Influence of a beam splitter on photon statistics

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The quantum analysis of the influence of a beam splitter on photodetection statistics is discussed. The link between second-order correlation functions and various experimental quantities obtained in photon counting and spectrum analysis is clarified. The introduction of a "vacuum field" is interpreted as a mathematical transformation between two sets of creation-annihilation operators, which can be used as a mathematical device for simplifying the calculations. However, its physical interpretation as a real field is shown to be potentially confusing. The present theoretical approach is used in the interpretation and the analysis of heterodyning experiments of a squeezed signal with a much stronger local oscillator in a coherent state.

I. INTRODUCTION

Quantum states of light (i.e., states whose properties are not explainable in classical terms[1]) had already been produced experimentally in the 1970s.[2,3] However, strong ones (i.e., states with a large number of photons) have been produced only recently.[4–6] The experimental creation and detection of such light implies that beams of nonclassical light are now available. This in turn calls for a careful quantum analysis of the basic components of the optical systems producing, transmitting, and detecting these states. Here we address the issue of the quantum analysis of the influence of a beam splitter (BS) on photon statistics.

This issue was first addressed by Aharonov *et al.*[7] To correspond to the splitting of the beam, they introduced the splitting of the operators into two parts in a way that conserves the commutation relations (CR's). This approach gives a clear physical picture of the process as the splitting of one state into two parts, between which exist quantum correlations (except for a coherent state for which the two parts are completely uncorrelated[7]). As the calculations of the correlations between the two parts are quite cumbersome, one can follow a different approach which has been given recently by Yuen and his colleagues.[8] The new approach introduces the concept of an additional "vacuum field" at the beam splitter. This method is useful as a mathematical device for simplifying the calculations, but its physical interpretation is more delicate. A proper use of this vacuum field by following the correct quantum description of the beam splitter, as presented very recently by Prasad, Scully, and Martienssen,[9] gives the same results as those obtained by the method of Aharonov *et al.*[7] However, by taking this vacuum field as a real field, many investigators consider it as the source of the noise and thus misinterpret the phenomena of homodyne and heterodyne detection.

The quantum characteristics of the BS appear explicitly only when one considers quantum light, as it not only splits the light beam into two, but also changes the photon statistics in each beam with respect to the initial ones. The simplest way to investigate these changes is by means of the second-order correlation function of the field.[1] This function also has the advantage of being directly linked to experimental quantities. A quantum analysis of a BS was given very recently by Ou *et al.*[10] and also by Fearn and Loudon.[11] Our approach, linking all quantities to the second-order correlation function, has the advantage of being applicable to any kind of state. We also emphasize the physical meaning of the splitting and its relationship to the "vacuum noise."

In Sec. II we review the theory of photon counting and the photocurrent spectrum and find that all experimental quantities related to the observations of quantum light are directly expressed in terms of the various second-order correlation functions at different space-time points.

In Sec. III we deal with the quantum analysis of a beam splitter. We follow here the approach presented recently by Prasad *et al.*[9] They have analyzed the mixing of two modes of the electromagnetic (e.m.) field entering the beam splitter from its two sides for the special case of two modes with the same frequency, the same polarization, and *exactly* at the same angle of incidence. However, their approach is quite general and can be used also for other cases. By considering the case where one of the modes is in the vacuum state (before entering the beam splitter) we see that the analysis is related to the approaches of Aharonov *et al.*[7] and Yuen and co-workers.[8] When we mix two beams, which are not at the same incidence angle, we can still follow their approach by using four modes [Fig. 1(c)]. The special case of exactly the same incidence angle can then be obtained by going to the limit of a pair of beams of equal **k**, as is usually done in interference experiments. In any case, as soon as we use two beams of different frequencies (heterodyning), we have no other choice but to use four modes.

In Sec. IV, we present some applications and, in particular, we analyze heterodyne detection. We use this analysis for the interpretation of recent experiments on squeezed states.[4–6]

II. PHOTODETECTION OF QUANTUM LIGHT

A. Photon counting

The theory of photon counting is well known. We assume a photomultiplier (PM) orthogonal to a monodirectional, stationary beam. The global multicoincidence rate is defined in a standard way,[12]

$$w^{(n)}(t_1,\ldots,t_n)=(\beta A)^n \mathrm{Tr}[\rho E^{(-)}(x_0,t_1)\cdots E^{(-)}(x_0,t_n)E^{(+)}(x_0,t_n)\cdots E^{(+)}(x_0,t_1)] , \tag{1}$$

where β is the efficiency of the detector and includes normalization and geometrical factors, A is the effective cross section of the beam, and x_0 is the position of the center of the beam.

