Bound solitary waves in a birefringent optical fiber

M. V. Tratnik and J. E. Sipe
Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7
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We present the first mixed-type solutions to the coupled nonlinear Schrödinger equations which govern optical pulse propagation in a birefringent fiber. These represent polarization-modulated pulses which, apart from the absolute phase, propagate unchanged in form. It is shown they are bound states of two solitary waves which separately have constant and uniform orthogonal linear polarizations. Furthermore, there exists a minimum-energy threshold for the formation of these bound states.

I. INTRODUCTION

The nonlinear Schrödinger equation[1] (NLSE) and its generalizations are applicable to a wide range of physical phenomena. In the field of optics the $U(1)$ NLSE describes pulses under the influence of group-velocity dispersion and a third-order nonlinearity, but with constant and uniform polarization; it was shown to be integrable by the inverse scattering transform.[2] The $U(2)$ NLSE which also takes into account polarization but with a relatively simple nonlinearity was also shown to be integrable by the same method.[3] In a circular optical fiber or a generic isotropic third-order nonlinear medium, the dynamics is governed by the $O(2)$ NLSE which was shown to fail the Painlevé integrability test.[4] In a birefringent optical fiber within the usual approximations, or in a homogeneous birefringent medium in one space dimension, the governing equations are[5,6]

$$i\frac{\partial \mathbf{e}}{\partial t}+\frac{\partial^2 \mathbf{e}}{\partial x^2}+\overleftrightarrow{\chi}\cdot\mathbf{e}+a(\mathbf{e}^*\cdot\mathbf{e})\mathbf{e}+b(\mathbf{e}\cdot\mathbf{e})\mathbf{e}^*=0\ . \qquad (1.1)$$

The slowly varying electric field amplitude $\mathbf{e}$ is a two-component complex vector, which describes the two independent polarizations of a transverse electromagnetic field with carrier frequency ω_0 and wave number $k_0=\sqrt{\epsilon_0}\omega_0/c$, where ϵ_0 is the isotropic dielectric constant and c the speed of light in vacuum. The group velocity and group-velocity dispersion are evaluated at the carrier frequency as are the nonlinear coefficients a and b. It is assumed that a and b are greater than or equal to zero, and the intensity has been rescaled so that $a+b=1$. The variables t and x are dimensionless time and space coordinates, respectively, in a reference frame moving at the group velocity. The derivative terms in (1.1) along with the first nonlinear term $a(\mathbf{e}^*\cdot\mathbf{e})\mathbf{e}$ comprise the $U(2)$ NLSE while also including the second nonlinear term $b(\mathbf{e}\cdot\mathbf{e})\mathbf{e}^*$, which is of a different qualitative nature, leads to the $O(2)$ NLSE. This second term is important in a generic optical medium and represents a polarization-dependent nonlinearity in contrast to the first nonlinear term which is isotropic with respect to polarization. The birefringence is described by the 2×2 matrix $\overleftrightarrow{\chi}$ which is real and symmetric and, the isotropic component having been incorporated into ϵ_0, traceless as well. This birefringence is crucial to a qualitatively new family of solitary waves which in its absence reduce to the single soliton family of essentially the $U(1)$ NLSE.

Without loss of generality we can choose a linear-polarization basis in which $\overleftrightarrow{\chi}$ takes the form

$$\overleftrightarrow{\chi}=\begin{bmatrix}\alpha^2 & 0\\ 0 & -\alpha^2\end{bmatrix},\quad \alpha^2>0 \qquad (1.2)$$

and then the equations of motion become

$$i\frac{\partial e_1}{\partial t}+\frac{\partial^2 e_1}{\partial x^2}+\alpha^2 e_1+a(|e_1|^2+|e_2|^2)e_1+b(e_1^2+e_2^2)e_1^*=0\ ,$$
$$i\frac{\partial e_2}{\partial t}+\frac{\partial^2 e_2}{\partial x^2}-\alpha^2 e_2+a(|e_1|^2+|e_2|^2)e_2+b(e_1^2+e_2^2)e_2^*=0\ , \qquad (1.3)$$

where e_1 and e_2 denote the amplitudes for linearly polarized fields lying along the birefringent axes. These are coupled nonlinear Schrödinger equations with no obvious symmetry apart from a constant phase transformation. It is not known whether these equations are integrable but for $\alpha=0$ they fail the Painlevé integrability test.[4]

