Quantum system driven by rapidly varying periodic perturbation

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(Received 29 February 1988)

A quantum system driven by a high-frequency periodic perturbation is studied. By using the asymptotic expansions in terms of inverse powers of the driving frequency, a class of unitary time-dependent canonical transformations is defined which renders the transformed Hamiltonians time independent. One representative of those Hamiltonians is the quasienergy operator, which is explicitly derived up to the fourth order. The classical limit of the theory and the possibility of separating the mean-motion Hamiltonian is discussed. It is shown that this separation can consistently be carried out to higher orders only in the case of a uniform external force. The application of the method to unperturbed systems with regular confining and singular potentials is discussed by considering the examples of harmonic oscillator, particle in the double-well potential, and particle in the Coulomb potential.

I. INTRODUCTION

In classical mechanics a common approximate way of describing the motion of a Hamiltonian system perturbed by a rapidly varying external periodic force is to try to separate the "mean motion" from the superimposed high-frequency oscillations caused by the external perturbations. This procedure involves the averaging of classical equations of motion over one period of external force and results in an effective, time-independent, mean-motion Hamiltonian function.[1–3]

In the present work we study the analogous quantum-mechanical problem. The theory is formulated in terms of time-dependent unitary canonical transformations and allows for a straightforward classical limit. As pointed out in Sec. II in the general case of periodically driven systems a class of unitary time-dependent canonical transformations exists which renders the new (transformed) Hamiltonian time independent. All time-independent Hamiltonians from this class have the same spectrum, and the quasienergy operator[4] belongs to this class. In the case of the high-frequency perturbation, and for a class of systems defined in Sec. III, an explicit construction of canonical transformations is presented by using the asymptotic expansions in inverse powers of the driving frequency. The quasienergy operator is derived up to the fourth order and the results are related to the Magnus expansion of the evolution operator.[4] The examination of the classical limit reveals that the separation of the mean motion can consistently be carried out to higher orders only in the case of a uniform external force. The latter case is studied in more detail in Sec. IV where the mean-motion Hamiltonian is derived up to the sixth order. It is also shown that in the case of a harmonic oscillator driven by a uniform periodic force the present theory gives the exact solution, but when applied to singular potentials it may face divergence problems. Our conclusions are summarized in Sec. V.

II. GENERAL CONSIDERATIONS AND FORMULATION OF THE PROBLEM

Let $p=\{p_1,p_2,\ldots,p_N\}$ and $q=\{q_1,q_2,\ldots,q_N\}$ be sets of canonically conjugate momentum and position operators of a quantum system described by a time-periodic Hamiltonian $H(p,q,t)=H(p,q,t+T)$. Then the well-known Floquet theorem applies (see, for example, Ref. 4, and references therein) and the unitary evolution operator of the system takes the form (a system of units with $\hbar=1$ is used throughout the work)

$$U(p,q,t)=W(p,q,t)e^{-iG(p,q)t}\,, \tag{1a}$$

$$W(p,q,0)=1,\quad W(p,q,t+T)=W(p,q,t)\,, \tag{1b}$$

where $G(p,q)$ is the Hermitian, so-called quasienergy operator and $W(p,q,t)$ is the unitary time-periodic operator. Various perturbative and nonperturbative methods of describing the evolution of periodically driven systems have been proposed in the past (see, for example, Refs. 4–6).

The method adopted in the present work, although particularly convenient for establishing the correspondence with classical treatments at high frequencies, is actually quite general. It is based on the construction of the time-dependent canonical transformation $(p,q)\to(P,Q)$ which renders the new Hamiltonian $K(P,Q)$ time independent. The transformation is defined in terms of a time-periodic Hermitian operator $S(p,q,t)=S(p,q,t+T)$ (which is a Hermitian functional form of canonically conjugate operators),

$$P_i(p,q,t)=e^{-iS(p,q,t)}p_i e^{iS(p,q,t)}\,, \tag{2a}$$

$$Q_i(p,q,t)=e^{-iS(p,q,t)}q_i e^{iS(p,q,t)}\,, \tag{2b}$$

$$K(P,Q,t)\equiv K(P,Q)=e^{iS(P,Q,t)}H(P,Q,t)e^{-iS(P,Q,t)} - ie^{iS(P,Q,t)}\frac{\partial e^{-iS(P,Q,t)}}{\partial t}\,. \tag{3}$$

The last formula[7] is analogous to the relation between "old" and "new" Hamiltonian functions in classical mechanics when canonical transformation (in its Lie-algebraic formulation[8]) is explicitly time dependent. In Eq. (3) $H(P,Q,t)$ and $S(P,Q,t)$ are the same functional forms as $H(p,q,t)$ and $S(p,q,t)$ but with old variables (p,q) replaced by new variables (P,Q). It should be emphasized that (as explicitly shown in Sec. III) the condition of time independence of the new Hamiltonian does not unambiguously determine the transformation (i.e., the operator S). There is actually a whole class of operators (S,K) satisfying condition (3).

