Bifurcations in a three-mode model of the Navier-Stokes equation

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We analytically follow the nonlinear evolution of three coupled modes of the two-dimensional Navier-Stokes equation in an externally driven and dissipative case. As the external force amplitude (control parameter) is increased, a node-to-node bifurcation appears, giving rise to new stable equilibria. We find that the most relevant set of three modes contributes to the energy and enstrophy cascades in the same directions as in fully developed turbulence. We model the coupling with the remaining modes as a white-noise contribution, deriving the equilibrium distribution function for the corresponding Fokker-Planck equation.

I. INTRODUCTION

The evolution of a viscous incompressible fluid is governed by the Navier-Stokes equation for its divergenceless velocity field.[1] This nonlinear equation, when linearized, gives purely damped modes,[2] due to the effect of viscosity. The nonlinear term turns out to be important when the viscosity is small (Reynolds number $R=uL/\nu\gg 1$) and represents the coupling among the modes. In such a case, energy may be transferred from one mode to another, but in the long run the total energy will decrease. In order to achieve a nontrivial stationary state it is necessary to force the system externally.

The relative importance of nonlinearity, dissipation, and external forcing may be analyzed in a very simple model of three modes. This approach has often been used in connection with different turbulences,[2–4] mainly as a way for determining the net flux of the corresponding ideal invariants in $\mathbf{k}$ space (cascades). For the two-dimensional Navier-Stokes turbulence, these invariants are the energy

$$W=\tfrac{1}{2}\int d^2x\ |\mathbf{u}|^2 \tag{1.1}$$

and the enstrophy

$$\Omega=\tfrac{1}{2}\int d^2x\ |\nabla\times\mathbf{u}|^2 \tag{1.2}$$

which, respectively, cascade to lower and higher wave numbers.[5] In this work we will follow the three-mode approach, modeling the effects of the remaining couplings as white noise.

The organization of the paper is as follows. In Sec. II we derive the basic general equations which have a trivial equilibrium solution. In Sec. III we analyze the linear stability of this equilibrium for the truncated three-mode problem. In Sec. IV we follow the nonlinear evolution of these modes when the old equilibrium turns linearly unstable, finding new exact equilibria. In Sec. V we study the effect of noise on these equilibria. In Sec. VI we discuss the conditions under which the evolution of a single set of three modes (triad) contains the relevant features of the whole system. We also analyze the contribution of this triad to the general energy and enstrophy cascade. Finally in Sec. VII the conclusions are summarized.

II. BASIC EQUATIONS

The hydrodynamic equations for a viscous and incompressible fluid are

$$\partial_t\mathbf{u}=-(\mathbf{u}\cdot\nabla)\mathbf{u}-\nabla p+\nu\nabla^2\mathbf{u}+\mathbf{F}\ , \tag{2.1a}$$

$$\nabla\cdot\mathbf{u}=0\ , \tag{2.1b}$$

where $\mathbf{u}(\mathbf{x},t)$ is the velocity field, p is the pressure per unit mass density, ν is the kinematic viscosity, and $\mathbf{F}$ is an external driver acting on the fluid.

In a two-dimensional fluid

$$\mathbf{u}=\mathbf{u}(x,y,t)\ , \tag{2.2a}$$

$$\mathbf{u}\cdot\hat{\mathbf{z}}=0\ . \tag{2.2b}$$

As a consequence of the incompressibility condition (2.1b) and the hypothesis (2.2), the velocity field can be expressed in terms of a scalar stream function $\psi(x,y,t)$

$$\mathbf{u}=\nabla\times(\hat{\mathbf{z}}\psi)\ . \tag{2.3}$$

Taking the curl of (2.1a) we get

$$\partial_t\nabla^2\psi=(\nabla\psi\times\hat{\mathbf{z}})\cdot\nabla(\nabla^2\psi)-\nu\nabla^4\psi+\nabla^2 f\ , \tag{2.4}$$

where we have written the vector field $\mathbf{F}(x,y,t)$ as

$$\mathbf{F}=\nabla\times(\hat{\mathbf{z}}f)+\nabla g\ . \tag{2.5}$$

Expressing the fields ψ and f in terms of their corresponding Fourier amplitudes

$$\psi(x,y,t)=\sum_{\mathbf{k}}\psi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}},\quad \psi_{\mathbf{k}}^*=\psi_{-\mathbf{k}}\ , \tag{2.6a}$$

$$f(x,y,t)=\sum_{\mathbf{k}}f_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}},\quad f_{\mathbf{k}}^*=f_{-\mathbf{k}}\ , \tag{2.6b}$$

Eq. (2.4) reduces to

$$\partial_t \psi_{\mathbf{k}} + \nu k^2 \psi_{\mathbf{k}} = f_{\mathbf{k}} + \sum_{\substack{\mathbf{k}',\mathbf{k}'' \\ (\mathbf{k}'+\mathbf{k}''=\mathbf{k})}} \Gamma^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''} \psi_{\mathbf{k}'} \psi_{\mathbf{k}''} , \tag{2.7}$$

where

$$\Gamma^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''} = \frac{\hat{\mathbf{z}}\cdot\mathbf{k}\times\mathbf{k}'}{2} \left[\frac{k''^2 - k'^2}{k^2} \right] . \tag{2.8}$$

We suppose a square $L \times L$ box with periodic boundary conditions, so that

$$\mathbf{k} = \frac{2\pi}{L}(n_x, n_y) \tag{2.9}$$

with n_x and n_y integer numbers.