If one of the two fields e_1 or e_2 vanishes the remaining equation is equivalent to the $U(1)$ NLSE. Thus each field in the absence of the other exhibits all the N-soliton solutions and multisoliton bound states of that equation. The single solitons corresponding to these decoupled cases are

$$e_1(x,t)=\sqrt{2}A\exp\{i[(A^2+\alpha^2-V^2/4)t+Vx/2+\psi_0]\}\times\text{sech}[A(x-x_0-Vt)]\ ,$$
$$e_2(x,t)=0 \qquad (1.4)$$

and

$$e_1(x,t)=0\ ,$$
$$e_2(x,t)=\sqrt{2}A\exp\{i[(A^2-\alpha^2-V^2/4)t+Vx/2+\psi_0]\}\times\text{sech}[A(x-x_0-Vt)]\ , \qquad (1.5)$$

where A is proportional to the integrated intensity, V is the velocity, x_0 is the location of the maximum, and ψ_0 is an arbitrary phase (recall that the intensity has been rescaled so that $a+b=1$). However, apart from the decoupled case, no analytic solutions to Eqs. (1.3) are known. Of special interest are "mixed-type" solutions or equivalently polarization-modulated pulses where the energy is exchanged between the two fields.

Blow, Doran, and Wood[5] have numerically investigated these equations and found a rich dynamical behavior. They tested the stability of decoupled solutions, including multisoliton bound states, as one varies the relative magnitude of birefringence and nonlinearity. Regions of instability were found where the initially decoupled soliton evolves, after shedding radiation, into a nondispersive mixed-type pulse in which the energy is exchanged between the two fields.

In this paper we present the first mixed-type analytic solutions to Eqs. (1.3). These represent polarization-modulated solitary waves that are bound states of the two decoupled solitons (1.4) and (1.5) and which appear to be those numerically observed by Blow *et al.*[5] We begin in Sec. II where we find an unexpectedly simple solution using a Stokes-vector formalism and in Sec. III, using a generalized Hirota technique, we extend these solutions to a four-parameter family.

II. STOKES PARAMETERS

The interpretation of Eqs. (1.3) is facilitated by introducing a Stokes-vector formalism[7] in which the polarization behavior of the field is clearly exhibited. The Stokes vector components are defined by

$$S_i \equiv e_j^*(\sigma_i)_{jk} e_k , \quad i=1,2,3 \tag{2.1}$$

where σ_i are the Pauli spin matrices. The magnitude of the Stokes vector is simply proportional to the intensity

$$S_0 \equiv |\mathbf{S}| = \mathbf{e}^* \cdot \mathbf{e} \tag{2.2}$$

and the polarization is specified by the direction $\hat{\mathbf{s}} \equiv \mathbf{S}/S_0$ of the Stokes vector. This correspondence between polarization and the direction of the Stokes vector is easiest seen on the Poincaré sphere.[8] The Stokes vector describes the intensity and polarization of the field but not the absolute phase. So corresponding to the four real fields in Eqs. (1.3) we consider the three Stokes parameters $\mathbf{S}$ and the absolute phase ψ. Furthermore we introduce spherical polar coordinates in Stokes-vector space,

$$\mathbf{S} = S_0(\sin\theta \sin\phi,\ \cos\theta,\ \sin\theta\cos\phi) , \tag{2.3}$$

where the polar axis $\theta=0$ corresponds to left circular polarization. The dynamical equations for the variables S_0, θ, ϕ, and ψ, as determined from Eqs. (1.3), are found to be

$$\begin{aligned}
\frac{\partial S_0}{\partial t} &= -\frac{\partial}{\partial x}\left[S_0\left[2\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial x}\cos\theta\right]\right] ,\\
\frac{\partial\theta}{\partial t} &= -\frac{1}{S_0}\frac{\partial}{\partial x}\left[S_0\frac{\partial\phi}{\partial x}\sin\theta\right]-2\frac{\partial\theta}{\partial x}\frac{\partial\psi}{\partial x}+2\alpha^2\sin\phi ,\\
\frac{\partial\phi}{\partial t} &= \frac{1}{S_0\sin\theta}\frac{\partial}{\partial x}\left[S_0\frac{\partial\theta}{\partial x}\right]-2\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+2\alpha^2\cos\phi\cot\theta\\
&\quad -2bS_0\cos\theta ,\\
\frac{\partial\psi}{\partial t} &= \frac{1}{2S_0}\frac{\partial^2 S_0}{\partial x^2}-\left[\frac{1}{2S_0}\frac{\partial S_0}{\partial x}\right]^2+\frac{\cot\theta}{2S_0}\frac{\partial}{\partial x}\left[S_0\frac{\partial\theta}{\partial x}\right]\\
&\quad +(a-b\cos 2\theta)S_0+\alpha^2\cos\phi\csc\theta-\left[\frac{1}{2}\frac{\partial\theta}{\partial x}\right]^2\\
&\quad -\left[\frac{1}{2}\frac{\partial\phi}{\partial x}\right]^2-\left[\frac{\partial\psi}{\partial x}\right]^2 .
\end{aligned} \tag{2.4}$$