The evolution operator in terms of (p,q) variables can now be expressed as

$$U(p,q,t)=e^{-iS(p,q,t)}e^{-iK(p,q)t}e^{iS(p,q,0)} . \tag{4}$$

The last formula represents the connection between evolution operators in (p,q) and (P,Q) "representations." By equating the right-hand side (rhs) of Eqs. (1a) and (4) and taking into account that $S(p,q,t)$ is periodic in time, after some simple algebra one derives

$$G(p,q)=e^{-iS(p,q,0)}K(p,q)e^{iS(p,q,0)} , \tag{5a}$$

$$W(p,q,t)=e^{-iS(p,q,t)}e^{iS(p,q,0)} . \tag{5b}$$

Therefore solving for any K and S is, in principle, equivalent to solving for G and W. In particular, from (5a), it follows that operators G and K are unitary transforms of each other and therefore have the same eigenspectrum. From (5a), it also follows that imposing the condition $S(p,q,0)=0$ [in addition to (3)] leads simply to $K=G$.

The question, however, arises, as to whether there are, and under what circumstances there would be, other conditions that could be imposed on S and which would lead to physically meaningful interpretation of the time-independent Hamiltonian $K(P,Q)$. This question will be explored in the following sections for the case of a high-frequency external perturbing force.

III. CONSERVATIVE SYSTEM DRIVEN BY A HIGH-FREQUENCY PERIODIC PERTURBATION

Let us be more specific and consider an N-degree-of-freedom system defined by a Hamiltonian

$$H(p,q,t)=H_0(p,q)+V(q)\sin(\omega t+\phi) , \tag{6a}$$

$$H_0(p,q)=\tfrac{1}{2}p^2+U(q) . \tag{6b}$$

In order to use $\omega=2\pi/T$, the frequency of periodic perturbation, as a large parameter in the theory, we shall assume that the exact wave function of the system represents a wave packet of bound eigenstates of H_0, such that $\omega \gg \omega_{ij}^0$, where ω_{ij}^0 is the Bohr frequency for the transition between any pair of eigenstates.

A. Construction of canonical transformation

Let us expand the generating operator S from Sec. II in an asymptotic series in inverse powers of ω,

$$S(P,Q,t)=\sum_{k=1}^{\infty}\frac{1}{\omega^k}S_k(P,Q,t) . \tag{7}$$

Then, by using operator equality

$$e^{iS}Ae^{-iS}=\sum_{j=0}^{\infty}\frac{(i)^j}{j!}[S,[S,\ldots[S,A]\cdots]] , \tag{8}$$

where the commutators are j-fold nested, and from Eq. (3) one derives the following asymptotic expansion for the new Hamiltonian:

$$K(P,Q)=\sum_{n=0}^{\infty}\frac{1}{\omega^n}K^{(n)}(P,Q) , \tag{9}$$

where

$$K^{(0)}(P,Q)+\frac{1}{\omega}\frac{\partial S_1(P,Q,t)}{\partial t}=H(P,Q,t) , \tag{10a}$$

and for $n\geq 1$,

$$K^{(n)}(P,Q)+\frac{1}{\omega}\frac{\partial S_{n+1}(P,Q,t)}{\partial t}=R^{(n)}(P,Q,t) , \tag{10b}$$

with

$$R^{(n)}(P,Q,t)=\sum_{j=1}^{n}\sum_{k_l\geq 1}\frac{(i)^j}{j!}[S_{k_1},\ldots[S_{k_j},H]\cdots]+\sum_{j=2}^{n+1}\sum_{n_l\geq 1}\frac{(i)^{j+1}}{j!}\left[S_{n_1},\ldots\left[S_{n_{j-1}},\frac{1}{\omega}\frac{\partial S_{n_j}}{\partial t}\right]\cdots\right] \tag{10c}$$

and

$$\sum_{l=1}^{j}k_l=n, \quad \sum_{l=1}^{j}n_l=n+1 . \tag{10d}$$

By using the condition that $K(P,Q)$ is time independent, Eqs. (10a) and (10b) can simultaneously be solved for $K^{(n)}(P,Q)$ and $S_{n+1}(P,Q,t)$, $n=0,1,2,\ldots$, by the following inductive procedure. Noting that S_{n+1} is periodic in time, hence

$$\int_t^{t+T}\frac{\partial S_{n+1}}{\partial t}dt=0 , \tag{11}$$

one easily determines from (10a) and (10b)

$$K^{(0)}(P,Q)=\frac{1}{T}\int_t^{t+T}H(P,Q,t)dt \ , \tag{12a}$$

$$\frac{1}{\omega}\frac{\partial S_1}{\partial t}=H(P,Q,t)-\frac{1}{T}\int_t^{t+T}H(P,Q,t)dt \ , \tag{12b}$$

and for $n \geq 1$,

$$K^{(n)}(P,Q)=\frac{1}{T}\int_t^{t+T}R^{(n)}(P,Q,t)dt \ , \tag{12c}$$

$$\frac{1}{\omega}\frac{\partial S_{n+1}}{\partial t}=R^{(n)}(P,Q,t)-\frac{1}{T}\int_t^{t+T}R^{(n)}(P,Q,t)dt \ , \tag{12d}$$

where $R^{(n)}(P,Q,t)$, as seen from (10c), depends on $S_1,S_2,\ldots,S_n$ and is therefore determined in the previous steps of the inductive procedure.

