{\partial P}{\partial t}=\int\mathcal{D}\mu_+\int dz\,P[(\Delta_{k,x}^{\mu}A_x^{\mu}+\Delta_{k,y}^{\mu}A_y^{\mu})+\tfrac{1}{2}(B_x^{\mu\sigma}B_x^{\nu\sigma}\Delta_{k,x}^{\mu}\Delta_{k',x}^{\nu}+B_x^{\mu\sigma}B_y^{\nu\sigma}\Delta_{k,x}^{\mu}\Delta_{k',y}^{\nu}$$
$$+B_y^{\mu\sigma}B_x^{\nu\sigma}\Delta_{k,y}^{\mu}\Delta_{k',x}^{\nu}+B_y^{\mu\sigma}B_y^{\nu\sigma}\Delta_{k,y}^{\mu}\Delta_{k',y}^{\nu})]\Lambda\ , \tag{4.26}$$

where we have defined the differential operators

$$\Delta_{k,x}^{\mu}\equiv\frac{1}{\sqrt{L}}\left\{\cos[\eta(\mu)kz]\frac{\partial}{\partial\alpha_{k,x}^{\mu}}-\sin[\eta(\mu)kz]\frac{\partial}{\partial\alpha_{k,y}^{\mu}}\right\} \tag{4.27}$$

and

$$\Delta_{k,y}^{\mu}\equiv\frac{1}{\sqrt{L}}\left\{\sin[\eta(\mu)kz]\frac{\partial}{\partial\alpha_{k,x}^{\mu}}+\cos[\eta(\mu)kz]\frac{\partial}{\partial\alpha_{k,y}^{\mu}}\right\}\ . \tag{4.28}$$

Now integrating by parts, and assuming that at least one solution is obtained by equating the integrands of the functional integrals, we find the functional Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\int dz\left[-\frac{\delta}{\delta\psi_x^{\mu}}A_x^{\mu}-\frac{\delta}{\delta\psi_y^{\mu}}A_y^{\mu}+\frac{1}{2}\left\{\frac{\delta^2}{\delta\psi_x^{\mu}\delta\psi_x^{\nu}}B_x^{\mu\sigma}B_x^{\nu\sigma}+\frac{\delta^2}{\delta\psi_x^{\mu}\delta\psi_y^{\nu}}B_x^{\mu\sigma}B_y^{\nu\sigma}+\frac{\delta^2}{\delta\psi_y^{\mu}\delta\psi_x^{\nu}}B_y^{\mu\sigma}B_x^{\nu\sigma}+\frac{\delta^2}{\delta\psi_y^{\mu}\delta\psi_y^{\nu}}B_y^{\mu\sigma}B_y^{\nu\sigma}\right\}\right]P\ , \tag{4.29}$$

where ψ_x^{μ} and ψ_y^{μ} are the real and imaginary parts of ψ^{μ}, and the real functional derivatives are defined by

$$\frac{\delta}{\delta\psi_x^{\mu}}\equiv\frac{1}{\sqrt{L}}\sum_{\mathbf{k}}\left\{c_k\frac{\partial}{\partial\alpha_{k,x}^{\mu}}-s_k\frac{\partial}{\partial\alpha_{k,y}^{\mu}}\right\} \tag{4.30}$$

and

$$\frac{\delta}{\delta\psi_y^{\mu}}\equiv\frac{1}{\sqrt{L}}\sum_{\mathbf{k}}\left\{s_k\frac{\partial}{\partial\alpha_{k,x}^{\mu}}+c_k\frac{\partial}{\partial\alpha_{k,y}^{\mu}}\right\}\ , \tag{4.31}$$

which follow directly from the definition of ψ^{μ} and the functional derivatives (3.13) and (3.14). We have also used the shorthand notation c_k and s_k for the cosine and sine functions in (4.27).

We now define the vectors

$$\boldsymbol{\psi} \equiv (\boldsymbol{\psi}_x, \boldsymbol{\psi}_y)^T, \quad \boldsymbol{\psi}_j \equiv (\psi_j, \psi_j^\dagger)^T, \quad j = x, y \ , \tag{4.32}$$

with similar definitions for the drift $\mathbf{A}$ and noise ξ vectors, and the semi positive-definite matrix

$$\mathcal{D} \equiv \begin{bmatrix} \mathbf{B}_x \mathbf{B}_x^T & \mathbf{B}_x \mathbf{B}_y^T \\ \mathbf{B}_y \mathbf{B}_x^T & \mathbf{B}_y \mathbf{B}_y^T \end{bmatrix} \equiv \mathcal{B}\mathcal{B}^T \ , \tag{4.33}$$

where

$$\mathcal{B} \equiv \begin{bmatrix} \mathbf{B}_x & 0 \\ \mathbf{B}_y & 0 \end{bmatrix} . \tag{4.34}$$

We assume the semipositivity of $\mathcal{D}$ is sufficient to enable us to write the equivalent Ito equations

$$\frac{\partial}{\partial t} \begin{bmatrix} \boldsymbol{\psi}_x \\ \boldsymbol{\psi}_y \end{bmatrix} = \begin{bmatrix} \mathbf{A}_x \\ \mathbf{A}_y \end{bmatrix} + \begin{bmatrix} \mathbf{B}_x \cdot \xi \\ \mathbf{B}_y \cdot \xi \end{bmatrix} \ , \tag{4.35}$$

which on combining real and imaginary parts may be written in the form

$$\frac{\partial}{\partial t} \psi^\mu(\mathbf{r}, t) = A^\mu[\boldsymbol{\psi}] + B^{\mu\nu}[\boldsymbol{\psi}] \xi^\nu(\mathbf{r}, t) \ . \tag{4.36}$$

The last equation is the same as that which we would have found by using the functional complex P distribution and the procedure of the previous section. Note that the drift is generally an operator valued vector on the function space, as was illustrated in the example above.

V. SUMMARY

In this paper we have presented a quantum propagation theory of slowly varying optical fields. After quasioptic approximations are made at the classical level, canonical quantization is applied. To illustrate the technique we considered an example which includes the usual complications associated with propagation in a dielectric medium: linear retarded response, nonlinearity and diffraction. To properly resolve the operator ordering issue associated with the nonlinearity, it is perhaps necessary to quantize the medium from a microscopic model.[20] We adopt normal ordering.

Having derived quantum field equations, the systems quantum statistical features may be investigated. This is conveniently done by imbedding the dynamics in a stochastic process defined on a suitable phase space. To this end we extended the coherent-state phase-space methods to cover spatially distributed systems.[13] Our treatment in terms of functional Fokker-Planck equations and stochastic partial differential equations is complementary to that recently developed by Drummond and Carter,[14] who consider guided wave propagation in an optical fiber, while we include diffractive effects. Diffraction may result in interesting quantum statistical properties for propagation in planar slab waveguides where the balance between self-focusing and diffraction allows stable transverse profiles to propagate.[29,30]

The methods described are applicable to a wide variety of nonlinear optical processes and configurations, and may be extended to propagation in anisotropic media. Quantum optical propagation theory has already provided interesting predictions on the squeezing of *scalar* solitons.[14,15] We have recently extended this to a vector theory of a class of anisotropic nonlinear dispersive media, for example, birefringent optical fibers or Kerr liquids with static-field-induced birefringence.[31] Our results predict wide band squeezing of quadrature fluctuations in regimes of modulational polarization instability.[32] This calculation illustrates the use of the formalism developed here.

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