Symmetry property of the Lyapunov spectra of a class of dissipative dynamical systems with viscous damping

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It is shown that the Lyapunov spectra of a class of dissipative dynamical systems with viscous damping are symmetric with respect to a constant determined by the dissipation of the system. A property of the associated linearized flow similar to the symplectic property is established. Furthermore, numerical computations of the characteristic exponents of 15 coupled damped Duffing oscillators driven by a van der Pol oscillator are presented.

I. INTRODUCTION

The numerical calculation of Lyapunov spectra of dynamical systems has become a standard but expensive tool in the investigation of dynamical systems. For Hamiltonian systems it is known[1,2] that the Lyapunov spectrum is symmetric with respect to zero because the linearization of the flow is symplectic (see *Definition* in Sec. III). In this case it is therefore sufficient to calculate only half of the Lyapunov exponents to determine the whole spectrum. It is the purpose of this paper to show that a certain class of dissipative systems also possesses a mirror symmetry of the Lyapunov spectrum.

Consider a periodically driven chain of damped Duffing oscillators. The dissipation is introduced as usual via a damping term proportional to the velocities of the oscillators with a proportionality factor d. It was in calculating numerically whole Lyapunov spectra with respect to a Poincaré hyperplane for such a system that we realized a symmetry of the Lyapunov spectra with respect to the point $-(d/2)$. Note that if the damping factor d goes to zero, the center of symmetry of the Hamiltonian case is recovered, as are the Hamiltonian equations of motion. It is therefore quite natural to conjecture that the observed symmetry of the Lyapunov spectra is not a numerical artifact or an oddity of the chosen Duffing potential but a general feature of dissipative dynamical systems which are non-Hamiltonian only due to viscous damping. In the following we establish for this class of dissipative systems a property of the associated linearized flow which is related to the symplectic property of Hamiltonian flows and implies the symmetry of the Lyapunov spectra.

II. EQUATIONS OF MOTION, DESCRIPTION OF THE COMPUTED MODEL

The purpose of this section is to put forward the notation and to prepare the general result by presenting numerically calculated Lyapunov spectra. First, we introduce the chain of coupled oscillators for which the above-mentioned symmetry of the Lyapunov spectrum has first been observed. It is a ring of 15 mass points with nearest-neighbor coupling where the interaction potential Φ (realized by a spring) is given by a single-well Duffing potential, in our case $\Phi(x)=(x^2/2)+(x^4/4)$. The chain is excited at one mass point by a periodic external force proportional to the space coordinate $y(t)$ of the limit cycle of a van der Pol oscillator obeying

$$\ddot{y}+a(y^2-1)\dot{y}+\omega_0^2 y=0, \quad a,\omega_0^2,y\in\mathbb{R} .$$

Thus we consider the following system of second-order differential equations:

$$\ddot{x}_i+d\dot{x}_i+h[\Phi'(x_i-x_{i-1})-\Phi'(x_{i+1}-x_i)]=fy\delta_{i1} , \quad i=1,\ldots,n=15 \tag{1}$$

with $x_i,h,d,f\in\mathbb{R}$ and with periodic boundary conditions (i.e., $x_0=x_n$, $x_{n+1}=x_1$). x_i denotes the displacement of the ith particle from its equilibrium position (the masses are absorbed in the parameters d, h, and f) and $\dot{x}_i$ its velocity.

Introducing the velocities $\dot{x}_i$ as new independent variables we obtain a nonautonomous system of the general form

$$\dot{\mathbf{x}}=\mathbf{v}(t,\mathbf{x})=\mathbf{v}(t+T(a,\omega_0),\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^{2n}, \; n=15 \tag{2}$$

where $T=T(a,\omega_0)$ is the period of the van der Pol oscillator.

The Poincaré map for nonautonomous systems depending periodically on time is given in the usual way[3] by looking at the equivalent autonomous system. This latter is obtained by adding the equation $\dot{\theta}=1$ to the nonautonomous equations (2), i.e., by introducing time as new explicit state variable:

$$\begin{aligned}\dot{\mathbf{z}}&=(\dot{\theta},\dot{\mathbf{x}})=\tilde{\mathbf{v}}(z)=(1,\mathbf{v}(z)) ,\\ z&=(\theta,\mathbf{x})\in S_T^1\times\mathbb{R}^{2n}=M .\end{aligned} \tag{3}$$

The circular component $S_T^1=\mathbb{R}(\mathrm{mod}\,T)$ of the new phase space M reflects the periodicity of the vector field $\mathbf{v}$ in time.

The Poincaré map of the system with respect to the Poincaré hyperplane (Poincaré plane for short) $\Sigma_{t_0}:=\{(t_0,\mathbf{x})\in M \mid \mathbf{x}\in\mathbb{R}^{2n}\}$ is then given by[3]

$$\mathbf{P}(\mathbf{x})=\pi\circ\phi^T(t_0,\mathbf{x}), \quad t_0\in\mathbb{R}, \; \mathbf{x}\in\mathbb{R}^{2n} \tag{4}$$

where π is the projection onto the second argument and ϕ^T is the flow map of the autonomous system (3) with time T.