Before discussing these equations we note that like the $U(1)$ NLSE (Ref. 9) Eq. (1.1) is invariant under a Galilean transformation to a moving reference frame. That is, if $\mathbf{e}(x,t)$ is a solution of (1.1) then

$$\mathbf{e}'(x,t)=\exp[iV(x-Vt/2)/2]\mathbf{e}(x-Vt,t) , \tag{2.5}$$

is also a solution where the velocity V is an arbitrary real parameter. We are here interested in a single solitary wave so it is sufficient to consider a stationary one since a corresponding solution with a given velocity can always be constructed using the Galilean transformation (2.5). For such a stationary solution, by which here we mean the intensity is independent of time, the first of Eqs. (2.4) requires

$$\frac{\partial}{\partial x}\left[S_0\left[2\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial x}\cos\theta\right]\right]=0 . \tag{2.6}$$

If we assume a finite pulse and well-behaved functions

$$S_0(x)\to 0,\quad \left|\frac{\partial\psi}{\partial x}\right|,\ \left|\frac{\partial\phi}{\partial x}\right|<\infty \quad \text{as } |x|\to\infty , \tag{2.7}$$

then (2.6) yields

$$\frac{\partial\psi}{\partial x}=\frac{1}{2}\frac{\partial\phi}{\partial x}\cos\theta . \tag{2.8}$$

To proceed further we consider the physical nature of the nonlinearity. It is a function of the polarization and specifically depends upon the magnitude of the circular component; all polarizations with the same circular component experience the same nonlinearity. Then since birefringence is present we consider a polarization that is everywhere linear but in a direction which can depend upon space and time. In Stokes-vector space this corresponds to taking $\theta=\pi/2$ and ϕ as of yet unrestricted. With $\theta=\pi/2$ Eq. (2.8) requires that $\partial\psi/\partial x=0$, and the third of Eqs. (2.4) then implies $\partial\phi/\partial t=0$. Thus ϕ and S_0 are only functions of x and ψ is only a function of t, in which case the last of Eqs. (2.4) is consistent only if $\partial\psi/\partial t$

is a constant, that is,

$$\psi(t)=\Lambda t+\psi_0 , \tag{2.9}$$

where Λ is a constant to be determined and ψ_0 is an arbitrary absolute phase. The remaining equations of motion then reduce to

$$\begin{aligned}&\phi''+\frac{S_0'}{S_0}\phi'-2\alpha^2\sin\phi=0 ,\\ &\frac{1}{2}\frac{S_0''}{S_0}-\frac{1}{4}\left[\frac{S_0'}{S_0}\right]^2+S_0-\left[\frac{\phi'}{2}\right]^2+\alpha^2\cos\phi-\Lambda=0 ,\end{aligned} \tag{2.10}$$

where the prime denotes differentiation with respect to x. Taking the derivative of the second of these equations, using the first to eliminate ϕ'', and then the second to eliminate $(\phi')^2$, gives a first-order equation for $\cos\phi$,

$$(\cos\phi)'+\frac{S_0'}{S_0}\cos\phi+\frac{1}{2\alpha^2 S_0}\left[\frac{1}{2}S_0''+\frac{3}{2}S_0^2-2\Lambda S_0\right]'=0 , \tag{2.11}$$

whose integrating factor is simply S_0. Upon integration we obtain

$$\cos\phi=-\frac{1}{2\alpha^2}\left[\frac{1}{2}\frac{S_0''}{S_0}+\frac{3}{2}S_0-2\Lambda\right] , \tag{2.12}$$

where the integration constant has been set to zero. Substituting this expression for $\cos\phi$ back into the second of Eqs. (2.10) gives a closed expression for the unknown intensity

$$\begin{aligned}2Z\left[\left[Z'+\frac{Z}{y}+\frac{3}{2}y-2\Lambda\right]'\right]^2 &=\left[4\alpha^4-\left[Z'+\frac{Z}{y}+\frac{3}{2}y-2\Lambda\right]^2\right]\\ &\quad\times\left[\left[\frac{Z}{y}\right]'+\frac{1}{2}\right] .\end{aligned} \tag{2.13}$$

Here $Z=[(\sqrt{S_0})']^2$, $y=S_0$, and the prime denotes differentiation with respect to y. One can easily verify that a solution is given by $Z(y)=A^2y-\frac{1}{2}y^2$, which can be integrated with respect to S_0 to yield the intensity profile

$$S_0(x)=2A^2\text{sech}^2[A(x-x_0)] , \tag{2.14}$$

where A and x_0 are arbitrary real constants. Substituting this profile back into (2.12) and then into (2.10) determines $\phi=0,\pi$ and $\Lambda=A^2\pm\alpha^2$, which are the single decoupled solitons (1.4) and (1.5), respectively. Another nontrivial solution to Eq. (2.13) is

$$\Lambda=\tfrac{5}{8}\alpha^2,\quad Z(y)=\tfrac{2}{3}\alpha^2y-\tfrac{1}{6}y^2 , \tag{2.15}$$

which upon integration also gives a hyperbolic-secant intensity profile, an unexpectedly simple result,

$$S_0(x)=4\alpha^2\text{sech}^2[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)] . \tag{2.16}$$

