Phase operators for SU(1,1): Application to the squeezed vacuum

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A phase-operator formalism is constructed for the Lie algebra of SU(1,1). Uncertainty relations, analogous to the usual number-phase relations, are also constructed. These relations are then evaluated for SU(1,1) coherent states, of which the squeezed-vacuum state of the electromagnetic field is an example. We argue that the new phase, in the case of single-mode fields, may be interpreted as the phase associated with the square of the field amplitude.

I. INTRODUCTION

Recently there has been some interest in the phase operators associated with the squeezed states of the electromagnetic field. Sanders *et al.*[1] use the phase operator formalism of Susskind and Glogower[2] (see also Refs. 3, 4, and 5) to show that squeezed states (actually squeezed-vacuum states exhibit phase sensitive fluctuations. The number-phase uncertainty relations do not approach the semiclassical uncertainty products as for the case of the ordinary coherent states. This would be as expected since there is no classical analog to the squeezed states. On the other hand, Lynch[6] has described a phase-operator formalism where the phase operator may be interpreted as the relative phase between the real and imaginary components of the field amplitude. The expectation values of these phase operators were found for squeezed states and, as expected, reduce to the classical value for small squeezing with large-field excitation. Finally, we mention that the phase operators for squeezed light produced from a nonabsorbing nonlinear medium modeled as an anharmonic oscillator also exhibit enhanced fluctuation.[7]

In this paper we consider an alternate approach to the formulation of phase operators for the squeezed-vacuum states. As has been shown,[8] the squeezed-vacuum states are an example of generalized coherent states (CS) associated with the noncompact Lie group SU(1,1). We therefore devise generalized phase operators given in terms of the elements of the su(1,1) Lie algebra. A similar set of phase operators has been considered for the SU(2) spin coherent states and studied in the case of high spin—the classical limit.[9] Our interest for the SU(1,1) case will be mainly for the squeezed-vacuum states of a single-mode field, where the results are nonclassical in the limit of high squeezing.

In Sec. II we briefly review the su(1,1) Lie algebra and the relevant representations of su(1,1). Coherent states are also reviewed. In Sec. III we present the generalized phase variables and define various uncertainty relations analogous to those of the usual Heisenberg algebra.[2–5] We evaluate these uncertainty products for the squeezed vacuum and discuss the limiting cases of high and low squeezing. In Sec. IV we argue that the new phase operator, in the case of single-mode quantum fields, is associated with the quadratures of the square of the field amplitude. In Sec. V we conclude with a brief summary.

II. su(1,1) LIE ALGEBRA AND COHERENT STATES

In this section we briefly review the su(1,1) Lie algebra and the associated coherent states. We use the Perelomov definition[10] of generalized coherent states.

The Lie algebra of SU(1,1) consist of three generators K_0 and $K_\pm$ satisfying the commutation relations

$$[K_0,K_\pm]=\pm K_\pm,\quad [K_-,K_+]=2K_0\ . \tag{2.1}$$

The Casimir operator is

$$C=K_0^2-\tfrac{1}{2}(K_+K_-+K_-K_+)\ . \tag{2.2}$$

We shall be interested only in the unitary irreducible representation known as the positive discrete series $\mathcal{D}^+(k)$, where k is the Bargmann index and the eigenvalue of C is $k(k-1)$. For the $\mathcal{D}^+(k)$ representations we denote the basis states as $|m,k\rangle$, where K_0 is diagonal according to $K_0|m,k\rangle=(m+k)|m,k\rangle$, $m=0,1,2,\ldots$ and $k>0$. The actions of K_+ and K_- are

$$\begin{aligned}K_+|m,k\rangle&=[(m+1)(m+2k)]^{1/2}|m+1,k\rangle\ ,\\ K_-|m,k\rangle&=[m(m+2k-1)]^{1/2}|m-1,k\rangle\ .\end{aligned} \tag{2.3}$$

A realization of this algebra relevant to single-mode fields is the oscillator realization given in terms of the operators a and a^-. We have

$$\begin{aligned}K_0&=\tfrac{1}{4}(a^\dagger a+aa^\dagger)=\tfrac{1}{2}(a^\dagger a+\tfrac{1}{2})\ ,\\ K_+&=\tfrac{1}{2}(a^\dagger)^2,\quad K_-=\tfrac{1}{2}a^2\ .\end{aligned} \tag{2.4}$$