It is well known that in a two-dimensional incompressible fluid, the energy [Eq. (1.1)] and enstrophy [Eq. (1.2)] are conserved when forcing and dissipation are absent.[5] In terms of the amplitudes $\psi_{\mathbf{k}}$, these global invariants take the form

$$W = \tfrac{1}{2} \sum_{\mathbf{k}} k^2 |\psi_{\mathbf{k}}|^2 , \tag{2.10a}$$

$$\Omega = \tfrac{1}{2} \sum_{\mathbf{k}} k^4 |\psi_{\mathbf{k}}|^2 . \tag{2.10b}$$

Moreover these quantities are the invariants of any truncation of the Fourier space. If viscosity is present, however small, these quantities decay in time unless an external force is applied to the system. In Sec. VI we study the behavior of these ideal invariants in the forced and dissipative case, for the reduced problem of three Fourier modes.

We look for the equilibria of Eq. (2.7) when a harmonic external force drives the system

$$f_{\mathbf{k}} = f \delta_{\mathbf{k},\mathbf{k}_0} \tag{2.11}$$

with $f = -f^*$ (any other particular choice for the phase of f may be absorbed by a redefinition of the coordinates origin to lead $f = -f^*$). There is a trivial equilibrium

$$\psi_{\mathbf{k}_0} = \frac{f}{\nu k_0^2}, \quad \psi_{\mathbf{k}} = 0 \quad \forall\ \mathbf{k} \neq \mathbf{k}_0 , \tag{2.12}$$

whose stability we study below. For this purpose we turn to a set of dimensionless variables using $x_0 = 2\pi/k_0$ and $t_0 = 1/\nu k_0^2$ as characteristic length and time, respectively. This is equivalent to take $\nu = 1$ and $k_0 = 1$.

III. LINEAR STABILITY AND THREE-MODE TRUNCATION

Once an equilibrium solution is found, the question about its stability immediately arises. For the moment we will only concentrate on the linear stability of solution (2.12). In order to do so, we perform an expansion of the form $\boldsymbol{\psi} = \boldsymbol{\psi} + \delta\boldsymbol{\psi}$ (where we denote by $\boldsymbol{\psi}$ a vector whose components are the Fourier amplitudes $\psi_{\mathbf{k}}$), insert it in the set (2.7) and retain terms up to first order in $\delta\boldsymbol{\psi}$. This procedure leads us to the following set of equations:

$$\delta\dot{\boldsymbol{\psi}} = \underline{A} \cdot \delta\boldsymbol{\psi} , \tag{3.1}$$

where the matrix $\underline{A}$ is

$$A_{ij} = \begin{cases} -k^2 & \text{if } i = j \\ \Gamma^{\mathbf{k}_i}_{\mathbf{k}_j,\mathbf{k}_0} f & \text{if } \mathbf{k}_i = \mathbf{k}_j + \mathbf{k}_0 \\ \Gamma^{\mathbf{k}_i}_{\mathbf{k}_j,-\mathbf{k}_0} f & \text{if } \mathbf{k}_i = \mathbf{k}_j - \mathbf{k}_0 \\ 0 & \text{otherwise} . \end{cases} \tag{3.2}$$

Let us consider now two modes $\mathbf{k}$ and $\mathbf{k}'$ such that $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{k}_0$ close a triangle as it is shown in Fig. 1 ($\mathbf{k} = \mathbf{k}_0 + \mathbf{k}'$). If the evolution of $\mathbf{k}$, $\mathbf{k}_0$, and $\mathbf{k}'$ is most affected by the couplings among themselves, we can neglect the interaction with modes outside this triangle and the problem will be reduced to that of the interaction among three modes. This is a standard simplifying procedure[2-4] and we will follow it in this case. However, we will take the influence of the other couplings into account through a noise term added to the evolution equations for the three modes of interest. We will not give particular values for $\mathbf{k}$ and $\mathbf{k}'$ until Sec. VI. It is necessary to stress that, due to the symmetries of the problem, there are always four similar triangles, the ones shown in Fig. 1. This is so because the coefficients Γ involved have the same absolute value. It is therefore necessary to analyze the evolution of ten modes simultaneously. However, the problem is not complicated since the matrix $\underline{A}$, which gives their linear evolution, is formed of two identical 1×1 blocks (the ones that determine the evolution of $\mathbf{k}_0$ and $-\mathbf{k}_0$) and four 2×2 blocks (each one corresponding to one of the triangles of Fig. 1). The last ones have the same pair of eigenvalues