Substituting this intensity profile back into (2.12) gives

$$\begin{aligned}&\cos\phi(x)=1-2\,\text{sech}^2[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)] ,\\ &\sin\phi(x)=2\,\text{sech}[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)]\tanh[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)] ,\end{aligned} \tag{2.17}$$

and from (2.3) and (2.17) we deduce the following expressions for the unit Stokes vector:

$$\begin{aligned}&\hat{s}_1(x)=2\,\text{sech}[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)]\tanh[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)] ,\\ &\hat{s}_2(x)=0 ,\\ &\hat{s}_3(x)=1-2\,\text{sech}^2[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)] .\end{aligned} \tag{2.18}$$

As x ranges from $-\infty$ to $+\infty$ the unit Stokes vector inscribes a great circle on the Poincaré sphere in the 1-3 plane. In other words, the direction of linear polarization rotates as a function of position in the pulse, and as one moves from the leading to the trailing edge it makes a complete rotation of 180° (in real space). Furthermore, in the far wings, the polarization lies along the birefringence axis defined by e_1, while at the peak it lies along the orthogonal birefringent axis. The intensity and polarization profiles, Eqs. (2.16) and (2.18), are depicted in Sec. III along with more general solitary wave solutions.

In terms of the electric field amplitudes this solitary wave becomes

$$\begin{aligned}e_1(x,t)&=2\alpha\exp[i(\tfrac{5}{3}\alpha^2t+\psi_0)]\text{sech}[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)]\\ &\quad\times\tanh[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)] ,\\ e_2(x,t)&=2\alpha\exp[i(\tfrac{5}{3}\alpha^2t+\psi_0)]\text{sech}^2[(\tfrac{2}{3})^{1/2}\alpha(x-x_0)] ,\end{aligned} \tag{2.19}$$

which is the first known mixed-type solution of Eqs. (1.3). Equations (2.19) represent a stationary solitary wave, but one can always apply the Galilean transformation (2.5) to obtain a solution propagating at an arbitrary velocity. In Sec. IV, using an alternate approach, we extend these solutions to a four-parameter family.

III. HIROTA APPROACH

The Hirota method which has been used in studying a number of nonlinear wave equations including the $U(1)$ NLSE (Ref. 1) can be extended to treat the coupled nonlinear Schrödinger equations (1.3).

Considering the solutions obtained in Sec. III we look for solitary waves of the form

$$\begin{aligned}&e_1(x,t)=\exp[i(A^2t+\psi_0)]e_1(x) ,\\ &e_2(x,t)=\exp[i(A^2t+\psi_0)]e_2(x) ,\end{aligned} \tag{3.1}$$

where A is a real constant and $e_1(x)$ and $e_2(x)$ are assumed to be real. Substituting Eqs. (3.1) into the equations of motion (1.3) we obtain

$$\frac{\partial^2 e_1}{\partial x^2}+(e_1^2+e_2^2)e_1=A_1^2 e_1\ ,$$
$$\frac{\partial^2 e_2}{\partial x^2}+(e_1^2+e_2^2)e_2=A_2^2 e_2\ , \qquad (3.2)$$

where we have defined

$$A_1^2\equiv A^2-\alpha^2,\quad A_2^2\equiv A^2+\alpha^2\ , \qquad (3.3)$$

and again recall that the intensity has been rescaled so that $a+b=1$. It is assumed $A^2\geq\alpha^2$ so that A_1 and A_2 are both real; A is otherwise arbitrary. Later we find this restriction on A implies a minimum energy threshold for the formation of these bound states.

To solve Eqs. (3.2) we make a change of dependent variables from the electric field components $e_1(x)$ and $e_2(x)$ to the functions $f(x)$, $g(x)$, and $h(x)$, where

$$e_1=\frac{g}{f},\quad e_2=\frac{h}{f}\ , \qquad (3.4)$$

and f satisfies the relation

$$e_1^2+e_2^2=2(\ln f)_{xx}\ , \qquad (3.5)$$

where hereon the subscript x denotes partial differentiation. From Eqs. (3.2), (3.4), and (3.5) we obtain the following coupled equations for f, g, and h:

$$fg_{xx}-2f_x g_x+gf_{xx}=A_1^2 fg\ ,$$
$$fh_{xx}-2f_x h_x+hf_{xx}=A_2^2 fh\ , \qquad (3.6)$$
$$g^2+h^2=2f^2(\ln f)_{xx}\ ,$$

and we look for solutions to these equations in the form of power series in a parameter ϵ,

$$f=1+\sum_{n=1}^{\infty}\epsilon^n f^{(n)},\quad g=\sum_{n=1}^{\infty}\epsilon^n g^{(n)},\quad h=\sum_{n=1}^{\infty}\epsilon^n h^{(n)}\ . \qquad (3.7)$$