Another useful concept to describe the evolution of the nonautonomous system (2) are the (t_1,t_2) advance mappings $g_{t_1}^{t_2}$ of the nonautonomous system which are defined[4] as

$$g_{t_1}^{t_2}(t_1,\mathbf{x})=(t_2,\boldsymbol{\psi}(t_2)), \quad \boldsymbol{\psi}(t_2)\in\mathbb{R}^{2n}, \; t_1,t_2\in\mathbb{R} \tag{5}$$

where $\boldsymbol{\psi}$ is a solution of the nonautonomous system (2) with $\boldsymbol{\psi}(t_1)=\mathbf{x}$. Thus $g_{t_1}^{t_2}$ maps a point $\mathbf{x}$ from Σ_{t_1} to Σ_{t_2}.

There is, of course, a strong connection between the flow map ϕ^t of the autonomous system and the (t_1,t_2) advance mappings $g_{t_1}^{t_2}$ of the corresponding nonautonomous system given by

$$\phi^t(z)=\phi^t(t_1,\mathbf{x})=g_{t_1}^{t_1+t}(t_1,\mathbf{x}) \quad \text{with } z=(t_1,\mathbf{x})\in M \; , \tag{6}$$

i.e., $\phi^t[\Sigma_{t_1}]=g_{t_1}^{t_1+t}:\Sigma_{t_1}\to\Sigma_{t_1+t}$, with $\phi^t[\Sigma_{t_1}]$ being the restriction of ϕ^t to Σ_{t_1}. In the following we choose the appropriate notation according to the circumstances.

We define the linearization of $\phi^t[\Sigma_{t_0}]$ with respect to $\mathbf{x}$ as

$$D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]=D_{\mathbf{x}}g_{t_0}^{t_0+t}:=D_{\mathbf{x}}(\pi\circ\phi^t\circ\pi_{t_0}^{-1}), \quad \mathbf{x}\in\mathbb{R}^{2n} \tag{7}$$

where $\pi_{t_0}^{-1}$ is given by $\pi_{t_0}^{-1}(\mathbf{x}):=(t_0,\mathbf{x})$ and $D_{\mathbf{x}}(\pi\circ\phi^t\circ\pi_{t_0}^{-1})$ is the Jacobian matrix of a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$, i.e., we just look at the action of $\phi^t[\Sigma_{t_0}]$ on the original phase space $\mathbb{R}^{2n}$ of the nonautonomous system, forgetting about the state variable corresponding to time.

The special construction of the autonomous system (i.e., including time as explicit state variable) prevents the existence of a fixed point. From the definition of the Lyapunov spectrum[1,5] it is well known then that one characteristic exponent has to be zero corresponding to the flow direction. To establish our symmetry property we will discard this zero exponent in the following. The remaining $2n$ characteristic exponents are determined by the linearization $D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]$ of $\phi^t[\Sigma_{t_0}]$ with respect to $\mathbf{x}$ for t going to infinity. We will call them characteristic exponents with respect to the Poincaré plane. Using the version of the multiplicative ergodic theorem of Oseledec in Ref. 1, they are given by the logarithms of the eigenvalues of the limit matrix $\Lambda_{\mathbf{x}}$,

$$\Lambda_{\mathbf{x}}=\lim_{t\to\infty}\{D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]^T D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]\}^{1/2t} \; , \tag{8}$$

i.e., letting μ_i, $i=1,\ldots,2n$, be the eigenvalues of the positive matrix $\Lambda_{\mathbf{x}}$ then $\lambda_i=\ln\mu_i$ are the characteristic exponents of the continuous dynamical system (3) with respect to the Poincaré plane. They are related to the Lyapunov exponents $\tilde{\lambda}_i$ of the discrete dynamical system corresponding to the first return map $\mathbf{P}=\pi\circ\phi^T\circ\pi_{t_0}^{-1}$ through $\lambda_i=\tilde{\lambda}_i/T$.[1] In the following, when speaking of the Lyapunov spectrum, we will only refer to the Lyapunov spectrum λ_i, $i=1,\ldots,2n$, with respect to the Poincaré plane.

Coming back to our model, we remark that $\operatorname{div}\mathbf{v}(\mathbf{x},t)=-nd$ holds independently of $\mathbf{x}$ and t. This being the case, the theorem of Liouville[4] implies for the determinant of the linearization of $\phi^t[\Sigma_{t_0}]=g_{t_0}^{t_0+t}$

$$\begin{aligned}\det D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]&=\det D_{\mathbf{x}}g_{t_0}^{t_0+t}\\&=\exp\left[\int_{t_0}^{t_0+t}\operatorname{div}\mathbf{v}(\mathbf{x},t')dt'\right]=e^{-ndt} \; .\end{aligned} \tag{9}$$

From this relation it follows that

$$\sum_{i=1}^{2n}\lambda_i=-nd \; . \tag{10}$$

If we label the characteristic exponents according to decreasing size $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{2n}$, the symmetry we observed calculating the characteristic exponents λ_i of the system (1) [see Figs. 1(a) and 1(b)] can be expressed as

$$\lambda_i-s=s-\lambda_{2n+1-i}, \quad i=1,\ldots,n \tag{11}$$

with $s=-(d/2)$. Note that $s=-(d/2)$ is the only symmetry point compatible with the relation (10) for the sum of the characteristic exponents.