Following Shapiro and Wagner,[13] we assume that the photodetector responds to photon flux, and we therefore use a modified "electric field,"[14] also called a detection operator,[15]

$$E(r,t)=i\sum_k \left[\frac{1}{2\epsilon_0 V}\right]^{1/2}[u_k(r)e^{-i\omega_k t}a_k-\text{c.c.}] . \tag{2}$$

This electric field differs from the usual one by the drop of the factor $(\hbar\omega_k)^{1/2}$ in the summation. V is the adopted volume of quantization. Since the detector responds to photon flux, the counting rate $w^{(1)}(t_1)$ should be proportional to the photon flux across surface A. We denote by $\langle n\rangle$ the average number of photons in our quantization volume V and get

$$w^{(1)}(t_1)=\alpha A\frac{\langle n\rangle c}{V} , \tag{3}$$

where α is the dimensionless quantum efficiency of the detector, i.e., the ratio between the flux of photons and the counting rate. Comparing (1) and (3) we find that $\beta=2\epsilon_0 c\alpha$. We now turn to photon-counting experiments. The factorial moments of the counts are then[16]

$$\langle m(m-1)\cdots(m-n+1)\rangle_T =\int\cdots\int dt_1\cdots dt_n w^{(n)}(t_1,\ldots,t_n) . \tag{4}$$

It is straightforward to derive the expressions for the number of counts

$$\langle m\rangle_T=\alpha\langle n\rangle\frac{T}{T_c} , \tag{5}$$

where $T_c\equiv V/Ac$ is a characteristic time of the light under investigation that we recently interpreted as the coherence time.[17] The variance is then

$$\langle\Delta m^2\rangle_T=\langle m\rangle_T +\langle m\rangle_T^2\left[\frac{2}{T}\int_0^T\left[1-\frac{\tau}{T}\right][g^{(2)}(\tau)-1]d\tau\right] , \tag{6}$$

where $g^{(2)}(\tau)$ is the second-order normalized correlation function,

$$g^{(2)}(\tau)\equiv\frac{w^{(2)}(t,t+\tau)}{w^{(1)}(t)w^{(1)}(t+\tau)} . \tag{7}$$

The first term is the standard quantum limit (SQL), i.e., the noise of measurement of a coherent state, for which $g^{(2)}(\tau)$ is equal to unity for all τ. The second one is always positive for classical light, so that the noise is larger than SQL (bunched light). It can become negative for quantum light, for which photons can be antibunched.

This formula (6) follows directly from the principles of photon counting and can be easily derived from any of the more well-known formulas.[12,16] The main advantage of (6) is that it directly shows the relation between experimental quantities $\langle\Delta m^2\rangle_T$ and $\langle m\rangle_T$ and the second-order correlation function of the field. For short counting times $(T\ll T_c)$ we can consider $g^{(2)}(\tau)$ as constant and get

$$\langle\Delta m^2\rangle_T=\langle m\rangle_T+\langle m\rangle_T^2[g^{(2)}(0)-1] . \tag{8}$$

B. Photocurrent spectrum

We follow here the approach given in the article of Kelley and Kleiner,[18] and define the random function representing the output of the photomultiplier by

$$S(t)=\sum_{i=0}^{N} eG\delta(t-t_i)y_i , \tag{9}$$

where the measurement time T is divided into N intervals $t_1,\ldots,t_N$ and N is chosen large enough to prevent two photoelectrons to be emitted at the same time; y_i is a random variable in the interval $[t_i,t_i+1]$; $y_i=1$ if there is count, $y_i=0$, if not; and $eG\delta(t-t_i)$ represents the intensity of the infinitely sharp pulse occurring when a detection takes place. (G is the gain of the PM.) To get the spectrum, we introduce the random variable correlation function,

$$C(\tau)\equiv\frac{1}{T}\int_0^T S(t)S(t+\tau)dt . \tag{10}$$

By using a procedure similar to that of Ou *et al.*,[19] we get the average value of $C(\tau)$,

$$\langle C(\tau)\rangle=\frac{e^2G^2}{T}\left[\int_0^T w^{(1)}(t)dt\right]\delta(\tau) +\frac{e^2G^2}{T}\left[\int_0^T w^{(2)}(t,t+\tau)dt\right] . \tag{11}$$

We now suppose stationary light and use (3) and (7) to get

$$\langle C(\tau)\rangle = eG\langle i\rangle\delta(\tau)+\langle i\rangle^2 g^{(2)}(\tau)\ , \tag{12}$$

where $\langle i\rangle \equiv eG\alpha\langle n\rangle T_c^{-1}$ is the average intensity. To get the intensity-fluctuation spectrum, we subtract $\langle i\rangle^2$ and Fourier transform Eq. (12),

$$N^2(\omega)=\frac{eG}{2\pi}\langle i\rangle+\frac{1}{2\pi}\langle i\rangle^2\int_{-\infty}^{\infty}[g^{(2)}(\tau)-1]e^{-i\omega\tau}d\tau\ , \tag{13}$$

where ω is an arbitrary frequency fixed by the spectrum analyzer. The first term corresponds once again to the SQL and is a "white noise." The second term, which can be frequency dependent, is again positive for classical light, and can be negative only for quantum light. We have recently calculated this frequency dependence for a homodyne experiment on a squeezed state.[20] Once again the experimental quantities are directly linked to the second-order correlation function.

C. Spatial correlations

One of the main advantages of the photocurrent spectrum is that it enables us, in a very simple way, to find the spatial correlations. One simply has to put two detectors at different places in the field and subtract the obtained currents before sending it to the frequency analyzer. For this case the correlation function becomes

$$\langle C(\tau)\rangle=\frac{1}{T}\int_0^T\langle[i_1(t)-i_2(t)]\times[i_1(t+\tau)-i_2(t+\tau)]\rangle dt\ . \tag{14}$$

One easily gets

$$N^2(\omega)=N_1^2(\omega)+N_2^2(\omega)-2\frac{\langle i_1\rangle\langle i_2\rangle}{2\pi}\int_{-\infty}^{\infty}[g^{(2)}(x_1,t;x_2,t+\tau)-1]e^{-i\omega\tau}d\tau\ . \tag{15}$$

The important point here is that we have shown that the second-order correlation function is all we need for analyzing all kinds of intensity correlation experiments.