Substituting (3.7) into (3.6) we deduce relations connecting the different $f^{(n)}$, $g^{(n)}$, and $h^{(n)}$ at each power of ϵ. At first order we obtain

$$f_{xx}^{(1)}=0,\quad g_{xx}^{(1)}=A_1^2 g^{(1)},\quad h_{xx}^{(1)}=A_2^2 h^{(1)}\ , \qquad (3.8)$$

for which we consider the following solution:

$$f^{(1)}=0,\quad g^{(1)}=2\sqrt{2}A_1\exp(\theta_1)\ ,$$
$$h^{(1)}=2\sqrt{2}A_2\exp(\theta_2)\ , \qquad (3.9)$$

where we have defined

$$\theta_1\equiv A_1(x-x_{10}),\quad \theta_2\equiv A_2(x-x_{20})\ , \qquad (3.10)$$

with x_{10} and x_{20} arbitrary constants. To second order in ϵ, taking into account (3.9), we find

$$f_{xx}^{(2)}=4A_1^2\exp(2\theta_1)+4A_2^2\exp(2\theta_2)\ ,$$
$$g_{xx}^{(2)}=A_1^2 g^{(2)},\quad h_{xx}^{(2)}=A_2^2 h^{(2)}\ , \qquad (3.11)$$

which gives

$$f^{(2)}=\exp(2\theta_1)+\exp(2\theta_2),\quad g^{(2)}=0,\quad h^{(2)}=0\ . \qquad (3.12)$$

To third order in ϵ, taking into account (3.9) and (3.12), we obtain

$$f_{xx}^{(3)}=0\ ,$$
$$g_{xx}^{(3)}=A_1^2 g^{(3)}+8\sqrt{2}A_1A_2(A_1-A_2)\exp(\theta_1+2\theta_2)\ , \qquad (3.13)$$
$$h_{xx}^{(3)}=A_2^2 h^{(3)}-8\sqrt{2}A_1A_2(A_1-A_2)\exp(\theta_2+2\theta_1)\ ,$$

to which corresponds the solution

$$f^{(3)}=0\ ,$$
$$g^{(3)}=2\sqrt{2}A_1\left[\frac{A_1-A_2}{A_1+A_2}\right]\exp(\theta_1+2\theta_2)\ , \qquad (3.14)$$
$$h^{(3)}=-2\sqrt{2}A_2\left[\frac{A_1-A_2}{A_1+A_2}\right]\exp(\theta_2+2\theta_1)\ ,$$

and in a similar manner we find at fourth order

$$f^{(4)}=\left[\frac{A_1-A_2}{A_1+A_2}\right]^2\exp(2\theta_1+2\theta_2)\ ,$$
$$g^{(4)}=0,\quad h^{(4)}=0\ . \qquad (3.15)$$

At this point we assume the series can be truncated, that is, we postulate a solution with all the higher orders set to zero. Then putting $\epsilon=1$ we have

$$f(x)=1+\exp(2\theta_1)+\exp(2\theta_2)+\left[\frac{A_1-A_2}{A_1+A_2}\right]^2\exp(2\theta_1+2\theta_2)\ ,$$
$$g(x)=2\sqrt{2}A_1\exp(\theta_1)\left[1+\left[\frac{A_1-A_2}{A_1+A_2}\right]\exp(2\theta_2)\right]\ , \qquad (3.16)$$
$$h(x)=2\sqrt{2}A_2\exp(\theta_2)\left[1-\left[\frac{A_1-A_2}{A_1+A_2}\right]\exp(2\theta_1)\right]\ .$$

Substituting (3.16) into Eqs. (3.6) verifies that this is indeed a solution, thus justifying our assumption. The electric field amplitudes according to (3.1) and (3.4) are then

$$e_1(x,t)=\frac{2\sqrt{2}A_1\exp(iA^2t+i\psi_0+\theta_1)\left[1+\left[\frac{A_1-A_2}{A_1+A_2}\right]\exp(2\theta_2)\right]}{1+\exp(2\theta_1)+\exp(2\theta_2)+\left[\frac{A_1-A_2}{A_1+A_2}\right]^2\exp(2\theta_1+2\theta_2)}\,,$$
$$e_2(x,t)=\frac{2\sqrt{2}A_2\exp(iA^2t+i\psi_0+\theta_2)\left[1-\left[\frac{A_1-A_2}{A_1+A_2}\right]\exp(2\theta_1)\right]}{1+\exp(2\theta_1)+\exp(2\theta_2)+\left[\frac{A_1-A_2}{A_1+A_2}\right]^2\exp(2\theta_1+2\theta_2)}\,, \quad (3.17)$$

which represent well-behaved solutions that go to zero as $|x|\to\infty$. As before, one can always apply the Galilean transformation (2.5) to obtain a solution propagating at an arbitrary velocity; for simplicity we discuss only the stationary case. Equations (3.17) represent a four-parameter ($\psi_0, A, x_{10}, x_{20}$) family of polarization modulated solitary waves. The first parameter is simply the initial absolute phase; the remaining three are best interpreted by considering some special cases.