The Lyapunov spectra were calculated with an algorithm described by Wolf *et al.*[6] We just sketch the method; for more details (and the FORTRAN code) see Ref. 6. The method is based on the fact[2,7] that the expansion rate of the k-dimensional volume of a k-dimensional parallelepiped evolving under the action of the linearized flow $D_x\phi^t[\Sigma_{t_0}]$ in tangent space for t going to infinity converges with probability 1 to the sum of the first k characteristic exponents $(\lambda_1\geq\cdots\geq\lambda_k)$ determined by $\Lambda_{\mathbf{x}}$. To express this with formulas we denote by $\mathbf{e}_i\in\mathbb{R}^{2n}$, $i=1,\ldots,k$, k linearly independent vectors in tangent space and by $\mathbf{e}_i(t)\in\mathbb{R}^{2n}$, $i=1,\ldots,k$, the vectors obtained by the action of $D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]$ on the vectors $\mathbf{e}_i$, $i=1,\ldots,k$, i.e.,

$$\mathbf{e}_i(t):=D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]\mathbf{e}_i, \quad \mathbf{e}_i\in\mathbb{R}^{2n}, \; i=1,\ldots,k \; . \tag{12}$$

Further, let $V_k(0)$ $[V_k(t)]$ be the k-dimensional volume of the parallelepiped spanned by the $\mathbf{e}_i$ $[\mathbf{e}_i(t)]$, $i=1,\ldots,k$. If the $\mathbf{e}_i\in\mathbb{R}^{2n}$ are chosen at random then

$$\lim_{t\to\infty}\frac{1}{t}\ln\frac{V_k(t)}{V_k(0)}=\sum_{j=1}^{k}\lambda_j, \quad k\in\{1,\ldots,2n\} \tag{13}$$

with probability 1, where $\lambda_1\geq\cdots\geq\lambda_k$ are the k greatest characteristic exponents determined by $\Lambda_{\mathbf{x}}$. Defining recursively

$$\begin{aligned}\lambda_1(t)&:=\frac{1}{t}\ln\frac{V_1(t)}{V_1(0)} \; ,\\ \lambda_k(t)&:=-\sum_{j=1}^{k-1}\lambda_j(t)+\frac{1}{t}\ln\frac{V_k(t)}{V_k(0)}, \quad k=2,\ldots,2n\end{aligned} \tag{14}$$