III. QUANTUM ANALYSIS OF THE INFLUENCE OF A BEAM SPLITTER ON PHOTON STATISTICS

We follow here the analysis given recently by Prasad *et al.*[9] They consider the beam splitter as a device for mixing two different modes incident on a BS with the same incidence angle. The effect of the beam splitter is demonstrated in Fig. 1. While in Fig. 1(a) there is no mixing between the modes **k** and **K**, the BS mixes these two beams as demonstrated in Fig. 1(b). The action of the BS is characterized by a unitary operator U acting on the states. The exact expression for U depends on the properties of the BS, and we shall treat a simple case (in accordance with our previous article[20]) where all transmission and reflection coefficients are real [case (a) of Ref. 9]. The transmission and reflection coefficients S and R are then defined by $S\equiv S(k)=S(K)$, $R\equiv R(k)=-R(K)$, and we do not include here the dependence on polarization.

We now write the two basic equations of transformation,

$$\begin{aligned} Ua_k U^\dagger &= a_k S - a_K R\ , \\ Ua_K U^\dagger &= a_K S + a_k R\ . \end{aligned} \tag{16}$$

As mentioned by Prasad *et al.*,[9] there are two possible ways of considering the splitting. In the first one, we consider that the operator U acts on the states, and mix the two modes **k** and **K**. This approach gives the best physical understanding of the process, especially if we consider that only one mode (e.g., **k**) is initially excited. However, the calculations are generally complicated. In the second approach, we deal only with quantum averages (i.e., measurable quantities) and we can therefore make U to act on the measured operators (e.g., $a_k^\dagger a_k$). Since the expression of these operators in terms of a's and $a^\dagger$'s is generally much simpler than the expression of the

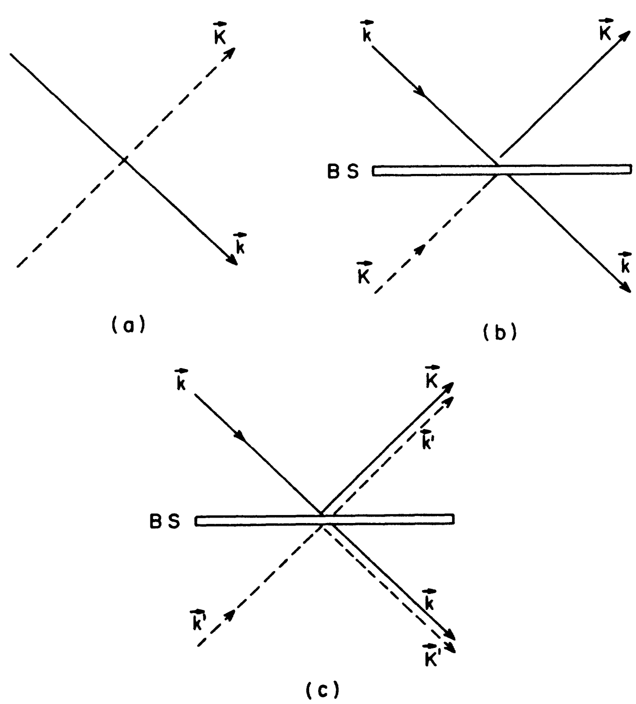

FIG. 1. (a) Modes **k** and **K** with no mixing. (b) Mixing between modes **k** and **K** by a beam splitter (BS). The mode **K** (**k**) incident on the BS produces a reflected mode, which coincides with the transmitted mode **k** (**K**). (c) Modes **k** and **k**′ with different frequencies incident on the BS from its two sides produce four modes: the transmitted **k** and **k**′ modes and the reflected **K** and **K**′ modes.

states, the calculations are simplified accordingly. Let us point out that the same analysis can be used for any device that separates a light beam into two. An example would be a calcite crystal that splits the beam according to the polarization of the photons. It is also correct for a device that attenuates the beam in a random way. In this case, we only have to consider that the photons in the second mode (e.g., **k**) are later absorbed. In such cases, there is no room for the introduction of a real vacuum field entering through the unused side of the BS.

We now start with the first approach and suppose that the field is in a coherent state $|\alpha\rangle$, with respect to mode **k**, and that the mode **K** is not excited [Fig. 1(b)].