Equations (12b) and (12d) determine each of S_{n+1} up to some additive time-independent Hermitian operator. Since S_{n+1} is periodic in time, it is equivalent to saying that its zero-frequency Fourier component

$$S_{n+1}^0(P,Q)=\frac{1}{T}\int_t^{t+T}S_{n+1}(P,Q,t)dt \tag{13}$$

remains undetermined. This is exactly the ambiguity mentioned in Sec. II, in the determination of the pair (S,K) solely from the condition of time independence of K. In order to remove it one has to impose additional conditions on $S(P,Q,t)$. Possible ways of doing this are discussed in Secs. III B and III C.

B. Quasienergy operator

As mentioned in Sec. II, the condition $S(P,Q,0)=0$ determines the new Hamiltonian as a quasienergy operator, i.e., $G(P,Q)=K(P,Q)$. Thus by solving (12a)–(12d) with the condition

$$S_{n+1}(P,Q,0)=0 \tag{14}$$

and with the time-dependent Hamiltonian as defined in Eq. (6a), one finds up to the fourth order in $1/\omega$,

$$G=\sum_{n=0}^{4}\frac{1}{\omega^n}G^{(n)} \ , \tag{15}$$

with

$$G^{(0)}=H_0 \ , \tag{16a}$$

$$G^{(1)}=i[V,H_0]\cos\phi \ , \tag{16b}$$

$$G^{(2)}=-[H_0,[H_0,V]]\sin\phi-[V,[V,H_0]](\tfrac{1}{4}+\tfrac{1}{2}\cos^2\phi) \ , \tag{16c}$$

$$G^{(3)}=i[H_0,[H_0,[V,H_0]]]\cos\phi-\frac{i}{8}[H_0,[V,[H_0,V]]]\sin2\phi-\frac{i}{4}[V,[H_0,[H_0,V]]]\sin2\phi \ , \tag{16d}$$

$$\begin{aligned}G^{(4)}=&-[H_0,[H_0,[H_0,[H_0,V]]]]\sin\phi+[[V,H_0],[V,[V,H_0]]](\tfrac{1}{4}\sin\phi-\tfrac{1}{8}\cos\phi\,\sin2\phi)\\&+[[V,H_0],[H_0,[V,H_0]]](-\tfrac{3}{4}-\tfrac{1}{2}\cos^2\phi)+[H_0,[V,[H_0,[H_0,V]]]](\tfrac{1}{4}+\tfrac{3}{16}\cos2\phi+\tfrac{1}{2}\cos^2\phi) \ .\end{aligned} \tag{16e}$$

The corresponding expansions for generator S and unitary operator W are given in Appendix A.

Nauts and Wyatt[4] have applied the Magnus expansion to the evolution operator corresponding to Hamiltonian (6a) in the special case $\phi=\pi/2$. Their result for the quasienergy operator obtained up to the second order is reproduced by formulas (16a)–(16c). This indicates that the two methods are closely related.

C. Classical limit and separation of mean motion

Up to now we have dealt with the quantum-mechanical system, but all of our formulas preserve their forms in the case of classical dynamics governed by the Hamiltonian function which corresponds to (6a) if only all commutators are replaced by Poisson brackets,

$$\frac{1}{i}[A(p,q,t),B(p,q,t)]\rightarrow\{\mathcal{A}(\not{p},\not{q},t),\mathcal{B}(\not{p},\not{q},t)\} \ , \tag{17}$$

where $\mathcal{A},\mathcal{B},\not{p},\not{q}$ are classical dynamical functions and canonical momentum and position variables. As for unitary canonical transformation

$$A(p,q,t)\rightarrow e^{iS(p,q,t)}A(p,q,t)e^{-iS(p,q,t)} \ , \tag{18}$$

it corresponds to the classical time-dependent canonical (Lie) transformation

$$\mathcal{A}(\not{p},\not{q},t)\rightarrow e^{-\tilde{\mathcal{S}}(\not{p},\not{q},t)}\mathcal{A}(\not{p},\not{q},t) \ , \tag{19}$$

where $\tilde{\mathcal{S}}(\not{p},\not{q},t)$ is a linear operator in the space of dynamical functions,[8]

$$\tilde{\mathcal{S}}(\not{p},\not{q},t)[\mathcal{A}(\not{p},\not{q},t)]=\{\mathcal{S}(\not{p},\not{q},t),\mathcal{A}(\not{p},\not{q},t)\} \ . \tag{20}$$

Bearing in mind the above facts we shall proceed to work with quantum-mechanical variables. In all the following formulas classical results are obtained by substitution (17).