(13) is equivalent to

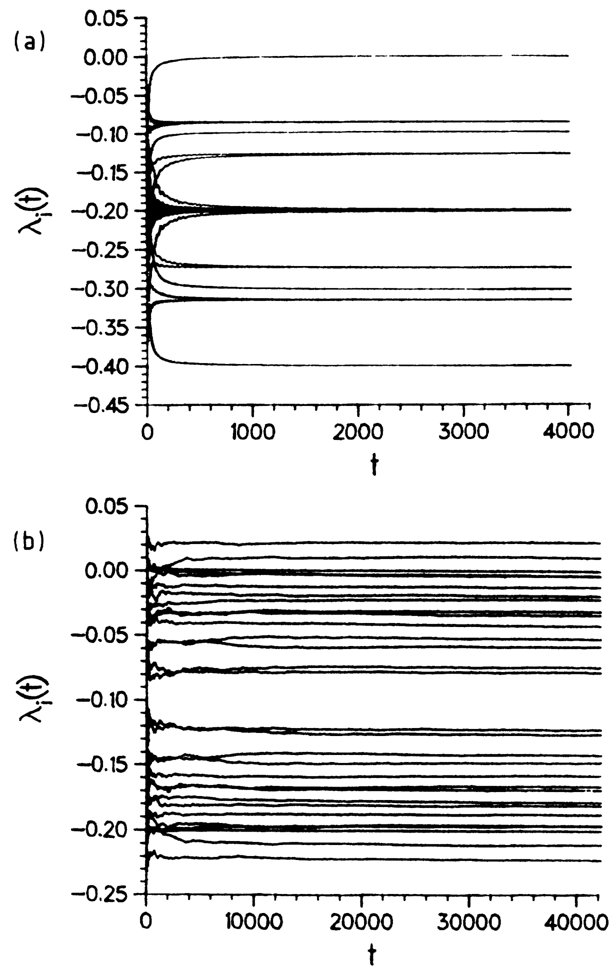

FIG. 1. For a chain of 15 coupled Duffing oscillators [interaction potential $\Phi(x)=x^2/2+x^4/4$] driven by a van der Pol oscillator we show for different parameter values of the equations of motion (1) the Lyapunov spectra (a) of a fixed point and (b) of a strange attractor of the Poincaré map. The Poincaré plane Σ_{t_0} was defined at the time t_0 when the van der Pol oscillator went through zero with positive velocity. The $\lambda_i(t)$, $i=1,\ldots,30$, defined in Eq. (14) are plotted vs t. For t going to infinity they converge to the characteristic exponents λ_i, $i=1,\ldots,30$, with respect to the Poincaré plane. In panel (a), for the parameter values $a=1$, $\omega_0=1.6$, $h=1$, $f=2$, and $d=0.4$, the Poincaré map has a fixed point. The $\lambda_i(t)$ reach numerically their limits after an integration time of about $t=3000$, i.e., after about 746 periods $T(a=1,\omega_0=1.6)=4.02$ of the van der Pol oscillator. A symmetry with respect to $-0.2=-(d/2)$ can be seen. The first characteristic exponent λ_1 being zero corresponds to the fact that the equations of motion (1) are translationally invariant. Therefore, a characteristic exponent equal to zero arises in the Lyapunov spectra with respect to the Poincaré plane for all parameter values. In panel (b), for the parameter values $a=1$, $\omega_0=1.82$, $h=1$, $f=4$, and $d=0.2$ of Eq. (1), chaotic behavior of the chain is found. As can be seen, two characteristic exponents are greater than zero. The $\lambda_i(t)$ reach their limits after an integration time of about 8500 periods $T(1,1.82)=3.51$ of the van der Pol oscillator, which corresponds to an integration time of about $t=30\,000$. This is, as expected, a longer time than that for a fixed point. As can be seen, the spectrum is again symmetric with respect to $-(d/2)=-0.1$.

$$\lim_{t\to\infty}\lambda_j(t)=\lambda_j,\quad j=1,\ldots,2n\ .$$

In order to determine $\lambda_j(t)$, $j=1,\ldots,2n$, all

$$V_k(t)=\exp[t\sum_{j=1}^{k}\lambda_j(t)]V_k(0),\quad k=1,\ldots,2n$$

have to be calculated. This is done by integrating the equations of motion (2) with some post-transient initial condition $(t_0,\mathbf{x})$ to obtain the orbit $\mathbf{x}(t)$ with $\mathbf{x}(t_0)=\mathbf{x}$. Simultaneously, $2n$ copies of the associated variational equations

$$\frac{d}{dt}\mathbf{u}(t)\,|_{t=\tau}=D_{\mathbf{x}(\tau+t_0)}\mathbf{v}(\tau+t_0,\mathbf{x}(\tau+t_0))\mathbf{u}(\tau)\ ,\qquad \mathbf{u}(t)\in\mathbb{R}^{2n}\qquad(15)$$

have been integrated with the standard basis $\{\mathbf{e}_i\}_{i=1,\ldots,2n}$ as initial conditions to obtain $\{\mathbf{e}_i(t)=D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]\mathbf{e}_i\}_{i=1,\ldots,2n}$. This is due to the fact that $\mathbf{u}(t)=D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]\mathbf{u}$ is a solution of (15) with $\mathbf{u}(0)=\mathbf{u}$. This can be seen by direct differentiation of (2) and reminding the definition (7) of $D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]$.

There are some numerical problems in determining $V_k(t)$ because of the divergence in magnitude of the vectors $\mathbf{e}_i(t)$, $i=1,\ldots,k$, and their trend to fall along the direction of most rapid growth, what causes the vectors to become numerically indistinguishable. These numerical difficulties can be circumvented by repeated use of Gram-Schmidt reorthonormalization. This is described in full detail in Refs. 2, 6, and 7.

In Fig. 1 we show the results of our calculations for the case of a fixed point [Fig. 1(a)] ($d=0.4$) and a strange attractor [Fig. 1(b)] ($d=0.2$) of the Poincaré map of the chain of oscillators (1) investigated. The $\lambda_i(t)$, $i=1,\ldots,2n$, defined in (14) are plotted versus t. In both cases, the $\lambda_i(t)$ reach numerically their limits for large t and exhibit a symmetry with respect to $s=-(d/2)$.

III. GENERAL EQUATIONS OF MOTION WITH SYMMETRIC LYAPUNOV SPECTRA

To explain the symmetry of the Lyapunov spectra for this and many other damped driven systems of oscillators we first rewrite the equations of motion in a more general form which makes their connection with the Hamiltonian case clear. Set $q_i:=x_i$ and $p_i:=\dot{x}_i$ and introduce a time-dependent Hamiltonian

$$\begin{aligned}H(t,q_1,\ldots,q_n,p_1,\ldots,p_n)&=H(t,\mathbf{q},\mathbf{p})\\&=\sum_{i=1}^{n}\frac{p_i^2}{2}+\sum_{i=1}^{n}h\Phi(q_i-q_{i-1})+fy(t)q_1\ ,\end{aligned}\qquad(16)$$

with $q_0=q_n$, $q_{n+1}=q_1$, and $\Phi(x)=x^2/2+x^4/4$. Let $y(t)$ be the space coordinate of the limit cycle of the van der Pol oscillator. With this substitution the equations of motion (1) of the chain of oscillators can be written as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(t,\mathbf{q},\mathbf{p}) \; ,$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}(t,\mathbf{q},\mathbf{p}) - dp_i, \quad i=1,\ldots,n \tag{17}$$

with $\mathbf{q},\mathbf{p}\in\mathbb{R}^n$, $d,q_i,p_i\in\mathbb{R}$, and with $H(t,\mathbf{q},\mathbf{p})=H(t+T,\mathbf{q},\mathbf{p})$ for some $T=T(a,\omega_0)\neq 0$.