$$\lambda_{\pm} = -\beta \pm \sqrt{\beta^2 - \gamma} , \tag{3.3}$$

where

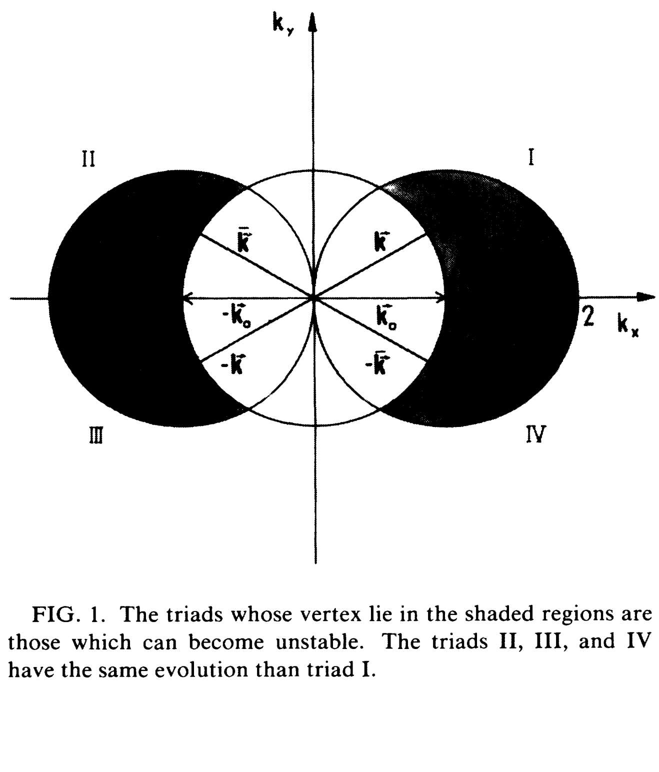

FIG. 1. The triads whose vertex lie in the shaded regions are those which can become unstable. The triads II, III, and IV have the same evolution than triad I.

$$\beta=\frac{k^2+k'^2}{2}, \quad \gamma=k^2k'^2\left[1-|f|^2\frac{\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}'}}{k^2k'^2}\right] \tag{3.4}$$

with eigenvectors of the form

$$\psi_{\pm}=\Gamma^{\mathbf{k}}f\psi_{\mathbf{k}'}-(\lambda_{\mp}+k^2)\psi_{\mathbf{k}} \ . \tag{3.5}$$

The eigenvalues of the matrix $\underline{A}$ are those in (3.3) and

$$\lambda_0=-1 \tag{3.6}$$

with eigenvectors $\delta\psi_{\mathbf{k}_0},\delta\psi_{-\mathbf{k}_0}$ which correspond to the viscous damping of the modes $\pm\mathbf{k}_0$.

It is clear from (3.3) and (3.4) that if $\beta^2-\gamma<0$, $\underline{A}$ has two complex eigenvalues λ_+ and λ_- ($\lambda_-=\lambda_+^*$) each of degeneracy four and a real eigenvalue $\lambda_0=-1$ of degeneracy two, all of them with negative real parts, meaning that the equilibrium (2.12) is stable. However, if $\beta^2-\gamma>0$, all the eigenvalues are real, but only λ_+ can become positive if and only if $\gamma<0$. From expression (3.4) we may see that γ can be negative only if $\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}'}$ is positive, since this a necessary condition for the existence of a positive real eigenvalue λ_+. We display in Fig. 1 the regions of the (k_x,k_y) plane for which the product $\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}'}$ is positive. An "unstable triangle," one corresponding to a block with positive λ_+, must necessarily have its vertex inside this region. Given one of these triangles, its instability will be decided by the value of $|f|$. We will therefore define $\mu=|f|$ as the control parameter of the problem. Given a μ value, those triangles having

$$\mu_c(k,k')=\frac{kk'}{(\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}'})^{1/2}} \tag{3.7}$$

less than μ will be unstable. Thus, if we only consider a set of three modes $\mathbf{k}$, $\mathbf{k}_0$, and $\mathbf{k}'$, the equilibrium will turn unstable if and only if μ exceeds the critical value $\mu_c(k,k')$.

IV. NONLINEAR SATURATION

Let us consider now the nonlinear evolution equations for the modes we chose in the preceding section (the ones shown in Fig. 1). Remembering that relation (2.6a) is always satisfied, it is necessary to take into account only five of them. The corresponding set of equations is

$$(\partial_t+1)\psi_{\mathbf{k}_0}=f+\Gamma^{\mathbf{k}_0}(\psi_{\mathbf{k}'}^*\psi_{\mathbf{k}}-\psi_{\bar{\mathbf{k}}'}\psi_{\bar{\mathbf{k}}}^*) \ , \tag{4.1}$$

$$(\partial_t+k^2)\psi_{\mathbf{k}}=\Gamma^{\mathbf{k}}\psi_{\mathbf{k}_0}\psi_{\mathbf{k}'} \ , \tag{4.2a}$$

$$(\partial_t+k'^2)\psi_{\mathbf{k}'}=\Gamma^{\mathbf{k}'}\psi_{\mathbf{k}_0}^*\psi_{\mathbf{k}} \ , \tag{4.2b}$$

$$(\partial_t+k^2)\psi_{\bar{\mathbf{k}}}=-\Gamma^{\mathbf{k}}\psi_{\mathbf{k}_0}^*\psi_{\bar{\mathbf{k}}'} \ , \tag{4.3a}$$

$$(\partial_t+k'^2)\psi_{\bar{\mathbf{k}}'}=-\Gamma^{\mathbf{k}'}\psi_{\mathbf{k}_0}\psi_{\bar{\mathbf{k}}} \ , \tag{4.3b}$$

where $\mathbf{k}=\mathbf{k}_0+\mathbf{k}'$ and $\bar{\mathbf{k}}=-\mathbf{k}_0+\bar{\mathbf{k}}'$. We may see that all five equations are coupled through the nonlinear term in (4.1). Nevertheless, it is possible to make further simplifications. It is clear that Eqs. (4.2) are similar to equations (4.3) if $\psi_{\mathbf{k}_0}=-\psi_{\mathbf{k}_0}^*$. Moreover, if $\psi_{\bar{\mathbf{k}}}=\psi_{\mathbf{k}}$ and $\psi_{\bar{\mathbf{k}}'}=\psi_{\bar{\mathbf{k}}'}$, as $f=-f^*$, $\psi_{\mathbf{k}_0}$ will always satisfy $\psi_{\mathbf{k}_0}=-\psi_{\mathbf{k}_0}^*$. Therefore, if at $t=0$, the following relations hold:

$$\psi_{\mathbf{k}_0}=-\psi_{\mathbf{k}_0}^* \ , \tag{4.3a'}$$

$$\psi_{\bar{\mathbf{k}}}=\psi_{\mathbf{k}} \ , \tag{4.3b'}$$

$$\psi_{\bar{\mathbf{k}}'}=\psi_{\mathbf{k}'} \ ; \tag{4.3c}$$

they will still hold for every time $t>0$. The problem reduces then, as in the linear case, to the study of the evolution of three coupled modes $\mathbf{k}_0$, $\mathbf{k}$, and $\mathbf{k}'$ governed by the equations

$$(\partial_t+k^2)\psi_{\mathbf{k}}=\Gamma^{\mathbf{k}}\psi_{\mathbf{k}_0}\psi_{\mathbf{k}'} \ , \tag{4.4a}$$

$$(\partial_t+k'^2)\psi_{\mathbf{k}'}=\Gamma^{\mathbf{k}'}\psi_{\mathbf{k}_0}^*\psi_{\mathbf{k}} \ , \tag{4.4b}$$

$$(\partial_t+1)\psi_{\mathbf{k}'}=f+\Gamma^{\mathbf{k}_0}(\psi_{\mathbf{k}'}^*\psi_{\mathbf{k}}-\psi_{\mathbf{k}'}\psi_{\mathbf{k}}^*) \ , \tag{4.4c}$$

where we have used relations (4.3) to obtain Eq. (4.4c). As we have already mentioned, we will take the coupling between $\psi_{\mathbf{k}}$ and $\psi_{\tilde{\mathbf{k}}}$ ($\tilde{\mathbf{k}}=\mathbf{k}+\mathbf{k}_0$) into account through a noisy term of the form $\Gamma'^{\mathbf{k}}_{\tilde{\mathbf{k}},-\mathbf{k}_0}\psi_{\mathbf{k}_0}^*\psi_{\tilde{\mathbf{k}}}$ added to Eq. (4.4a) and the coupling between $\psi_{\mathbf{k}'}$ and $\psi_{\tilde{\mathbf{k}}'}$ ($\tilde{\mathbf{k}}'=\mathbf{k}'-\mathbf{k}_0$) through a term $\Gamma^{\mathbf{k}'}_{\tilde{\mathbf{k}}',\mathbf{k}_0}\psi_{\mathbf{k}_0}\psi_{\tilde{\mathbf{k}}'}$ added to Eq. (4.4b). We will consider in both cases white noise of correlation

$$\langle\psi_{\tilde{\mathbf{k}}}(t)\psi_{\tilde{\mathbf{k}}}^*(t')\rangle=\tilde{Q}\delta(t-t')=\langle\psi_{\tilde{\mathbf{k}}'}(t)\psi_{\tilde{\mathbf{k}}'}^*(t')\rangle \ . \tag{4.5}$$

Couplings others than the ones mentioned above will be neglected. These terms of noise will not be added until Sec. V.

The set of nonlinear equations (4.4) has the equilibrium solution (2.12). There exist new equilibria for values of μ greater than the critical one (μ_c). This occurs simultaneously to the destabilization of (2.12) (see Fig. 2). We are therefore facing at a bifurcation from one node to another node.[6] The new equilibria are defined by

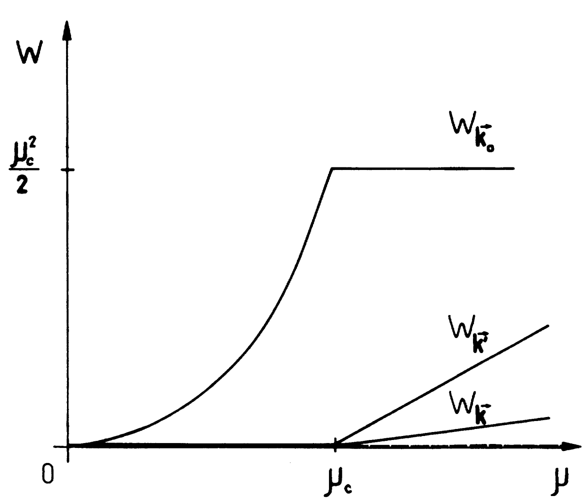

FIG. 2. Energy of the three modes as a function of the control parameter μ. μ_c indicates the bifurcation point.

$$|\psi_{\mathbf{k}}|^2=\frac{(\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}'})^{1/2}}{(-\Gamma^{\mathbf{k}'}\Gamma^{\mathbf{k}_0})}\frac{k'}{2k}(\mu-\mu_c)\ , \tag{4.6a}$$

$$|\psi_{\mathbf{k}'}|^2=\frac{(\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}'})^{1/2}}{(-\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}_0})}\frac{k}{2k'}(\mu-\mu_c)\ , \tag{4.6b}$$

$$|\psi_{\mathbf{k}_0}|^2=\frac{k^2k'^2}{\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}'}}\ , \tag{4.6c}$$

$$\varphi_{\mathbf{k}_0}=\varphi_f,\quad \varphi_f=\pm\frac{\pi}{2}\ , \tag{4.7a}$$

$$\varphi_{\mathbf{k}}-\varphi_{\mathbf{k}'}=\varphi_f\,\mathrm{sgn}(\Gamma^{\mathbf{k}})\ , \tag{4.7b}$$

where the φ's are the corresponding phases of the modes and $(-\Gamma^{\mathbf{k}}\Gamma^{\mathbf{k}_0})$, $(-\Gamma^{\mathbf{k}'}\Gamma^{\mathbf{k}_0})$ are both positive for triangles lying on the shaded region of Fig. 1.