In order to establish the connection with classical treatments[1–3] of the problem a deeper insight into the structure of the transformations inverse to (2a) and (2b) is necessary. Using again formula (8) and expression (7) one finds

$$p_i(P,Q,t)=e^{iS(P,Q,t)}P_i e^{-iS(P,Q,t)}$$
$$=P_i+\sum_{n=1}^{\infty}\frac{1}{\omega^n}p_i^{(n)}(P,Q,t)\ , \quad (21a)$$
$$q_i(P,Q,t)=e^{iS(P,Q,t)}Q_i e^{-iS(P,Q,t)}$$
$$=Q_i+\sum_{n=1}^{\infty}\frac{1}{\omega^n}q_i^{(n)}(P,Q,t)\ , \quad (21b)$$

where

$$p_i^{(n)}(P,Q,t)=\sum_{j=1}^{n}\sum_{k_l\geq 1}\frac{(i)^j}{j!}[S_{k_1},[S_{k_2},\ldots[S_{k_j},P_i]\cdots]]\ , \quad (22a)$$

$$q_i^{(n)}(P,Q,t)$$
$$=\sum_{j=1}^{n}\sum_{k_l\geq 1}\frac{(i)^j}{j!}[S_{k_1},[S_{k_2},\ldots[S_{k_j},P_i]\cdots]]\ , \quad (22b)$$

$$\sum_{l=1}^{j}k_l=n\ . \quad (22c)$$

In the previous classical treatments[1,2] of the system described by (6a) an effective time-independent Hamiltonian function $\mathcal{H}(\mathcal{P},\mathcal{Q})$ has been derived up to the order $1/\omega^2$ by requiring that, when averaged over one period of fast oscillations, original variables reduce to new variables,

$$\frac{1}{T}\int_t^{t+T}q_i\,dt=\mathcal{Q}_i\ , \quad (23a)$$

$$\frac{1}{T}\int_t^{t+T}p_i\,dt=\mathcal{P}_i\ . \quad (23b)$$

In Ref. 3 the claim is made that the method has been extended to order $1/\omega^4$. If the conditions (23a) and (23b) are fulfilled the Hamiltonian function $\mathcal{H}(\mathcal{P},\mathcal{Q})$ can be said to describe "the mean" (i.e., smoothed over fast oscillations) motion of the system.

In our formulation, as seen from (21a) and (21b), conditions (23a) and (23b) correspond to requirements that the zero-frequency Fourier components of $q_i^{(k)}$ and $p_i^{(k)}$, up to a given order n, all vanish,

$$\int_t^{t+T}q_i^{(k)}(P,Q,t)dt=0\ , \quad (24a)$$

$$\int_t^{t+T}p_i^{(k)}(P,Q,t)=0\ . \quad (24b)$$

In each order $k=1,2,\ldots,n$ conditions (24a) and (24b), via relations (22a) and (22b), impose restrictions upon the zero-frequency Fourier components S_k^0, Eq. (13), of the generating operator S, which were otherwise undetermined. Clearly, it may happen at certain order, that conditions (24a) and (24b) overdetermine single operator $S_k^0(P,Q)$. Indeed, when applied to Hamiltonian (6a) the above procedure of determining the canonical transformations goes smoothly up to the order $1/\omega^4$ where difficulties are encountered. The resulting asymptotic expansions for S, p_i, and q_i are listed in Appendix B, whereas the time-independent Hamiltonian is given by expansion (9) with

$$K^{(0)}=H_0\ , \quad (25a)$$
$$K^{(1)}=0\ , \quad (25b)$$
$$K^{(2)}=-\tfrac{1}{4}[V,[V,H_0]]\ , \quad (25c)$$
$$K^{(3)}=0\ , \quad (25d)$$
$$K^{(4)}=i[S_4^0,H_0]-\tfrac{1}{4}[V,[H_0,[H_0,[V,H_0]]]]\ . \quad (25e)$$

Now, S_4^0 is to be determined in such a way that conditions (24a) and (24b) for $n=4$ are fulfilled. However, as shown in Appendix B, this is possible only if the coupling term $V(Q)$ obeys the condition (hereafter we assume summation over the repeated indices),

$$\frac{\partial^2V(Q)}{\partial Q_i\partial Q_l}\frac{\partial^2V(Q)}{\partial Q_j\partial Q_l}=0\ , \quad (26)$$

i.e., only when V is a linear function of the coordinates, and in which case one should take $S_4^0=0$.

Thus, for the general case of an arbitrary function $V(q)$ in Eq. (6a), it is not possible to separate the mean-motion Hamiltonian, in the classical sense of Eqs. (23a) and (23b), of order higher than $1/\omega^3$. With H_0 given by (6b) the commutators in (25c) can easily be calculated to give

$$K^{(2)}=\frac{1}{4}\frac{\partial V}{\partial Q_i}\frac{\partial V}{\partial Q_i}\ , \quad (27)$$

which is the same as the classical result.[1,2] On the other hand, the classical result quoted in Ref. 3,

$$\mathcal{K}^{(4)}=\frac{1}{4}\frac{\partial\mathcal{V}}{\partial\mathcal{Q}_j}\frac{\partial\mathcal{V}}{\partial\mathcal{Q}_k}\frac{\partial^2\mathcal{U}}{\partial\mathcal{Q}_j\partial\mathcal{Q}_k}\ , \quad (28)$$

is meaningful only if the condition (26) is fulfilled (uniform external force), in which case it follows from (25e) with $S_4^0=0$.