Introducing the matrix

$$J=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{18}$$

where each entry in J is an $n\times n$ block and 1 is the $n\times n$ identity matrix, the equations (17) can be written in a more compact form as

$$\dot{\mathbf{x}} = J\nabla_{\mathbf{x}} H(t,\mathbf{x}) - (0,d\mathbf{p})^T = \mathbf{v}(t,\mathbf{x}) \; ,$$
$$0,\mathbf{q},\mathbf{p}\in\mathbb{R}^n, \quad \mathbf{x}\in\mathbb{R}^{2n} \; . \tag{19}$$

with $\mathbf{x}=(\mathbf{q},\mathbf{p})$ and $\nabla_{\mathbf{x}}H$ denoting the gradient of H with respect to $\mathbf{x}$.

The presence of **J** is the hallmark of the "symplectic" property.

Definition. A $2n\times 2n$ matrix A is said to be symplectic if

$$A\,J\,A^T = J$$

holds. J is called the symplectic tensor.

IV. A RELATION FOR THE LINEARIZED FLOW $D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]$

Since the symmetry of the Lyapunov spectrum in the Hamiltonian case is strongly related to the symplectic property of the linearized Hamiltonian flow, a similar geometrical property of the linearized flow of the dissipative system introduced above is expected. In fact, it will turn out that this latter flow is symplectic "up to a factor" [see Eq. (20) below]. Guided by the proof for the symplectic property of the linearization of canonical transformations[8] (because the Hamiltonian flow map is just a special one), we apply the method of infinitesimal transformations to dissipative systems of the form (19) to prove the following proposition.

Proposition. Let a nonautonomous system be given through [Eq. (19)]

$$\dot{\mathbf{x}} = J\nabla_{\mathbf{x}} H(t,\mathbf{x}) - (0,d\mathbf{p})^T = \mathbf{v}(t,\mathbf{x}) \; ,$$
$$0,\mathbf{q},\mathbf{p}\in\mathbb{R}^n, \quad \mathbf{x}=(\mathbf{q},\mathbf{p})\in\mathbb{R}^{2n} \; .$$

Then the linearization $D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]$ of the flow ϕ^t of the equivalent autonomous system with respect to a time constant plane Σ_{t_0} satisfies the relation

$$D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]J(D_{\mathbf{x}}\phi^t[\Sigma_{t_0}])^T = Je^{-dt} \; . \tag{20}$$

Proof. In order to investigate $\phi^t[\Sigma_{t_0}]$ by the method of infinitesimal transformations it is more convenient for notational purposes to look at $g_{t_0}^{t_0+t}=\phi^t[\Sigma_{t_0}]$. Our aim is to write $g_{t_0}^{t_0+t}$ as a composition of infinitesimal transformations $g_t^{t+\epsilon}$, where ϵ is thought to be infinitesimal. Thus, when calculating $D_{\mathbf{x}}g_t^{t+\epsilon}$ only first-order terms in ϵ have to be retained and all higher-order terms in ϵ can be neglected. In this way we obtain

$$g_t^{t+\epsilon}(t,\mathbf{x}) = (t+\epsilon,\mathbf{x}+\mathbf{v}(t,\mathbf{x})\epsilon) \; , \tag{21}$$

and with the definition (7) of the linearization of $g_t^{t+\epsilon}$ with respect to $\mathbf{x}$ it follows that

$$D_{\mathbf{x}}g_t^{t+\epsilon} = 1_{2n} + D_{\mathbf{x}}\mathbf{v}(t,\mathbf{x})\epsilon$$
$$= 1_{2n} + D_{\mathbf{x}}[J\nabla_{\mathbf{x}}H(t,\mathbf{x})]\epsilon + \begin{bmatrix} 0 & 0 \\ 0 & -d1 \end{bmatrix}\epsilon \; , \tag{22}$$

where 1_{2n} is the $2n\times 2n$ identity matrix and 1 and 0 are $n\times n$ matrices.