We write

$$|\alpha_k\rangle = D(a_k,\alpha)|0_k\rangle \equiv \exp(\alpha a_k^\dagger - \alpha^* a_k)|0_k\rangle , \quad (17)$$

where $D(a_k,\alpha)$ is the displacement operator creating a coherent state when acting on the vacuum. By acting with the BS, the state $|\alpha_k,0_K\rangle$ transforms into

$$U^\dagger|\alpha_k,0_K\rangle = [U^\dagger D(a_k,\alpha)U]U^\dagger|0_k,0_K\rangle . \quad (18)$$

But $U^\dagger|0_k,0_K\rangle = |0_k,0_K\rangle$, i.e., the vacuum is not changed and it is straightforward to show that

$$U^\dagger D(a_k,\alpha)U = D(a_k,S\alpha)D(a_K,R\alpha) , \quad (19)$$

so that

$$U^\dagger|\alpha_k,0_K\rangle = |S\alpha_k\rangle\otimes|R\alpha_K\rangle . \quad (20)$$

The coherent state is transformed into two independent (direct product) coherent states, one in each mode. It has been shown by Aharonov *et al.*[7] that this is a unique property of the coherent state, i.e., any other state would be mixed into two *correlated* states. We can, for example, analyze the number state case. So we write the initial state as $|n_k,0_K\rangle$. It is transformed into

$$U^\dagger|n_k,0_K\rangle = \left[U^\dagger\frac{(a_k^\dagger)^n}{(n!)^{1/2}}U\right]|0_k,0_K\rangle . \quad (21)$$

We get, by using the basic formula (16),

$$U^\dagger|n_k,0_K\rangle = \sum_{p=0}^{n} S^p(R)^{n-p}\begin{bmatrix} n \\ p\end{bmatrix}|p_k\rangle\otimes|(n-p)_K\rangle . \quad (22)$$

This state is not a direct product of a state in mode **k** and a state in mode **K**, so that the two beams are now correlated.

If we perform only experiments on *one side* (e.g., **k**) of the BS, we need to describe the state by a density matrix and perform a partial trace on the part related to **K** (cf. Cohen-Tannoudji *et al.*[21]). We get

$$\rho_k = \sum_{p=0}^{n} S^{2p}(R)^{2(n-p)}\begin{bmatrix} n \\ p\end{bmatrix}|p_k\rangle\langle p_k| . \quad (23)$$

This operator fully describes the field, after splitting, for the kth mode. For example, we get

$$\langle N_k\rangle \equiv \mathrm{Tr}(\rho_k a_k^\dagger a_k) = S^2 n , \quad (24)$$

and for the noise,

$$\langle \Delta N_k^2\rangle = S^2R^2 n , \quad (25)$$

where, for a pure number state, we should have

$$\langle \Delta N^2\rangle = 0 . \quad (26)$$

We get also for the second-order correlation function

$$g^{(2)}(0) - 1 = -\frac{1}{n} = -\frac{S^2}{S^2 n} , \quad (27)$$

whereas for a number state *with* S^2n *photons,*

$$g^{(2)}(0) - 1 = -\frac{1}{S^2 n} , \quad (28)$$

so that the antibunched character of this light is decreased by the beam splitter. We therefore have the straightforward physical interpretation of the process. The number state is split into two correlated states. They are not number states anymore, but only a mixture of various number states. Of course, the same approach can be used for any kind of state, but the calculations are then much more complicated. We introduce, for example, squeezed states (SS) defined by

$$|a_k,\alpha,r,\varphi\rangle \equiv D(a_k,\alpha)S(a_k,r,\varphi)|0_k\rangle , \quad (29)$$

where $D(a_k,\alpha)$ is the displacement operator defined in (17) and

$$S(a_k,r,\varphi) \equiv \exp[r(a_k^2 e^{-2i\varphi} - a_k^{\dagger 2}e^{2i\varphi})] \quad (30)$$

is the squeeze operator.[22] To find explicitly the transformed state, we need to calculate

$$US(a_k,r,\varphi)U^\dagger = \sum_{n=0}^{\infty}\frac{r^n}{n!}U(a_k^2 e^{-2i\varphi} - a_k^{\dagger 2}e^{2i\varphi})^n U^\dagger , \quad (31)$$

and this is quite complicated. We therefore turn to the second method.

This method relies on the fact that we do not really need to calculate how a state transforms. The only possible result of a measurement is always a quantum average, so that the kind of terms we need to calculate are, for a state $|Q\rangle$,

$$\langle Q|UF(a_k,a_k^\dagger,a_K,a_K^\dagger)U^\dagger|Q\rangle . \quad (32)$$

Now, in general, the function F is expressed in terms of $a,a^\dagger$'s in a much simpler way than the state, so that we may prefer to calculate the effect of U and $U^\dagger$ on the operator F rather than on the state $|Q\rangle$. (This is described by Prasad *et al.*[9] as the Heisenberg approach.)

An interesting point appears here when one considers only the splitting of *one* mode of the e.m. field incident on the BS, e.g., **k**. In this case the state $|Q\rangle$ is expressed only in terms of creation operators $a_k^\dagger$. Therefore, *as long as the operator* $F(a_k,a_k^\dagger,a_K,a_K^\dagger)$ *is normal ordered,* all the contributions from the $a_K,a_K^\dagger$ are canceled.