The fact that Eq. (4.7b) defines only the phase difference between the complex amplitudes $\psi_{\mathbf{k}}$ and $\psi_{\mathbf{k}'}$, implies that relations (4.6) and (4.7) represent a continuum of equilibria, each one labeled by a different value of $\varphi_{\mathbf{k}}$ (or $\varphi_{\mathbf{k}'}$). Although the set of equations (4.4) and the trivial equilibrium solution (2.12) are invariant against changes of the form $\psi_{\mathbf{k}}\to\psi_{\mathbf{k}}e^{i\phi}$, $\psi_{\mathbf{k}'}\to\psi_{\mathbf{k}'}e^{i\phi}$, this does not remain true for the new equilibria (4.6) and (4.7), showing that the bifurcation gives rise to a breaking of symmetry.

We are interested now in the temporal evolution of the system for values of $\mu\gtrsim\mu_c$. Writing Eqs. (4.4) in terms of $\delta\psi_{\mathbf{k}_0}$ and of the normal modes ψ_+ and ψ_- [defined in (3.5)] we obtain

$$\partial_t\psi_\pm=\lambda_\pm\psi_\pm-\frac{\delta\psi_{\mathbf{k}_0}}{f(\lambda_+-\lambda_-)}[(\lambda_++k^2)(\lambda_\mp+\lambda_-+2k^2)\psi_+-(\lambda_-+k^2)(\lambda_\mp+\lambda_++2k^2)\psi_-]\ , \tag{4.8a}$$

$$\partial_t\delta\psi_{\mathbf{k}_0}=-\delta\psi_{\mathbf{k}_0}-\frac{\Gamma^{\mathbf{k}_0}}{(\lambda_+-\lambda_-)^2f\Gamma^{\mathbf{k}}}[2(\lambda_++k^2)|\psi_+|^2+2(\lambda_-+k^2)|\psi_-|^2+(k'^2-k^2)(\psi_+\psi_-^*-\psi_+^*\psi_-)]\ . \tag{4.8b}$$

Defining the small parameter

$$\epsilon=(\mu-\mu_c)^{1/2} \tag{4.9}$$

we make the perturbative expansion

$$\psi_\pm=\sum_{i=1}\psi_\pm^{(i)}\epsilon^i\ , \tag{4.10a}$$

$$\delta\psi_{\mathbf{k}_0}=\sum_{i=1}\delta\psi_{\mathbf{k}_0}^{(i)}\epsilon^i\ . \tag{4.10b}$$

We will stretch the time coordinate as

$$\tau=\epsilon^2t\ . \tag{4.11}$$

As the characteristic growth time of the instability is λ_+^{-1}, taking the scaling (4.11) means that we are following the evolution at the slow characteristic time λ_+^{-1}.

From (4.8a) at order ϵ^3, we obtain

$$\psi_-^{(3)}=\frac{\delta\psi_{\mathbf{k}_0}^{(2)}}{f\lambda_-^{(0)}(\lambda_+^{(0)}-\lambda_-^{(0)})}[(\lambda_+^{(0)}+k^2)(\lambda_+^{(0)}+\lambda_-^{(0)}+2k^2)\psi_+^{(1)}] \tag{4.12}$$

and from (4.8c) at order ϵ^2 we obtain

$$\delta\psi_{\mathbf{k}_0}^{(2)}=\frac{-2\Gamma^{\mathbf{k}_0}}{(\lambda_+^{(0)}-\lambda_-^{(0)})^2f\Gamma^{\mathbf{k}}}[(\lambda_+^{(0)}+k^2)|\psi_+^{(1)}|^2]\ . \tag{4.13}$$

These are the lower-order nonvanishing contributions in the expansions of ψ_- and $\delta\psi_{\mathbf{k}_0}$. Equations (4.12) and (4.13) could have been obtained by application of the so-called "slaving principle" in Ref. 6 with ψ_- and $\delta\psi_{\mathbf{k}_0}$ the "slaves" and ψ_+ the order parameter.

The evolution equation for $\psi_+=|\psi_+|\exp(i\varphi_+)$ at order ϵ^3 is

$$\partial_\tau|\psi_+^{(1)}|=\lambda_+^{(2)}|\psi_+^{(1)}|-b|\psi_+^{(1)}|^3\ , \tag{4.14a}$$

$$\partial_\tau\varphi_+=0\ , \tag{4.14b}$$

where

$$b=\frac{(-4\Gamma^{\mathbf{k}_0}\Gamma^{\mathbf{k}'})k^2}{(k^2+k'^2)^3}\ . \tag{4.15}$$

Equation (4.14a) is well known[6] and may be trivially integrated leading to

$$|\psi_+^{(1)}(\tau)|^2=\frac{\lambda_+^{(2)}}{b}\frac{1}{1-\left[1-\dfrac{\lambda_+^{(2)}}{b|\psi_+(0)|^2}\right]e^{-2\lambda_+^{(2)}\tau}}\ . \tag{4.16}$$