Using simple matrix manipulations and the fact that $D_{\mathbf{x}}(\nabla_{\mathbf{x}}H)$ is a symmetric and J an antisymmetric matrix we find that $D_{\mathbf{x}}g_t^{t+\epsilon}$ satisfies for infinitesimal ϵ

$$D_{\mathbf{x}}g_t^{t+\epsilon}J(D_{\mathbf{x}}g_t^{t+\epsilon})^T = (1-d\epsilon)J \; . \tag{23}$$

By means of the group property of $g_{t_1}^{t_2}$, i.e., $g_{t_2}^{t_3}\circ g_{t_1}^{t_2}=g_{t_1}^{t_3}$, $g_{t_0}^{t_0+t}$ can be expressed as

$$g_{t_0}^{t_0+t} = \lim_{k\to\infty} g_{t_0+(k-1)\epsilon}^{t_0+k\epsilon}\circ\cdots\circ g_{t_0+\epsilon}^{t_0+2\epsilon}\circ g_{t_0}^{t_0+\epsilon}$$
$$\text{with } \epsilon=\frac{t}{k} \; , \tag{24}$$

and therefore (23) implies for $D_{\mathbf{x}}g_{t_0}^{t_0+t}$

$$D_{\mathbf{x}}g_{t_0}^{t_0+t}J(D_{\mathbf{x}}g_{t_0}^{t_0+t})^T = e^{-dt}J$$

or [cf. (7)] [Eq. (20)]

$$D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]J(D_{\mathbf{x}}\phi^t[\Sigma_{t_0}])^T = e^{-dt}J. \quad \text{Q.E.D.}$$

Note that for d being zero we immediately obtain the Hamiltonian symplectic property. The relation (20) shows the strong connection between flows of simple damped systems as considered here (chains of oscillators with linear damping) and Hamiltonian flows.

Another approach to establish this connection and to prove the relation (20) without the use of infinitesimal transformations is based on a paper of Steeb and Kunick,[9] who showed that the equation of motion of a damped oscillator can be derived from a time-dependent Hamiltonian. Applying their ideas to dissipative dynamical systems of the form (19), we define the following time-dependent transformation of coordinates which can be viewed as a time-dependent scale transformation:

$$\tilde{q}_i(t) = q_i(t) \; ,$$
$$\tilde{p}_i(t) = p_i(t)e^{dt}, \quad i=1,\ldots,n, \; t,q_i,p_i,\tilde{q}_i,\tilde{p}_i\in\mathbb{R} \tag{25}$$

with $\mathbf{x}(t)=(\mathbf{q}(t),\mathbf{p}(t))$ being a solution of (19). Then $\tilde{\mathbf{x}}(t)=(\tilde{\mathbf{q}}(t),\tilde{\mathbf{p}}(t))$ is a solution of the canonical equations of motion

$$\dot{\tilde{q}}_i = \frac{\partial\tilde{H}}{\partial\tilde{p}_i}(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}}), \quad \tilde{\mathbf{q}},\tilde{\mathbf{p}}\in\mathbb{R}^n \tag{26}$$

$$\dot{\tilde{p}}_i=-\frac{\partial\tilde{H}}{\partial\tilde{q}_i}(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}}),\quad i=1,\ldots,n$$

with a time-dependent Hamiltonian $\tilde{H}$ given by

$$\tilde{H}(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}})=H(t,\mathbf{q},\mathbf{p})\,e^{dt}\ . \tag{27}$$

To prove (20) we go back to the symplectic notation and rewrite the transformation of coordinates (25) as

$$\tilde{\mathbf{x}}(t)=A(t)\mathbf{x}(t),\quad t\in\mathbb{R},\ \mathbf{x},\tilde{\mathbf{x}}\in\mathbb{R}^{2n} \tag{28}$$

with

$$A(t)=\begin{bmatrix}1 & 0\\ 0 & e^{dt}1\end{bmatrix} \tag{29}$$

where 1 and 0 are $n\times n$ matrices.

Denoting by $\tilde{g}\,^t_{t_0}$ the (t_0,t) advance mapping of the Hamiltonian system (26) and by $g^t_{t_0}$ the (t_0,t) advance mapping of the original system (19), Eq. (28) can be written [with the notation introduced in (4)–(6)] as

$$\pi\circ\tilde{g}\,^t_{t_0}(t_0,\tilde{\mathbf{x}})=A(t)\pi\circ g^t_{t_0}(t_0,A(t_0)^{-1^{-1}}\tilde{\mathbf{x}}),\quad \tilde{\mathbf{x}}\in\mathbb{R}^{2n}\ . \tag{30}$$

Differentiation of (30) with respect to $\tilde{\mathbf{x}}$ gives for the linearizations of the (t_0,t)-advance mappings of (19) and (26) the relation:

$$D_{\tilde{\mathbf{x}}}\tilde{g}\,^t_{t_0}=A(t)D_{\mathbf{x}}g^t_{t_0}A(t_0)^{-1},\quad \tilde{\mathbf{x}}\in\mathbb{R}^{2n},\ \mathbf{x}=A(t_0)^{-1}\tilde{\mathbf{x}}\ . \tag{31}$$