If we write

$$F = \sum_{n,m,p,q} C_{n,m,p,q}\,a_k^{\dagger n}a_K^{\dagger m}a_k^p a_K^q , \quad (33)$$

then

$$UFU^\dagger = \sum_{n,m,p,q} C_{n,m,p,q}(Sa_k^\dagger - Ra_K^\dagger)^n (Sa_K^\dagger + Ra_k^\dagger)^m \times (Sa_k - Ra_K)^p (Sa_K + Ra_k)^q \ . \quad (34)$$

Now

$$\langle Q \mid UFU^\dagger \mid Q \rangle = \left\langle Q \middle| \sum_{n,m,p,q} C_{n,m,p,q}(Sa_k^\dagger)^n (Ra_k^\dagger)^m \times (Sa_k)^p (Ra_k)^q \middle| Q \right\rangle = \langle Q \mid F(Sa_k, Sa_k^\dagger, Ra_k, Ra_k^\dagger) \mid Q \rangle \quad (35)$$

(all terms with $a_K, a_K^\dagger$ are zero); so that we see that, in this case, we only need to replace the operators as follows:

$$a_k \to Sa_k \ , \qquad a_K \to Ra_k \ . \quad (36)$$

When the operator F is not normal ordered (no), then the terms $a_K, a_K^\dagger$ can give a contribution via the commutation relations. This contribution is generally understood (cf. Yuen and co-workers[8]) as a vacuum noise. We now understand that it comes from the CR of the vacuum field. However, we have seen that a much more direct interpretation of the splitting is given by the splitting of a state. A coherent state is split into two uncorrelated coherent states, and no noise is added in the process. A number state, on the other hand, is split into two correlated states, none of them being in a number state anymore. Therefore, in this case, the splitting process adds noise. However, this does not depend in any way on the introduction of a real vacuum field at the "unused port" of the BS. This is even clearer for another kind of splitting of an absorbing device for which there is no unused port. Moreover, we have shown in the previous discussion that we can always use normally ordered expressions and, more specifically, the second-order correlation function to get the measurable quantities. In this case there is no vacuum noise introduced by the beam splitter.

IV. APPLICATIONS

A. Attenuation of a state

Let us take a field and attenuate it by means of a BS. The BS not only reduces the intensity as it would do classically, but also changes the photon distribution. We shall characterize this by means of the second-order correlation function. Since we work with normal-ordered expressions, we simply replace the creation operator $a^\dagger$ by $Sa^\dagger$, and immediately see that the normalized function is unchanged. However, noting that the average number of photons in the field has been reduced, we have to compare the counting statistics of the attenuated field with the ones obtained with a field of equal intensity [cf. Eqs. (27) and (28)]. We shall analyze, as a typical example of quantum light, the case of the number state. We start with a number state $\mid n \rangle$. After splitting, we have seen that the number of photons in the transmitted beam is $\langle n_s \rangle = S^2 \langle n \rangle$. So we now compare this field (A) with a field (B) in a number state $\mid S^2 n \rangle$. We use Eqs. (5) and (8) to get, for field A,

$$\langle \Delta m_A^2 \rangle_T = \langle m_A \rangle_T \left[1 - S^2 \alpha \frac{T}{T_c} \right] , \quad (37a)$$

and for field B,

$$\langle \Delta m_B^2 \rangle_T = \langle m_B \rangle_T \left[1 - \alpha \frac{T}{T_c} \right] . \quad (37b)$$

So we now see that for $S^2 \to 0$ (strong attenuation) the noise tends to the SQL. Indeed, whenever we split a quantum state, we lose part of the possible noise reduction.

B. Determination of the SQL for the intensity-fluctuation spectrum

A standard way of determining the SQL for the intensity-fluctuation spectrum is by means of "a balanced scheme" [Fig. 2(a)]. We subtract the two currents to get

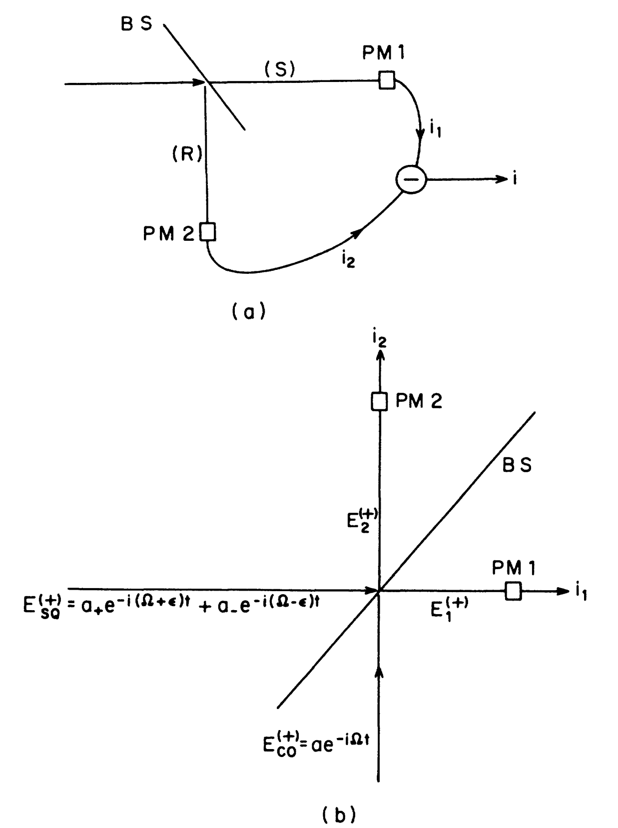

FIG. 2. (a) Light beam incident on the BS is partially transmitted (with transmittivity coefficient S) and partially reflected (with reflectivity coefficient R). The currents i_1 and i_2 are measured by PM 1 and PM 2 and are subtracted by the frequency analyzer giving the SQL noise. (b) Coherent field with $E_{\rm co}^{(+)} = ae^{-i\Omega t}$ and squeezed field with $E_{\rm sq}^{(+)} = a_+ e^{-i(\Omega+\epsilon)t} + ae^{-i(\Omega-\epsilon)t}$ incident on the BS from its two sides produce the two fields with $E_1^{(+)}$ and $E_2^{(+)}$, which are measured by PM's 1 and 2 to produce the currents i_1 and i_2, respectively.