Its asymptotic value $(t\to\infty)$ is

$$|\psi_+(\infty)|^2=\frac{\lambda_+^{(2)}}{b}\ . \tag{4.17}$$

V. NOISE EFFECTS AND A LYAPUNOV FUNCTION

As it has been said above, we are going to take into account the effect of the remaining modes over the triad by the addition of noise terms. If these noise terms are retained from (4.4) throughout all the subsequent calculations, the evolution equation for ψ_+ [see (4.14)] transforms into the following Langevin equation:

$$\partial_\tau\psi_+=\lambda_+\psi_+-b|\psi_+|^2\psi_++\xi(t)\ , \tag{5.1}$$

where $\xi(t)$ is a stochastic variable which is a linear combination of the amplitudes $\psi_{\tilde{\mathbf{k}}}$ and $\psi_{\tilde{\mathbf{k}}'}$ and as a consequence also behaves as white noise

$$\langle \xi(t)\xi^*(t')\rangle = Q\delta(t-t') , \tag{5.2a}$$

$$Q=[(k^2-k'^2)^2(\Gamma^{\mathbf{k}}_{\tilde{\mathbf{k}},-\mathbf{k}_0})^2|\mu_c|^2 + (\Gamma^{\mathbf{k}}_{\mathbf{k}_0,\mathbf{k}'}\Gamma^{\mathbf{k}'}_{\tilde{\mathbf{k}}',\mathbf{k}_0})^2|\mu_c|^4]\tilde{Q} . \tag{5.2b}$$

For this result to be true we have made the hypothesis that the noisy amplitudes $\psi_{\tilde{\mathbf{k}}}$ and $\psi_{\tilde{\mathbf{k}}'}$ do not have first-order terms in their expansion in ϵ. This is a very reasonable assumption since we are interested in fluctuations whose amplitudes are much smaller than the order parameter $|\psi_+|$. We want to remark that under this approximation, the "enslaving" of ψ_- and $\delta\psi_{\mathbf{k}_0}$ (4.12) and (4.13) still remains valid.

In this case, we can associate a Fokker-Planck equation for the probability distribution $P(\psi_+,\tau)$ defined on the phase space of ψ_+:

$$\partial_\tau P = -\frac{\partial}{\partial\psi_+}[(\lambda_+\psi_+ - b|\psi_+|^2\psi_+)P] + \frac{1}{2}\frac{\partial^2}{\partial\psi_+\partial\psi_+^*}[QP] . \tag{5.3}$$

The asymptotic solution (for $t\to\infty$) of this equation is[7]

$$P(\psi_+,\infty) = N\exp\left[\frac{2S(|\psi_+|)}{Q}\right] , \tag{5.4}$$

where N is a normalization factor and

$$S(|\psi_+|) = \frac{\lambda_+}{2}|\psi_+|^2 - \frac{b}{4}|\psi_+|^4 . \tag{5.5}$$

In the absence of noise it is evident that

$$\dot{S} = \frac{\partial S}{\partial|\psi_+|^2}(\dot{\psi}_+\psi_+^* - \psi_+\dot{\psi}_+^*) = |\lambda_+\psi_+ - b|\psi_+|^2\psi_+|^2 \geq 0 . \tag{5.6}$$

Therefore, S is a Lyapunov function[8] in the noiseless limit, whose maxima correspond to the stable equilibrium configurations. The function S has a clear thermodynamic interpretation. According to the information theory, the entropy associate to $P(\psi_+,t)$ is

$$\mathcal{S} = -\int d\psi_+ d\psi_+^*(P\ln P) , \tag{5.7}$$

where it can easily be seen that

$$\mathcal{S}\to\langle S\rangle \quad \text{for } t\to\infty , \tag{5.8a}$$

$$\langle S\rangle = \int d\psi_+ d\psi_+^* S(|\psi_+|)P(\psi_+,\infty) . \tag{5.8b}$$

It is interesting to notice that for sufficiently low $|\psi_+|$, S can be expressed as a linear combination of the ideal invariants (the energy W and the enstrophy Ω). This can readily be seen through the following argument. The invariants defined in (2.10) can be put in terms of $\psi_\pm$ by the application of (3.5), obtaining terms $|\psi_+|^2$ and $\psi_+\psi_-^* \propto |\psi_+|^4$ by using the slaving condition (4.12) and (4.13). On the other hand, $|\psi_{\mathbf{k}_0}|^2$ develops into constant terms ($|f|^2$) plus terms $f^*\delta\psi_{\mathbf{k}_0} \propto |\psi_+|^2$ due to (4.13) and finally $|\delta\psi_{\mathbf{k}_0}|^2 \propto |\psi_+|^4$. Thus both invariants become a linear combination of $|\psi_+|^2$ and $|\psi_+|^4$ as well as the entropy [see (5.5)]. So we can always calculate a couple of constants T_W and T_Ω such that

$$S = -\frac{Q}{2}\left[\frac{W}{T_W} + \frac{\Omega}{T_\Omega}\right] . \tag{5.9}$$

Thus even for our forced and dissipative case

$$P_{(\psi_+,\infty)} = N\exp\left[-\frac{W}{T_W} - \frac{\Omega}{T_\Omega}\right] , \tag{5.10}$$

where T_W and T_Ω are fictitious temperatures associated to the equilibrium distribution. We have computed these temperatures as a function of μ for the most unstable triad (see Sec. VI) and obtained that both are positive being $T_W \gg T_\Omega$. We want to remark that the equilibrium distribution function we obtained, which resembles a distribution for a canonical ensemble (with two invariants) has been derived in our simple three-mode model without any assumption about thermodynamic equilibrium. Moreover, external forces and dissipation are simultaneously allowed.