Now we use the fact[8] that $\pi\circ\tilde{g}\,^t_{t_0}\circ\pi_{t_0}^{-1}$ defines a canonical transformation of the coordinates $\tilde{\mathbf{x}}$. Hence, the linearization of $\tilde{g}\,^t_{t_0}$ is symplectic, i.e., $D_{\tilde{\mathbf{x}}}\tilde{g}\,^t_{t_0}$ satisfies

$$D_{\tilde{\mathbf{x}}}\tilde{g}\,^t_{t_0}J(D_{\tilde{\mathbf{x}}}\tilde{g}\,^t_{t_0})^T=J\ . \tag{32}$$

With (31) and (29) the symplectic property (32) of $D_{\tilde{\mathbf{x}}}\tilde{g}\,^t_{t_0}$ leads immediately to

$$D_{\mathbf{x}}g^t_{t_0}J(D_{\mathbf{x}}g^t_{t_0})^T=e^{-d(t-t_0)}J,\quad \mathbf{x}\in\mathbb{R}^{2n},\ t\in\mathbb{R} \tag{33}$$

or [cf. (7)]

$$D_{\mathbf{x}}\phi^{t'}[\Sigma_{t_0}]J(D_{\mathbf{x}}\phi^{t'}[\Sigma_{t_0}]^T=e^{-dt'}J,\quad t'=t-t_0 \tag{20'}$$

i.e., the desired equation (20) for the linearized flow map of the original system (19).

V. SYMMETRY OF THE LYAPUNOV SPECTRUM

The symmetry of the Lyapunov spectrum is now an immediate consequence of (20) and, in fact, it follows like in the Hamiltonian case. We just sketch the arguments. In the same manner as for symplectic matrices[10] it can be shown that if $\mu\in\mathbb{C}$ is an eigenvalue of a $2n\times 2n$ matrix A fulfilling

$$A\,JA^T=\lambda J,\quad \lambda\neq 0 \tag{34}$$

then $\tilde{\mu}=(1/\mu)\lambda$ is an eigenvalue of A as well. For a positive matrix A obeying (34) this implies that its $2n$ eigenvalues μ_i ordered according to $\mu_1\geq\mu_2\geq\cdots\geq\mu_{2n}$ satisfy

$$\mu_{2n-i+1}=\frac{1}{\mu_i}\lambda,\quad i=1,\ldots,n \tag{35a}$$

or, equivalently, that the $2n$ eigenvalues can be written as

$$\mu_1\geq\cdots\geq\mu_n\geq\frac{1}{\mu_n}\lambda\geq\cdots\geq\frac{1}{\mu_1}\lambda\ . \tag{35b}$$

In order to determine the Lyapunov spectrum of $\phi^t[\Sigma_{t_0}]$ we are interested in the eigenvalues of

$$\{(D_{\mathbf{x}}\phi^t[\Sigma_{t_0}])^TD_{\mathbf{x}}\phi^t[\Sigma_{t_0}]\}^{1/2t}\ .$$

As a consequence of (20) the positive matrix

$$\Lambda_{\mathbf{x}}(t)^{2t}:=\{(D_{\mathbf{x}}\phi^t[\Sigma_{t_0}])^TD_{\mathbf{x}}\phi^t[\Sigma_{t_0}]\}$$

satisfies

$$\Lambda_{\mathbf{x}}(t)^{2t}J(\Lambda_{\mathbf{x}}(t)^{2t})^T=e^{-2dt}\underline{J},\quad t\in\mathbb{R}\ . \tag{36}$$

In accordance with (35b), the $2n$ positive eigenvalues $\mu_i(t)$ of $\Lambda_{\mathbf{x}}(t)^{2t}$ are

$$\mu_1(t)\geq\cdots\geq\mu_n(t)\geq\frac{1}{\mu_n(t)}e^{-2dt}\geq\cdots\geq\frac{1}{\mu_1(t)}e^{-2dt}\ . \tag{37}$$

The eigenvalues $\tilde{\mu}_i(t)$ of

$$\Lambda_{\mathbf{x}}(t):=\{(D_{\mathbf{x}}\phi^t[\Sigma_{t_0}])^TD_{\mathbf{x}}\phi^t[\Sigma_{t_0}]\}^{1/2t}$$

are then given by $\tilde{\mu}_i(t)=\mu_i(t)^{1/2t}$ and the theorem of Oseledec yields for the characteristic exponents λ_i

$$\lim_{t\to\infty}\ln\tilde{\mu}_i(t)=\lim_{t\to\infty}\frac{1}{2t}\ln\mu_i(t)=\lambda_i,\quad i=1,\ldots,2n\ . \tag{38}$$

Finally, the relation (37) together with (38) gives for the characteristic exponents

$$\lambda_1\geq\cdots\geq\lambda_n\geq-\lambda_n-d\geq\cdots\geq-\lambda_1-d\ , \tag{39}$$

i.e., the symmetry condition

$$\lambda_i-s=s-\lambda_{2n-i+1},\quad i=1,\ldots,n\ \text{with}\ s=-\frac{d}{2} \tag{40}$$

is proven.