If we take the limit $x_{20}\to\infty$ with x_{10} constant, Eqs. (3.17) reduce to

$$e_1(x,t)=\sqrt{2}A_1\exp\{i[(A_1^2+\alpha^2)t+\psi_0]\}\times\operatorname{sech}[A_1(x-x_{10})]\,,\quad e_2(x,t)=0\,, \quad (3.18)$$

which is just the single decoupled soliton (1.4) with amplitude A_1 and position of maximum x_{10}. Taking the limit $x_{10}\to\infty$ with x_{20} constant we find

$$e_1(x,t)=0\,,\quad e_2(x,t)=\sqrt{2}A_2\exp\{i[(A_2^2-\alpha^2)t+\psi_0]\}\times\operatorname{sech}[A_2(x-x_{20})]\,, \quad (3.19)$$

which is the other decoupled soliton (1.5) with amplitude A_2 and position of maximum x_{20}. In the limit $x_{20}\to-\infty$ with constant x_{10} we obtain

$$e_1(x,t)=\sqrt{2}A_1\exp\{i[(A_1^2+\alpha^2)t+\psi_0+\pi]\}\times\operatorname{sech}[A_1(x-x_{10}-\Delta)]\,,\quad e_2(x,t)=0\,, \quad (3.20)$$

which is again (1.4) but with a phase shift of π and a displacement of Δ in the position of the maximum, where

$$\Delta=-\frac{1}{A_1}\ln\left[\frac{A_2-A_1}{A_2+A_1}\right]. \quad (3.21)$$

Similarly in the limit $x_{10}\to-\infty$ with x_{20} constant we find

$$e_1(x,t)=0\,,\quad e_2(x,t)=\sqrt{2}A_2\exp\{i[(A_2^2-\alpha^2)t+\psi_0]\}\times\operatorname{sech}[A_2(x-x_{20}-\Delta')]\,, \quad (3.22)$$

where

$$\Delta'=-\frac{1}{A_2}\ln\left[\frac{A_2-A_1}{A_2+A_1}\right], \quad (3.23)$$

which is again the other decoupled soliton (1.5). Such a phase shift and displacement of the maximum is typical behavior for multisoliton solutions. It is then clear that (3.17) describes a bound state of the two solitons (1.4) and (1.5). Then one can think of A_1 and A_2 as the amplitudes of the two constituent solitons and likewise x_{10} and x_{20} as their respective "locations." The definition of the locations of the individual solitons is however somewhat ambiguous due to the displacements (3.21) and (3.23).

Let us consider further special cases. If we set $A^2=\frac{5}{3}\alpha^2$,

$$x_{10}=-(\tfrac{3}{2})^{1/2}\frac{1}{2\alpha}\ln3+x_0,\quad x_{20}=-(\tfrac{3}{2})^{1/2}\frac{1}{4\alpha}\ln3+x_0\,,$$

we obtain the solutions (2.19) discussed in Sec. II, apart from the phase transformation $e_1\to-e_1$ under which Eqs. (1.3) are invariant. If $A^2=\alpha^2$ then $A_1^2=0$, and we again obtain a single decoupled soliton

$$e_1(x,t)=0\,,\quad e_2(x,t)=2\alpha\exp[i(\alpha^2t+\psi_0)]\times\operatorname{sech}[\sqrt{2}\alpha(x-x_{20})]\,. \quad (3.24)$$

In the limit $\alpha\to0$, $A_1\to A_2=A$, and we obtain

$$\mathbf{e}(x,t)=\sqrt{2}A\hat{\mathbf{e}}_0\exp[i(A^2t+\psi_0)]\operatorname{sech}[A(x-x_0)]\,, \quad (3.25)$$

where x_0 and $\hat{\mathbf{e}}_0$ are defined by

$$\exp(-2Ax_0)\equiv\exp(-2Ax_{10})+\exp(-2Ax_{20})\,,\quad \hat{\mathbf{e}}_0\equiv(\exp[A(x_0-x_{10})],\exp[A(x_0-x_{20})])\,, \quad (3.26)$$

and $\hat{\mathbf{e}}_0$ describes a linearly polarized wave.