As a consequence of noise, the phase of ψ_+ (which was a constant in the deterministic case) can fluctuate. In the deterministic case, once μ exceeds μ_c we cannot find the system at the trivial equilibrium (2.12). However, when noise is introduced, there is a nonvanishing probability $P(0,\infty)$ for the system to be in this state. Moreover, sufficiently close to the bifurcation point, $P(0,\infty)$ is nearly as high as the probabilities associated to the new equilibria (4.6) and (4.7). Thus, the concept of bifurcation point becomes meaningless and must be replaced by the notion of bifurcation region.[9]

VI. DISCUSSION

As is well known, when forcing and dissipation are allowed, the ideal invariants (energy and enstrophy) cascade in the Fourier space. In the stationary regime of the Navier-Stokes turbulence, the energy is transferred to shorter wave numbers (inverse cascade) while the enstrophy is transported to greater wave numbers (direct cascade).

We are interested in computing the contribution of the three-mode interaction to these cascades. In Fig. 3 we show the cascade direction of energy and enstrophy for each triad. Once μ surpasses μ_c we let the system evolve towards its new equilibrium and then compare the energy (enstrophy) in the $\mathbf{k}$ mode ($k>1$) with the energy (enstrophy) in $\mathbf{k}'$ ($k'<1$) to decide the direction of the energy (enstrophy) cascade.

At this point we decide which is the most unstable triad (MUT). For a given μ, we look for the triad whose λ_+ is maximum. According to this criterium, we numerically computed the MUT for different values of μ as is shown in Fig. 3. It can be seen that the position of this triad in the $\mathbf{k}$ plane is a weak function of μ. We have also checked that λ_+, $|\psi_+(\infty)|$ and $\epsilon=\sqrt{\mu-\mu_c}$ monotonically increase with μ. Thus increasing the amplitude of the external force, the new modes $\mathbf{k}$ and $\mathbf{k}'$ will grow faster and saturate higher. However, at the same time the

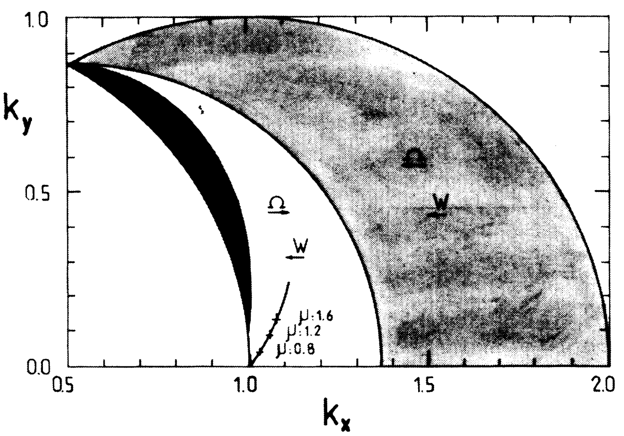

FIG. 3. The arrows indicate the corresponding cascade direction. Triads in the dark region contribute with direct cascade of both energy and enstrophy. The central region corresponds to the inverse cascade of energy and the direct cascade of enstrophy. In the grey region both cascades are inverse. The continuous curve indicate the position of the most unstable triad (MUT) as the control parameter μ is varied from 0 to 2.28 (where $\epsilon=1$).

perturbative expansion in ϵ [see Eqs. (4.10)] will be less accurate. For $\mu=2.28$ (where the curve of Fig. 3 cuts off) will be $\epsilon=1$ and the perturbative expansion will definitely be wrong.

We want to remark at this point the importance of calculating the MUT in the way we did (maximum λ_+ for each μ). Hasegawa and Kodama,[3] considering that

$$\lambda_+ = -\frac{k^2+k'^2}{2} + \left[\frac{(k^2-k'^2)}{4} + (\Gamma^k\Gamma^{k'})\mu^2\right]^{1/2}, \tag{6.1}$$

define the MUT as the one whose $\Gamma^k\Gamma^{k'}$ is maximum. This occurs for $\mathbf{k}=(1,(\sqrt{2}-1)^{1/2})=(1,0,64)$ (Ref. 2) and is very far from our MUT's (see Fig. 3). The difference between these results comes from the fact that $\Gamma^k\Gamma^{k'}$ maximum does not imply that λ_+ (which is function of k and k' not only through $\Gamma^k\Gamma^{k'}$) is also maximum.

Once the MUT is chosen, it is important to know the conditions under which the effect of the other triads can be neglected. Suppose that the box that contains the fluid is infinitely large ($L\to\infty$). Then the spectrum of eigenmodes will densely fill the $\mathbf{k}$ plane. Thus it is inevitable that for any value of μ, a continuum of triads in the surroundings of the MUT also become unstable. In Fig. 4 we show the set of unstable modes for different values of μ and we also give an idea of their instability rates. We also show the saturation values of these neighboring triads. If on the contrary the length of the box is finite, the Fourier eigenmodes are a discrete set [see (2.9)] being $2\pi/L$ their minimum separation in the $\mathbf{k}$ plane. It can readily be seen that if the box size is sufficiently small, the MUT may be the only excited one or at least it will grow with a far greater rate.

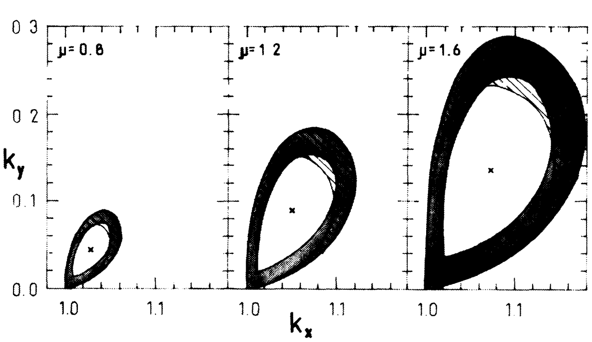

FIG. 4. For different values of μ, the x indicates the position of the MUT. In the shaded region lie the unstable triads whose growth rates (λ_+) range from 50% to 0% of the MUT. In the hatched region lie the unstable triads whose saturation levels [$|\psi_+(\infty)|$] range from 50% to 0% of the MUT.