Regarding the quantum Hamiltonian defined by (25a)–(25e), it can be seen that it has a much simpler structure than the quasienergy operator (16a)–(16e) but, as noted in Sec. II, still the same spectrum. Thus, at least in principle, it can be used for determination of the quasienergy spectrum (with $S_4^0=0$, for example), although any clear physical interpretation, even in the classical limit, of that Hamiltonian is lost. The result for the quasienergy operator, as given by Eqs. (15) and (16a)–(16e), can of course be rederived by using relation (5a) and Eqs. (25a)–(25e), independently on the choice of S_4^0.

IV. UNIFORM EXTERNAL FORCE. EXAMPLES

All the formulas of the preceding sections considerably simplify in the case of a uniform external force,

$$V(q)=f_iq_i\ . \quad (29)$$

In particular, the conditions (24a) and (24b) can be satisfied at least up to the order $1/\omega^6$ with the resulting time-independent Hamiltonian,

$$K(P,Q)=H_0(P,Q)+\frac{1}{4\omega^2}f_if_i+\frac{1}{4\omega^4}f_if_j\frac{\partial^2 U}{\partial Q_i\partial Q_j}+\frac{1}{4\omega^6}f_if_j\frac{\partial^2 U}{\partial Q_i\partial Q_k}\frac{\partial^2 U}{\partial Q_j\partial Q_k}\ . \tag{30}$$

The corresponding asymptotic expansions for S, p_i, and q_i are listed in Appendix C. The classical result derived up to the order $1/\omega^4$ by Nadezhdin and Oks[3] coincides with the first three terms in Eq. (30). The time-independent Hamiltonian $K(P,Q)$ can be used to determine the approximation to the quasienergy spectrum at large frequencies of the external perturbation. We next consider a few examples by specifying the potential $U(q)$ in Eq. (6b).

A. Harmonic oscillator

This is an exactly solvable problem[9] and therefore a good test for an approximate theory. All the expansions used in the previous sections are in general case asymptotic, i.e., divergent; however, for this particular case, as shown, they all converge towards the exact solution.

We take the potential of the harmonic oscillator in N dimensions to be of the form $(q^2\equiv q_iq_i)$

$$U(q)=\frac{\omega_0^2}{2}q^2\ . \tag{31}$$

Then, from Eqs. (12a)–(12d), one finds for the generating operator $S(P,Q,t)$, which fulfills the conditions (24a) and (24b),

$$S_{2k-1}=-f_iQ_i\omega_0^{2k-2}\cos(\omega t+\phi)-(1-\delta_{k,1})\tfrac{1}{8}f^2\omega_0^{2k-4}\sin(2\omega t+2\phi)\ , \tag{32a}$$

$$S_{2k}=f_iP_i\omega_0^{2k-2}\sin(\omega t+\phi)\ , \tag{32b}$$

with $k=1,2,\ldots$. If $\omega_0^2/\omega^2<1$, the sum (7) is convergent so that

$$S(P,Q,t)=\frac{1}{\omega^2-\omega_0^2}[f_iP_i\sin(\omega t+\phi)-\omega f_iQ_i\cos(\omega t+\phi)]-\frac{1}{8}\frac{f^2}{\omega}\sin(2\omega t+2\phi)\ . \tag{33}$$

The time-independent Hamiltonian and canonical transformation between original and new variables are given by

$$K(P,Q)=\frac{P^2}{2}+\frac{\omega_0^2}{2}Q^2+\frac{f^2}{4(\omega^2-\omega_0^2)}\ , \tag{34}$$

$$p_i=P_i+\frac{\omega f_i}{\omega^2-\omega_0^2}\cos(\omega t+\phi)\ , \tag{35a}$$

$$q_i=Q_i+\frac{f_i}{\omega^2-\omega_0^2}\sin(\omega t+\phi)\ . \tag{35b}$$

Thus the quasienergy spectrum, i.e., the spectrum of the Hamiltonian (34) is that of the harmonic oscillator. The quasienergy operator derived from Eq. (5a) is

$$G(P,Q)=\frac{1}{2}\left[P_i-\frac{\omega}{\omega^2-\omega_0^2}f_i\cos\phi\right]^2+\tfrac{1}{2}\omega_0^2\left[Q_i-\frac{1}{\omega^2-\omega_0^2}f_i\sin\phi\right]^2+\frac{f^2}{4(\omega^2-\omega_0^2)}\ . \tag{36}$$

B. Double-well potential

As an example of a one–degree-of-freedom system in confining potential, we consider the double-well potential

$$U(q)=-\alpha q^2+\beta q^4\ , \tag{37}$$

with $\alpha,\beta>0$. Then, up to the sixth order in $1/\omega$, one finds from Eq. (30),

$$K(P,Q)=\frac{P^2}{2}+A_0+A_2Q^2+A_4Q^4\ , \tag{38a}$$

with

$$A_0=\frac{f^2}{4\omega^2}-\frac{\alpha f^2}{2\omega^4}+\frac{\alpha^2f^2}{\omega^6}\ , \tag{38b}$$

$$A_2=-\alpha+\frac{3\beta f^2}{\omega^4}-\frac{12\alpha\beta f^2}{\omega^6}\ , \tag{38c}$$

$$A_4=\beta+\frac{36\beta^2f^2}{\omega^6}\ . \tag{38d}$$

The constant A_0 does not affect the dynamics, and the effective time-independent potential (which also determines the quasienergy spectrum), is either a double $(A_2<0)$ or a single well $(A_2>0)$.