Returning to the general bound-state solutions (3.17) we deduce from (3.1), (3.5) and the first of Eqs. (3.16),

$$\int_{-\infty}^{\infty}(|e_1|^2+|e_2|^2)dx=4(A_1+A_2)=4[(A^2-\alpha^2)^{1/2}+(A^2+\alpha^2)^{1/2}]\,, \quad (3.27)$$

and so these are finite energy pulses, and since these solutions are only defined for $A^2 \geq \alpha^2$ we find a minimum energy threshold

$$\int_{-\infty}^{\infty} (|e_1|^2+|e_2|^2)dx \geq 4\sqrt{2}\alpha \quad (\alpha>0) . \tag{3.28}$$

The intensity profile described by Eqs. (3.17) can exhibit interesting forms including a double peak structure depending on the parameters A, x_{10}, and x_{20}. This is easily understood since these solutions are a superposition of the two solitons (1.4) and (1.5) with possibly different locations. Nevertheless the polarization profile can still be viewed in a simple manner. In terms of spherical polar coordinates in Stokes-vector space the polarization profile is described by

$$\theta=\frac{\pi}{2}, \quad \phi(x)=2\arctan\left[\frac{A_2(A_1+A_2)\exp[-A_1(x-x_{10})]-A_2(A_1-A_2)\exp[A_1(x-x_{10})]}{A_1(A_1+A_2)\exp[-A_2(x-x_{20})]+A_1(A_1-A_2)\exp[A_2(x-x_{20})]}\right], \tag{3.29}$$

and, as with the simpler solutions discussed in Sec. II, this profile inscribes a great circle on the Poincaré sphere in the 1-3 plane as x ranges from $-\infty$ to ∞. At the trailing edge the polarization lies along the birefringence axis of e_1, and as one moves through the pulse the plane of polarization rotates until at some point it lies along the orthogonal birefringent axis defined by e_2. Proceeding through the pulse the rotation continues until at the leading edge the polarization is once again along the first birefringence axis. In Figs. 1(a)–1(d) are depicted intensity and polarization profiles for $A_2=2A_1=2$ and $x_{10}=-x_{20}\equiv\delta$, with differing values for δ. The solid lines indicate the intensity profiles while the dotted lines the angle of rotation from the e_1 birefringence axis in real space; this is one half the rotation angle in Stokes-vector space, the second of Eqs. (3.29). In Fig. 1(a), where $\delta=3.00$, the two constituent solitons are widely separated. The polarization is essentially constant throughout each pulse but orthogonal for the two pulses, lying along each of the two birefringence axes. Only in the wings is the polarization modulated, and as discussed previously, goes through a full cycle $0°\rightarrow 180°$ as x ranges from $-\infty$