$\langle i_1 \rangle - \langle i_2 \rangle = \langle i \rangle$ and perform a frequency analysis. As explained in Sec. II, we need to calculate two kinds of correlation functions,

$$g^{(2)}(x_1,t;x_1,t+\tau), \quad g^{(2)}(x_1,t;x_2,t+\tau) ,$$

where x_1 and x_2 refer to the locations of PM's 1 and 2. Using the results of Sec. III to express these functions in terms of creation and annihilation operators, it is easy to see that these two functions are equal, and also equal to the correlation function of the beam before splitting, which we shall denote by $g^{(2)}(\tau)$. Then, by the results of Sec. II, we immediately get

$$N^2(\omega)=\frac{eG}{2\pi}(\langle i_1 \rangle + \langle i_2 \rangle) + \frac{(\langle i_1 \rangle - \langle i_2 \rangle)^2}{2\pi}\int_{-\infty}^{\infty}[g^{(2)}(\tau)-1]e^{-i\omega\tau}d\tau . \tag{38}$$

So we see that when we subtract the photocurrents, the SQL noise in the two branches has still to be added [which is fully consistent with the fact that the splitting of a coherent state (CS), which has the SQL noise, gives two independent CS's]. However, the "excess noise," which can be negative for quantum light, is subtracted. For a 50-50 BS, for example, we find that

$$N^2(\omega)=\frac{eG}{2\pi}(\langle i_1 \rangle + \langle i_2 \rangle) . \tag{39}$$

Remembering that by a direct measurement of the field we would get a photocurrent $\langle i \rangle = \langle i_1 \rangle + \langle i_2 \rangle$, we therefore see that this noise is the SQL of the field under investigation (i.e., the noise that we should get by a direct detection of a coherent state of identical intensity). Let us, however, emphasize that in the first case (50-50 BS and subtraction) the average current $\langle i \rangle$ is zero, so that we have only noise and no signal, whereas by direct measurement we have a signal $\langle i \rangle$ and a noise, whose corresponding SQL is the previously obtained one.

C. Heterodyning

We are interested in heterodyning a squeezed signal and a much stronger LO in a coherent state. The squeezed signal is assumed to be a two-frequency signal $(\Omega+\varepsilon, \Omega-\varepsilon)$, where the squeezing introduces correlations between the two frequencies.[23] The LO is in frequency Ω. We are going to use only n.o. expressions, so that there is no need to introduce any vacuum field operator. Writing only the relevant part of the operators on each side of the BS as described schematically in Fig. 2(b), we get

$$\begin{aligned} E_1^{(+)}(t)&=S(a_+e^{-i(\Omega+\varepsilon)t}+a_-e^{-i(\Omega-\varepsilon)t}) + Re^{-i\Omega t}a , \\ E_2^{(+)}(t)&=-R(a_+e^{-i(\Omega+\varepsilon)t}+a_-e^{-i(\Omega-\varepsilon)t}) + Se^{-i\Omega t}a . \end{aligned} \tag{40}$$

These expressions and their complex conjugates (CC's) are valid *only* when we use them in n.o. expressions. For our purpose we need to calculate $g^{(2)}(\tau)-1$ (same space point) and $g^{(2)}(x_1,t;x_2,t+\tau)-1$. We assume a strong coherent LO and calculate only terms that are of second order in a and $a^\dagger$. (Terms which are of orders three or four in the LO operators give vanishing contributions, while terms which are of first or zeroth order in a and $a^\dagger$ lead to very small contributions, which are neglected within our approximations.) The calculations for $g^{(2)}(\tau)-1$ are very similar to those performed in our previous work[20,23] [using Eqs. (4.16) of Ref. 20]. For the field E_1 we get

$$g_1^{(2)}(\tau)-1=\frac{2S^2}{R^2}\frac{\cos(\varepsilon\tau)}{\langle n \rangle}\mathcal{S} , \tag{41}$$

where

$$\mathcal{S}\equiv 2\sinh^2 r - 2\cosh r \sinh r \cos[2(\phi_c-\phi)] . \tag{42}$$

$\langle n \rangle$ is the number of photons in the strong coherent state, $\phi_c-\phi$ is the difference between the LO phase ϕ_c and the squeezed field phase ϕ, and r is the squeeze factor. It is easy to see that $\mathcal{S}$ can take any value between $-(1-e^{-2r})$ and $(e^{2r}-1)$ when we change the relative phase $(\phi_c-\phi)$ so that there exists a purely quantum domain of variation $g_1^{(2)}(0)-1<0$.

Equation (41) is correct only under the approximation that the reflected coherent field is much stronger than the transmitted squeezed field. For the field E_2 we get a similar result for $g_2^{(2)}(\tau)-1$, with S^2/R^2 replaced by R^2/S^2. This result will be correct under the condition that the transmitted coherent field is much stronger than the reflected squeezed field.