C. Coulomb potential

This problem has been treated semiclassically (up to the order $1/\omega^4$) by Nadezhdin and Oks[3] as a model for a highly excited hydrogen atom driven by a linearly polarized electric field. It is of interest to compare their results with quantum treatment. The time-periodic Hamiltonian is taken to be

$$H(\mathbf{p},\mathbf{r},t)=\tfrac{1}{2}p^2-\frac{1}{r}+fz\sin(\omega t+\phi)\ . \tag{39}$$

Then, by using notation $Q\equiv\mathbf{R}=\{X,Y,Z\}$ one finds from (30)

$$K(\mathbf{P},\mathbf{R})=\frac{P^2}{2}-\frac{1}{R}+\frac{f^2}{4\omega^2}+\frac{f^2}{4\omega^4}\frac{1-3\cos^2\Theta}{R^3}+\frac{f^2}{4\omega^6}\frac{1+3\cos^2\Theta}{R^6}\ , \tag{40}$$

with $\cos\Theta=Z/R$. Leaving aside for the moment the term $K^{(6)}$ of order $1/\omega^6$ in Eq. (40), the quasienergy spectrum can be estimated by using perturbation theory. The nontrivial shifts to the hydrogenic spectrum come from the term $K^{(4)}$, so that one finds (correct zeroth-order wave functions are those in spherical coordinates)

$$\mathcal{K}_{nlm}=-\frac{1}{2n^2}+\frac{f^2}{4\omega^2}+\frac{1}{\omega^4}\mathcal{K}_{nlm}^{(4)}\ , \tag{41a}$$

where

$$\mathcal{K}^{(4)}_{nlm} = \begin{cases} \dfrac{f^2}{3n^3}, \quad l=0 & \text{(41b)} \\ \dfrac{f^2[3m^2-l(l+1)]}{n^3 l(l+1)(2l-1)(2l+1)(2l+3)}, \quad l=1,2,\ldots,n-1 & \text{(41c)} \end{cases}$$

and n,l,m are the usual spherical quantum numbers. Expressions (41b) and (41c) can be compared with the semiclassical result[3]

$$\mathcal{K}^{(4)sc}_{nlm} = \frac{f^2[3m^2-(l+1/2)^2]}{8n^3(l+1/2)^5} . \tag{42}$$

For large values of quantum numbers l, expressions (41c) and (42) converge to each other.

As seen from Eq. (40) the term $K^{(6)}$ is even more singular than $K^{(4)}$, and in an attempt to include the contribution of the order $1/\omega^6$ to the quasienergy spectrum, one is faced with the problem that diagonal matrix elements between s states of $K^{(6)}$ diverge. This points out a general limitation of the present method when applied to singular potentials. As indicated by Eq. (30), higher-order terms contain products of higher derivatives of the potential, thus becoming increasingly singular. Nevertheless, bearing in mind the asymptotic nature of the expansions involved, the results such as (41a)–(41c) or (42) can still be meaningful.

V. CONCLUSIONS

We have studied the quantum system perturbed by rapidly varying external periodic force. By using asymptotic expansions in inverse powers of driving frequency, a class of time-dependent unitary canonical transformations, which renders the transformed Hamiltonian time independent, can be defined. Additional conditions on transformation can be imposed, either to identify the time-independent Hamiltonian with quasienergy operator or to interpret it as a mean-motion Hamiltonian in the classical limit. In the latter case the interpretation can be carried out to higher orders only in the case of a uniform external force. The method reproduces the exact solution in the case of a harmonic oscillator driven by a uniform force and faces divergence problems when applied to singular potentials.

APPENDIX A

From (12a)–(12d), with conditions (14), apart from (16a)–(16e), one finds, for the first four terms in expansion (7),

$$S_1 = V[\cos\phi - \cos(\omega t+\phi)] , \tag{A1}$$

$$S_2 = i[V,H_0][\sin\phi - \sin(\omega t+\phi)] , \tag{A2}$$

$$S_3 = [H_0,[H_0,V]][\cos\phi - \cos(\omega t+\phi)] - [V,[V,H_0]][\tfrac{1}{8}\sin 2\phi - \tfrac{1}{2}\sin\omega t - \tfrac{1}{8}\sin(2\omega t+2\phi)] , \tag{A3}$$

$$S_4 = i[H_0,[H_0,[V,H_0]]][\sin\phi - \sin(\omega t+\phi)] - \tfrac{3}{16} i[V,[H_0,[V,H_0]]][\cos 2\phi - \cos(2\omega t+2\phi)] . \tag{A4}$$