VI. REMARKS

While the relation (20) for the linearization $D_{\mathbf{x}}\phi^t[\Sigma_{t_0}]$ of the flow with respect to the time constant plane Σ_{t_0} holds for all systems (19) irrespectively of the periodicity in time of the Hamiltonian H, the symmetry of the Lyapunov spectrum with respect to the Poincaré plane can only be established for Hamiltonians periodic in time. This is so because the existence of the limit matrix $\Lambda_{\mathbf{x}}$ [and thus the validity of Eq. (38)] can only be guaranteed in the periodic case in which $\phi^t[\Sigma_{t_0}]$ is related to a Poincaré map $\mathbf{P}$ via $\mathbf{P}=\pi\circ\phi^T\circ\pi_{t_0}^{-1}$. As a matter of fact, the symmetry (40) of the characteristic exponents λ_i of the continuous dynamical system with respect to $-(d/2)$ is equivalent to a symmetry of the Lyapunov spectrum of the Poincaré map with respect to $-(Td/2)$.

When proving the relation (20) we had in mind the explanation of the symmetry of the Lyapunov spectra numerically found for a periodically driven chain of damped

Duffing oscillators. Therefore the statement was formulated for the case of nonautonomous dissipative systems related to a time-dependent Hamiltonian through (19). So we stressed the concept of the linearization of the flow with respect to a time constant plane Σ_{t_0} to establish the symplectic property up to a factor [see (20)]. For dissipative autonomous systems given by (19) with a time-independent Hamiltonian the statement (20), of course, holds for the linearized flow $D_x\phi^t$ itself.

A deficiency is that this result can only be applied at the moment to systems with uniform damping, i.e., homogeneous systems (particles with equal masses). For unequal dissipation rates d_i the analytical expression for the infinitesimal transformation $D_x g_t^{t+\epsilon}$, i.e., Eq. (23), has to be replaced by

$$D_x g_t^{t+\epsilon} J (D_x g_t^{t+\epsilon})^T = (1-\tilde{\mathbf{D}}\epsilon) J \ , \tag{41}$$

$\tilde{D}$ being a $2n\times 2n$ diagonal matrix of the form

$$\tilde{D} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} , \tag{42}$$

where each entry is a $n\times n$ block and D is a diagonal matrix with diagonal elements d_i, $i=1,\dots,n$. Because the factor in front of J on the rhs of (41) is not a scalar, it is not possible to continue in the same way as in the proof of (20) to get an expression for $D_x g_t^{t+t_0}$ for finite t_0. The second approach given through the time-dependent scale transformation (25) fails also because for general H there is no connection [as (27)] to a Hamiltonian $\tilde{H}$, i.e., the equations of motion for the new coordinates are not Hamiltonian equations.

VII. SUMMARY AND CONCLUSION

Dissipative dynamical systems were considered which differ from Hamiltonian systems only by a uniform viscous damping [see (19)]. This relationship is manifested by the fact that the equations of motion become Hamiltonian by a time-dependent transformation of coordinates (25) which can be viewed as a time-dependent scale transformation. No constraint is made for the considered Hamiltonian, even time-dependent Hamiltonians (i.e., forced systems) are allowed. It is shown that the linearization of the flow of these dissipative systems fulfills (20), which up to a time-dependent factor is the symplectic property. As in the Hamiltonian case, this derived structure of the linearized flow of the dissipative system implies that the Lyapunov spectrum is symmetric with respect to a constant, which for dissipative systems is negative and determined by the damping constant.

Our result can be applied to systems of coupled oscillators. This is so because these systems are derived from Hamiltonian mechanics. To model physical systems it is common to introduce dissipation through a uniform viscous damping term. Therefore the equations of motion for these systems are often given through (19) independently of the chosen interaction, i.e., of the Hamiltonian.

If such systems are investigated the costs of the calculation of whole Lyapunov spectra can be reduced by one half. This could be of advantage if one is interested in the Lyapunov dimension D_L of a strange attractor, since D_L requires knowledge of all the positive and some of the negative Lyapunov exponents. Thus depending on the distribution of the Lyapunov exponents it might be necessary to know more than half of the Lyapunov exponents to determine the Lyapunov dimension which is related by the Kaplan-Yorke conjecture to the information dimension.[1]

But even if one is not interested in whole Lyapunov spectra, the symplectic property up to a factor (20) reveals much of the structure of the dynamics of these systems and manifests their strong relation to Hamiltonian systems. In this case, results for Hamiltonian systems can be applied to dissipative systems independently of the strength of the dissipation. For example, for these systems it might be possible to calculate rotation numbers which for high-dimensional systems are only defined if the linearized flow is symplectic or symplectic times a scalar (for details see Ruelle[11]). Therefore (20) could be used to extend the results of Parlitz and Lauterborn[12] on the classification of resonances and local bifurcations of driven dissipative oscillators through torsion numbers to systems of coupled oscillators and so to explain their resonance structure.[13]

To the knowledge of the author this is the first observation that the Hamiltonian structure can be generalized to encompass dissipative systems. The result reported here may not be a peculiarity and is expected to extend the applicability of Hamiltonian dynamics.

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