VII. CONCLUSIONS

We have derived the evolution of a system of three coupled modes of the two-dimensional Navier-Stokes equation in the externally driven and dissipative case. We have shown that the external force amplitude μ acts as a control parameter of the problem. When μ is increased, the trivial equilibrium solution (2.12) turns unstable giving rise to the existence of new stable equilibria (node-to-node bifurcation with symmetry breaking). By a standard perturbative expansion of the quantities of interest, we have obtained the nonlinear evolution equations for a time scale of the order of the growth time of the instability (λ_+^{-1}). This procedure turned to be equivalent to the application of the slaving principle.[6] We have then added white-noise terms to the evolution equations in order to model the couplings among the modes of interest and the remaining ones. We have therefore been led to a Langevin equation and, in this approximation, we have derived the equilibrium distribution function $P(\psi_+,\infty)$ of the corresponding Fokker-Planck equation. We have found that $P=N\exp(2S/Q)$, where S is the Lyapunov function of the noiseless evolution equation, and may be set as

$$S=-\frac{Q}{2}\left[\frac{W}{T_W}+\frac{\Omega}{T_\Omega}\right],$$

with W the energy and Ω the enstrophy of the system. This equilibrium distribution resembles that of a canonical ensemble (with two invariants) and has been derived without the hypothesis of thermodynamical equilibrium. We have then discussed the conditions under which the evolution of a single set of three modes (triad) contains the relevant features of the whole system. We define the most unstable triad (MUT) as the one which maximizes λ_+. We have obtained that the most relevant set of three modes contributes to the energy and enstrophy cascades in the same directions as the fully developed turbulence ones.

ACKNOWLEDGMENTS

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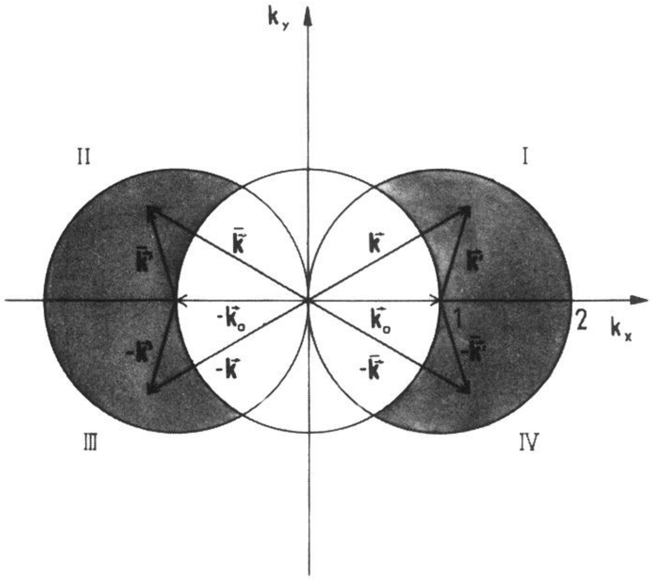
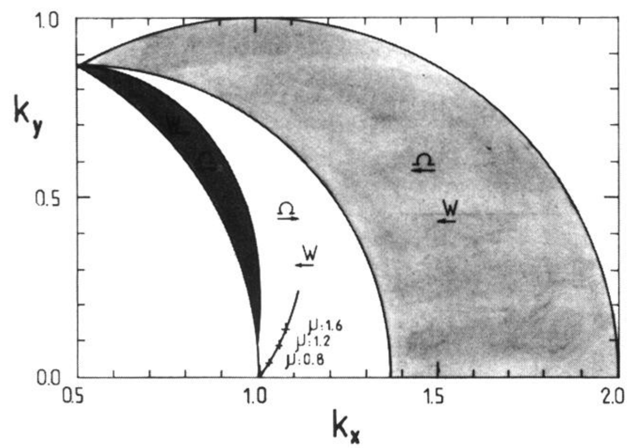


FIG. 1. The triads whose vertex lie in the shaded regions are those which can become unstable. The triads II, III, and IV have the same evolution than triad I.



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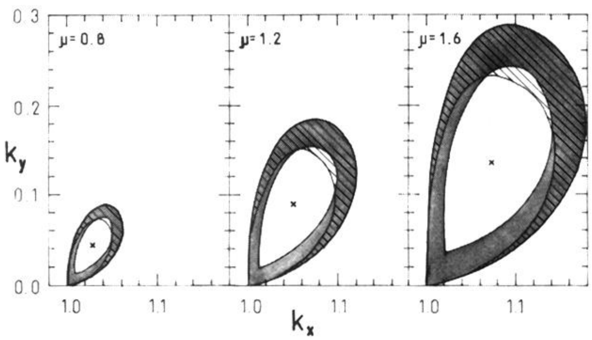

FIG. 4. For different values of μ, the x indicates the position of the MUT. In the shaded region lie the unstable triads whose growth rates (λ_+) range from 50% to 0% of the MUT. In the hatched region lie the unstable triads whose saturation levels [$|\psi_+(\infty)|$] range from 50% to 0% of the MUT.