As defined in Eq. (5b), the unitary periodic operator W can be found by exponentiating the operator S. Up to the third order in $1/\omega$ one obtains

$$W = \sum_{n=0}^{3} \frac{1}{\omega^n} W^{(n)} , \tag{A5}$$

$$W^{(0)} = 1 , \tag{A6}$$

$$W^{(1)} = -iV[\cos\phi - \cos(\omega t+\phi)] , \tag{A7}$$

$$W^{(2)} = [V,H_0][\sin\phi - \sin(\omega t+\phi)] - \tfrac{1}{2}V^2[\cos\phi - \cos(\omega t+\phi)] , \tag{A8}$$

$$W^{(3)} = -i[H_0,[H_0,V]][\cos\phi - \cos(\omega t+\phi)] + i[V,[V,H_0]][\tfrac{1}{8}\sin 2\phi - \tfrac{1}{8}\sin(2\omega t+2\phi) - \tfrac{1}{2}\sin\omega t] - \frac{i}{2}(V[V,H_0]+[V,H_0]V)[\cos\phi - \cos(\omega t+\phi)][\sin\phi - \sin(\omega t+\phi)] + \frac{i}{6}V^3[\cos\phi - \cos(\omega t+\phi)]^3 . \tag{A9}$$

APPENDIX B

From (12a)–(12d), with conditions (24a) and (24b), simultaneously with (25a)–(25e) one finds, for the first four terms in expansion (7),

$$S_1=-V\cos(\omega t+\phi)\ , \tag{B1}$$

$$S_2=i[H_0,V]\sin(\omega t+\phi)\ , \tag{B2}$$

$$S_3=[H_0,[V,H_0]]\cos(\omega t+\phi)+\tfrac{1}{8}[V,[V,H_0]]\sin(2\omega t+2\phi)\ , \tag{B3}$$

$$S_4=S_4^0-i\tfrac{3}{16}[V,[H_0,[H_0,V]]]\cos(2\omega t+2\phi)+i[H_0,[H_0,[H_0,V]]]\sin(\omega t+\phi)\ . \tag{B4}$$

The corresponding expansions (21a) and (21b) for momentum and position operators are given by

$$p_i^{(1)}=i[P_i,V]\cos(\omega t+\phi)\ , \tag{B5}$$

$$p_i^{(2)}=-[P_i,[V,H_0]]\sin(\omega t+\phi)\ , \tag{B6}$$

$$p_i^{(3)}=i[P_i,[H_0,[H_0,V]]]\cos(\omega t+\phi)-\frac{i}{8}[P_i,[V,[H_0,V]]]\sin(2\omega t+2\phi)+\frac{i}{2}[[H_0,V],[V,P_i]]\sin(2\omega t+2\phi)\ , \tag{B7}$$

$$\begin{aligned}p_i^{(4)}=&\,i[S_4^0,P_i]-\tfrac{1}{4}[V,[P_i,[H_0,[V,H_0]]]]+\tfrac{1}{4}[[H_0,V],[P_i,[V,H_0]]]\\&+\tfrac{1}{4}[[V,P_i],[H_0,[H_0,V]]]-[P_i,[H_0,[H_0,[V,H_0]]]]\sin(\omega t+\phi)\\&+(\tfrac{1}{2}[[V,P_i],[H_0,[H_0,V]]]-\tfrac{1}{4}[[H_0,V],[P_i,[V,H_0]]]-\tfrac{1}{16}[P_i,[V,[H_0,[V,H_0]]]])\cos(2\omega t+2\phi)\ ,\end{aligned} \tag{B8}$$

$$q_i^{(1)}=0\ , \tag{B9}$$

$$q_i^{(2)}=-[Q_i,[V,H_0]]\sin(\omega t+\phi)\ , \tag{B10}$$

$$q_i^{(3)}=i[Q_i,[H_0,[H_0,V]]]\cos(\omega t+\phi)\ , \tag{B11}$$

$$\begin{aligned}q_i^{(4)}=&\,i[S_4^0,Q_i]-\tfrac{1}{4}[V,[Q_i,[H_0,[V,H_0]]]]+\tfrac{1}{4}[[H_0,V],[Q_i,[V,H_0]]]\\&-(\tfrac{1}{16}[Q_i,[V,[H_0,[V,H_0]]]]+\tfrac{1}{4}[[H_0,V],[Q_i,[V,H_0]]])\cos(2\omega t+2\phi)-[Q_i,[H_0,[H_0,[V,H_0]]]]\sin(\omega t+\phi)\ .\end{aligned} \tag{B12}$$

Now, the conditions that the zero-frequency Fourier component of $p_i^{(4)}$ and $q_i^{(4)}$ in Eqs. (B8) and (B12) vanish, obviously fix the commutators $[S_4^0,P_i]$ and $[S_4^0,Q_i]$. On the other hand, from the Jacobi identity it follows that

$$[[S_4^0,P_i],Q_j]=[[S_4^0,Q_j],P_i],\quad i,j=1,2,\ldots,N\ . \tag{B13}$$

Upon substitution, and after some algebra with repeated application of the Jacobi identity, the condition (B13) gives

$$[Q_j,[[V,P_i],[[H_0,V],H_0]]]=[[P_i,[H_0,V]],[Q_j,[H_0,V]]]\ . \tag{B14}$$

Substituting expression (6b) for H_0 in (B14) and explicitly calculating commutators we obtain condition (26) quoted in the text.