For a 50-50 BS we get, for both E_1 and E_2, the same result,

$$\begin{aligned} g^{(2)}(\tau)-1 &\equiv g_1^{(2)}(\tau)-1 \equiv g_2^{(2)}(\tau)-1 \\ &=2\frac{\cos(\varepsilon\tau)}{\langle n \rangle}\mathcal{S} , \end{aligned} \tag{43}$$

This result is similar to that derived in our previous work,[20] with the following two differences.

(a) We do not obtain here factors like $(\Omega+\varepsilon/\Omega)^{1/2}$ $(\varepsilon \ll \Omega)$ in the field, since we use the field of Eq. (3), dropping the factor $(\hbar\omega_k)^{1/2}$.

(b) The result given in Eq. (50) is smaller by a factor of 2, since we did not include there the effect of the BS on photon statistics. Indeed, the noise obtained in our previous work corresponds to the calculation of the direct interference between the SS and the LO without the BS.

The possibility of reducing the quantum noise up to zero for a unity quantum efficiency detector follows from the subtraction scheme, where one measures the quantum noise of i_1-i_2 and where the noise should be calculated according to Eq. (15). $N_1^2(\omega)$ [or $N_2^2(\omega)$] is calculated according to Eq. (13), replacing $\langle i \rangle$ by $\langle i_1 \rangle$ (or $\langle i_2 \rangle$) and substituting the expression for $g^{(2)}(\tau)-1$ from (41) or (43). To calculate the full noise $N^2(\omega)$, we now need

$g^{(2)}(x_1,t;x_2,t+\tau)$.

We get, by straightforward calculations,

$$g^{(2)}(x_1,t;x_2,t+\tau)-1=-2\frac{\cos(\varepsilon\tau)}{\langle n\rangle}\mathcal{S} . \quad (44)$$

We find here that the previous quantum domain [$\mathcal{S}<0$ corresponding to $g^{(2)}(0)<1$] is now replaced by

$$g^{(2)}(x_1,t;x_2,t)>1 . \quad (45)$$

This has been emphasized recently by Ou, Hong, and Mandel,[24] who have shown that these positive correlations at two different points exist only for nonclassical states.

In Eqs. (41) and (44) the time τ enters only through the term $\cos(\varepsilon\tau)$. This, of course, is not completely satisfactory, since we know that $g^{(2)}(\tau)-1$ has to go to 0 for large times ($\tau>T_c$).[17] We may introduce this dependence in our expression in the following two ways.

(i) We may artificially introduce a cutoff in the Fourier transform (FT) for an interval T_c,

$$\int_{-\infty}^{\infty}\cos(\varepsilon\tau)e^{-i\omega\tau}d\tau\to\int_{-T_c/2}^{T_c/2}\cos(\varepsilon\tau)e^{-i\omega\tau}d\tau =\frac{T_c}{2}\{j_0[(\omega+\varepsilon)T_c] +j_0[(\omega-\varepsilon)T_c]\} . \quad (46)$$

(ii) In a better way we may multiply the integrand by an exponentially decreasing factor $e^{-2|\tau|/T_c}$,

$$\int_{-\infty}^{\infty}\cos(\varepsilon\tau)e^{-i\omega\tau}e^{-2|\tau|/T_c}d\tau =\frac{T_c}{2}\left[\frac{1}{1+[(\omega+\varepsilon)T_c]^2}+\frac{1}{1+[(\omega-\varepsilon)T_c]^2}\right] . \quad (47)$$

These two expressions give the same kind of dependence,

$$\int_{-\infty}^{\infty}[g^{(2)}(\tau)-1]e^{-i\omega\tau}d\tau =\frac{\mathcal{S}}{\langle n\rangle}T_c[f((\omega-\varepsilon)T_c)+f((\omega+\varepsilon)T_c)] , \quad (48)$$

where

$$f(x)=j_0(x)x \quad\text{or}\quad \frac{1}{1+x^2} , \quad (49)$$

or more generally, a function that is 1 for $x=0$ and is rapidly decreasing to 0 for $|x|>1$.

We can now easily calculate the noise. Using Eqs. (13), (15), (41), and (44) we get

$$N^2(\omega)=\frac{eG}{2\pi}(\langle i_1\rangle+\langle i_2\rangle) +\frac{1}{2\pi}\frac{\mathcal{S}}{\langle n\rangle}T_c[f((\omega-\varepsilon)T_c)+f((\omega+\varepsilon)T_c)] \times\frac{(S^2\langle i_1\rangle+R^2\langle i_2\rangle)^2}{R^2S^2} . \quad (50)$$

For a 50-50 BS,

$$\langle i_1\rangle=\langle i_2\rangle=\frac{\langle i\rangle}{2} , \quad (51)$$

and by the relation $\langle i\rangle=eG\alpha\langle n\rangle/T_c$ we obtain

$$N^2(\omega)=\frac{eG}{2\pi}\langle i\rangle[1+\alpha\mathcal{S}(f((\omega-\varepsilon)T_c)+f((\omega+\varepsilon)T_c))] . \quad (52)$$

We immediately see that if $(\omega+\varepsilon)T_c\gg1$, the corresponding term $f((\omega+\varepsilon)T_c)$ is negligible and the second one is important only for $(\omega-\varepsilon)T_c\ll1$. We obtain once again the result of our previous article,[20] that for a nondegenerate squeezed state there is a noise reduction only close to the modulation frequency ε. However, we can now get a more quantitative result: *close to* means that $(\omega-\varepsilon)T_c\ll1$. One should notice, however, that to get the cutoff at T_c, we need to assume that $T>T_c$, i.e., the opening time of the detector has to be long enough, otherwise the FT has to be taken in the limits of $\int_{-T/2}^{T/2}$, since it cannot correlate the intensity for times larger than T.