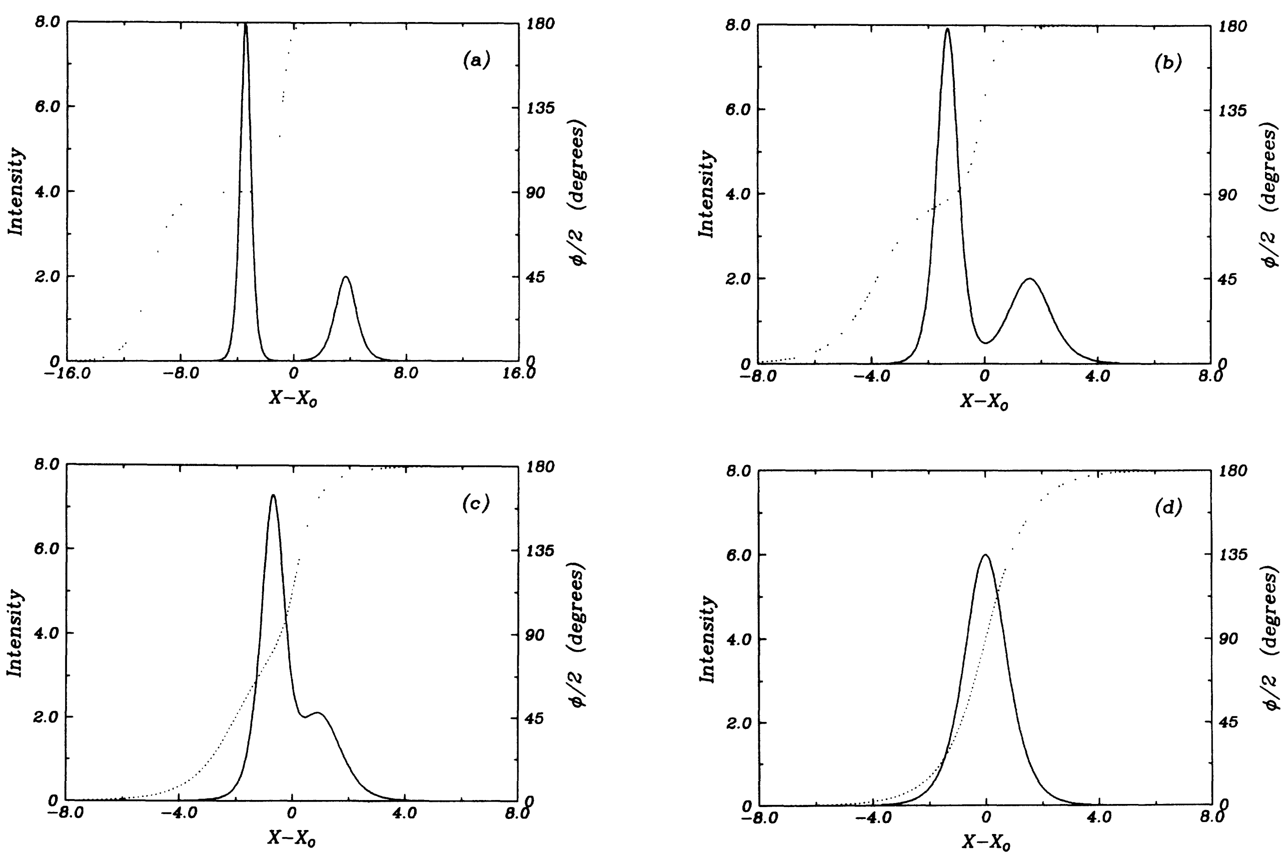

FIG. 1. Dimensionless (normalized by $a+b$) intensity (solid line) and polarization (dotted line) profiles of bound-state solitary waves with varying separation δ of the constituent solitons. (a) $\delta=3.00$, (b) $\delta=0.90$, (c) $\delta=0.30$, (d) $\delta=-0.14$.

to ∞. In Fig. 1(b), on a different horizontal scale, the separation is decreased to $\delta=0.90$. The constituent solitons begin to overlap and the polarization becomes modulated in this overlap region. In Fig. 1(c), where the separation is $\delta=0.30$, one still observes the double peak structure but a smoother polarization modulation. The last figure, with $\delta=-0.14$, which is the case discussed in Sec. II, exhibits a symmetric intensity profile and a kink-like polarization modulation.

It is also interesting to note that Eqs. (3.2) can be viewed as a quantum-mechanical problem for a bounded potential $-(e_1^2+e_2^2)$. In this interpretation $e_1(x)$ and $e_2(x)$ are energy eigenfunctions with eigenvalues $-A_1^2$ and $-A_2^2$, respectively. Then in the usual manner one can deduce the orthogonality relation

$$\alpha^2\int_{-\infty}^{\infty} dx\, e_1(x)e_2(x)=0\ , \tag{3.30}$$

and so the integral vanishes for $\alpha\neq 0$.

IV. DISCUSSION

Using a Stokes-vector formalism and then a generalized Hirota technique we have obtained the first mixed-type solutions to the coupled nonlinear Schrödinger equations which govern optical pulse propagation in a birefringent fiber. These describe polarization modulated solitary waves whose polarization profiles are conveniently pictured in Stokes-vector space where they simply inscribe a great circle on the Poincaré sphere. It was shown these are bound states of two solitons which separately have constant and uniform orthogonal polarizations along the two birefringent axes. Furthermore there exists a minimum energy threshold for the formation of these bound states.

Of considerable interest is the question of stability. The numerical work of Blow *et al.*[5] has shown that the two decoupled solitons (1.4) and (1.5) are both separately stable for A^2 less than about α^2, which is below the energy threshold for the formation of bound states. However, for A^2 greater than about α^2, above the threshold, they observe an instability in one of the two decoupled solitons, and for certain initial conditions find that the pulses evolve into mixed-type nondispersive solitary waves. It would be interesting to see if the energy threshold exactly corresponds to this bifurcation and if the analytic solutions presented here are stable.

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[1] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic, New York, 1982).

[2] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys.—JETP **34**, 62 (1972)].

[3] S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys.-JETP **38**, 248 (1974)].

[4] R. Sahadevan, K. M. Tamizhmani, and M. Lakshmanan, J. Phys. A **19**, 1783 (1986).

[5] K. J. Blow, N. J. Doran, and D. Wood, Opt. Lett. **12**, 202 (1987).

[6] C. R. Menyuk, IEEE J. Quantum. Electron. **QE-23**, 174 (1987).

[7] M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1959).

[8] M. V. Tratnik and J. E. Sipe, Phys. Rev. A **35**, 2965 (1987); **35**, 2976 (1987).

[9] J. Satsuma and N. Yajima, Suppl. Prog. Theor. Phys. **55**, 284 (1974).