APPENDIX C

In the case of a uniform external force the separation of mean motion can be carried out at least up to the sixth order. The corresponding expansion coefficients in Eqs. (7), (21a), and (21b) are (summation over repeated indices is assumed)

$$S_1=-f_iQ_i\cos(\omega t+\phi)\ , \tag{C1}$$

$$S_2=f_iP_i\sin(\omega t+\phi)\ , \tag{C2}$$

$$S_3=-f_i\frac{\partial U}{\partial Q_i}\cos(\omega t+\phi)-\tfrac{1}{8}f^2\sin(2\omega t+2\phi)\ , \tag{C3}$$

$$S_4=\frac{i}{2}\left[P^2,f_i\frac{\partial U}{\partial Q_i}\right]\sin(\omega t+\phi)\ , \tag{C4}$$

$$S_5=\tfrac{1}{2}\left[\left[P^2,f_i\frac{\partial U}{\partial Q_i}\right],\frac{P}{2}+U\right]\cos(\omega t+\phi)-\tfrac{1}{8}f_if_j\frac{\partial^2 U}{\partial Q_i\partial Q_j}\sin(2\omega t+2\phi)\ , \tag{C5}$$

$$S_6=i\left[f_if_j\frac{\partial^2U}{\partial Q_i\partial Q_j},P^2\right][\tfrac{3}{8}+\tfrac{5}{16}\cos(2\omega t+2\phi)]+i\left[\frac{P^2}{2}+U,\left[\frac{P^2}{2}+U,\left[\frac{P^2}{2},f_i\frac{\partial U}{\partial Q_i}\right]\right]\right]\sin(\omega t+\phi)\ , \tag{C6}$$

$$p_i^{(1)}=f_i\cos(\omega t+\phi)\ , \tag{C7}$$

$$p_i^{(2)}=0\ , \tag{C8}$$

$$p_i^{(3)}=f_j\frac{\partial^2U}{\partial Q_i\partial Q_j}\cos(\omega t+\phi)\ , \tag{C9}$$

$$p_i^{(4)}=i\left[f_j\frac{\partial^2U}{\partial Q_i\partial Q_j},\frac{P^2}{2}\right]\sin(\omega t+\phi)\ , \tag{C10}$$

$$p_i^{(5)}=\frac{i}{2}\left[P_i,\left[\frac{P^2}{2}+U,\left[P^2,f_j\frac{\partial U}{\partial Q_j}\right]\right]\right]\cos(\omega t+\phi)+\tfrac{1}{8}f_jf_k\frac{\partial^3U}{\partial Q_i\partial Q_j\partial Q_k}\sin(2\omega t+2\phi)\ , \tag{C11}$$

$$p_i^{(6)}=\frac{i}{16}\left[\frac{P^2}{2},f_jf_k\frac{\partial^3U}{\partial Q_i\partial Q_j\partial Q_k}\right]\cos(2\omega t+2\phi)-\left[P_i,\left[\frac{P^2}{2}+U,\left[\frac{P^2}{2}+U,\left[f_j\frac{\partial U}{\partial Q_j},\frac{P^2}{2}\right]\right]\right]\right]\sin(\omega t+\phi)\ , \tag{C12}$$

$$q_i^{(1)}=0\ , \tag{C13}$$

$$q_i^{(2)}=f_i\sin(\omega t+\phi)\ , \tag{C14}$$

$$q_i^{(3)}=0\ , \tag{C15}$$

$$q_i^{(4)}=f_j\frac{\partial^2U}{\partial Q_i\partial Q_j}\sin(\omega t+\phi)\ , \tag{C16}$$

$$q_i^{(5)}=\frac{i}{2}\left[Q_i,\left[\frac{P^2}{2},\left[P^2,f_j\frac{\partial U}{\partial Q_j}\right]\right]\right]\cos(\omega t+\phi)\ , \tag{C17}$$

$$q_i^{(6)}=-\tfrac{1}{16}f_kf_j\frac{\partial^3U}{\partial Q_i\partial Q_j\partial Q_k}\cos(2\omega t+2\phi)-\left[Q_i,\left[\frac{P^2}{2}+U,\left[\frac{P^2}{2}+U,\left[f_j\frac{\partial U}{\partial Q_j},\frac{P^2}{2}\right]\right]\right]\right]\sin(\omega t+\phi)\ . \tag{C18}$$

Note that $S_n^0=0$ for $n=1,2,\ldots,5$, but $S_6^0\neq 0$ in order to vanish the zero-frequency Fourier components of $q_i^{(6)}$ and $p_i^{(6)}$.

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