It is interesting to examine what is happening for $(\omega+\varepsilon)T_c\ll1$. This condition is not only a condition on the frequency of analysis, but it also implies that $\varepsilon T_c\ll1$. In this case the frequency difference in the two parts of the signal is so small that we can indeed consider it to be a one-frequency signal, and take

$$g^{(2)}(\tau)-1=\frac{2\mathcal{S}}{\langle n\rangle} \quad (53)$$

[since this relation is only correct for $\tau\ll T_c$, we take $\cos(\omega\tau)=1$]. Now Eq. (52) shows that T_c^{-1} is a cutoff frequency: For $\omega\ll T_c^{-1}$, noise can be reduced under the SQL. It goes back to it for $\omega\gg T_c^{-1}$. We also see that ε does not play any role anymore. Therefore we now have a quantitative definition of degenerate and nondegenerate squeezed states, and it shows that the important parameter is not only the frequency difference between the two parts, but also the coherence time T_c. Indeed, for $\varepsilon T_c\ll1$, the squeezed state can be considered as degenerate, whereas for $\varepsilon T_c\gg1$, it is nondegenerate. These conclusions seem to be consistent with recent experimental observations.

The observations by Slusher *et al.*[4] correspond to a nondegenerate case: $\varepsilon=421.5$ MHz, noise reduction close to ε, in a bandwidth of 3 MHz. The observations by Wu *et al.*[5] correspond to the degenerate case: observation of noise reduction at $\omega=1.8$ MHz with no indication of frequency dependence (for $\omega<T_c^{-1}$).

In our previous article[20] we have explained the spectral width in the noise reduction by assuming a band of frequencies, which are squeezed. Here we obtain the bandwidth by the coherence time T_c. For sufficiently monochromatic light the concept of coherence time T_c is useful. When the squeezing really introduces many frequencies in a band $\Omega\pm\varepsilon$, then the approach of our previous article[20] is more suitable.

We now analyze further the nondegenerate case, where the only nonzero component is $f((\omega-\varepsilon)T_c)$. For a 50-50 BS the use of Eq. (52) gives

$$N^2(\omega)=\frac{eG}{2\pi}\langle i\rangle[1+\alpha f((\omega-\varepsilon)T_c)] . \quad (54)$$

This result is interesting as, by the use of two-port homodyning, the quantum noise can tend to zero for large squeezing ($\mathcal{S}=-1$), for a unity quantum efficiency and $\omega\simeq\varepsilon$. The noise obtained by (54) is the quantum noise we would have gotten by direct interference between the SS and the LO without BS. However, the advantage of the two-port homodyning is that, due to the subtraction scheme, we eliminate classical noise that we would get by the direct interference experiment.

We compare (54) with the result for one-port measurement obtained by the use of Eqs. (13), (41), (48) and by the use of the relation $\langle i_1\rangle=eG\alpha\langle n\rangle T_c^{-1}R^2$,

$$N_1^2(\omega)=\frac{eG}{2\pi}\langle i_1\rangle[1+S^2\alpha\mathcal{S}f((\omega-\varepsilon)T_c)] . \quad (55)$$

In the 50-50 BS one-port measurement, the noise can be reduced for $\omega\simeq\varepsilon$ only to $\frac{1}{2}$ of the SQL.

The subtraction scheme is not useful for applications in the transmission of information, since the signal $\langle i_1\rangle-\langle i_2\rangle$ is zero for the 50-50 BS, and we obtain only noise.

For one-port homodyning with $S^2\to 1$, but with $R^2\langle n\rangle$ still quite large (relative to the SS), we get the best quantum noise reduction with a beam, which is directly utilizable. However, one needs to have perfect intensity stability to avoid classical noise. Since the LO is much stronger than the SS, even a small excess noise would be enough to cancel the possible noise reduction. For this reason this scheme has not been used experimentally.

V. CONCLUSIONS

We have analyzed the influence of a BS on the photon statistics for different cases and have shown that the BS may affect the states of radiation in one approach and how it may affect the operators in an alternative approach. By using n.o. expressions it is found that there is no need to insert the concept of vacuum noise, which seems to be quite confusing. The present analysis is in agreement with the basic formulas presented previously in Refs. 7 and 9.

We have shown in the present article how the measurement of quantum noise in different experiments is related to second-order correlation functions. In particular, we have analyzed heterodyne detection experiments explaining the result of recent experiments.[4–6] Since in our previous work[20] we have not considered the effects of the BS on the photon statistics, this work is complementary to the previous one, following the same approach that the measurements are related to second-order correlation functions and not to the measurement of quadratures.[20]

Our work can be also considered as complementary to the work of Ou *et al.*,[10] who adopted a very similar presentation, but restricted themselves to different cases. It can be also related to the paper of Fearn and Loudon,[11] who adopted a different point of view by quantizing the field in terms of the spatial modes of the complete system. However, their approach is also restricted to two modes at the same frequency, so that they do not analyze the nondegenerate squeezed state.

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