In this case we obtain $C=-\frac{3}{16}$, so that $k=\frac{1}{4},\ \frac{3}{4}$. The $k=\frac{1}{4}$ representation corresponds to states with even numbers of photons, while the $k=\frac{3}{4}$ states contain odd numbers of photons. This is easily seen by noting that the number operator $N=a^\dagger a$ may be written, from Eq. (2.4), as

$$N=2K_0-\tfrac{1}{2}\ . \tag{2.5}$$

Coherent states for SU(1,1), following Perelomov,[10] are given as

$$|\xi,k\rangle=S(\alpha)|0,k\rangle\ , \tag{2.6}$$

where

$$S(\alpha)=\exp(\alpha K_+ -\alpha^* K_-) \quad (2.7)$$

and where $\alpha=\frac{1}{2}\theta e^{-i\varphi}$. The variables θ and φ are group parameters and have the ranges $-\infty<\theta<\infty$ and $0\le\varphi\le 2\pi$. Expanding in terms of the $\mathcal{D}^+(k)$ basis we have

$$|\xi,k\rangle=(1-|\xi|^2)^k \sum_{m=0}^{\infty}\left[\frac{\Gamma(m+\frac{1}{2})}{m!\Gamma(\frac{1}{2})}\right]^{1/2}\xi^m|m,k\rangle , \quad (2.8)$$

where $\xi=-\tanh(\frac{1}{2}\theta)e^{-i\varphi}$. Properties of these states may be found in Refs. 10 and 8 and references therein.

Now consider the single-mode field again with the Lie algebra realized as in Eq. (2.4). We define the quadrature operators

$$X_1=\tfrac{1}{2}(a+a^\dagger),\quad X_2=\frac{1}{2i}(a-a^\dagger) , \quad (2.9)$$

such that

$$[X_1,X_2]=\frac{i}{2} , \quad (2.10)$$

from which follows the uncertainty product

$$(\Delta X_1)^2(\Delta X_2)^2\ge\tfrac{1}{16} . \quad (2.11)$$

Squeezing[11] exists if $(\Delta X_1)^2<\frac{1}{4}$ or $(\Delta X_2)^2<\frac{1}{4}$. For the SU(1,1) CS we have $\langle X_i\rangle=0$, $i=1,2$, so that the variance, in the X_1 quadrature, for example, may be written as[8]

$$(\Delta X_1)^2=\langle K_0\rangle+\tfrac{1}{2}\langle K_+ +K_-\rangle . \quad (2.12)$$

Now for the SU(1,1) CS for the squeezed-vacuum state, with $k=\frac{1}{4}$, the average number of photons in the state is

$$n=\langle N\rangle=\frac{1}{2}\left[\frac{1+|\xi|^2}{1-|\xi|^2}\right]-\frac{1}{2} . \quad (2.13)$$

With the phase $\varphi=0$ the variance of X_1 becomes[8]

$$(\Delta X_1)^2=\tfrac{1}{2}(n+\tfrac{1}{2})-\tfrac{1}{2}[n(n+1)]^{1/2}\le\tfrac{1}{4} . \quad (2.14)$$

The greater the average number of photons, the greater the degree of squeezing. This kind of squeezed-vacuum state may be produced by the degenerate parametric amplifier.[8]

III. PHASE OPERATORS FOR su(1,1)

We define the exponential operator

$$\hat{e}=(K_-K_+)^{-1/2}K_-,\quad \hat{e}^\dagger=K_+(K_-K_+)^{-1/2} . \quad (3.1)$$

Since $[K_0,K_-K_+]=0$, one can show that

$$[K_0,\hat{e}]=-\hat{e},\quad [K_0,\hat{e}^\dagger]=\hat{e}^\dagger . \quad (3.2)$$

We note that

$$\hat{e}|m,k\rangle=|m-1,k\rangle,\quad \hat{e}^\dagger|m,k\rangle=|m+1,k\rangle . \quad (3.3)$$

In analogy to the usual case, we define sine and cosine operators

$$\hat{S}=\frac{1}{2i}(\hat{e}-\hat{e}^\dagger),\quad \hat{C}=\frac{1}{2}(\hat{e}+\hat{e}^\dagger) . \quad (3.4)$$

The commutation relations

$$[K_0,\hat{C}]=-i\hat{S} , \quad (3.5a)$$

$$[K_0,\hat{S}]=i\hat{C} , \quad (3.5b)$$

lead to the respective uncertainty relations

$$(\Delta K_0)(\Delta C)\ge\tfrac{1}{2}|\langle\hat{S}\rangle| . \quad (3.6a)$$

$$(\Delta K_0)(\Delta S)\ge\tfrac{1}{2}|\langle\hat{C}\rangle| . \quad (3.6b)$$

Thus one can form generalizations of the uncertainty relations proposed by Caruther and Nieto,[3] namely,

$$U_1=(\Delta K_0)^2(\Delta C)^2/\langle\hat{S}\rangle^2\ge\tfrac{1}{4} , \quad (3.7a)$$

$$U_2=(\Delta K_0)^2(\Delta S)^2/\langle\hat{C}\rangle^2\ge\tfrac{1}{4} , \quad (3.7b)$$

and the symmetrical form

$$U_3=(\Delta K_0)^2\frac{(\Delta S)^2+(\Delta C)^2}{(\langle\hat{S}\rangle^2+\langle\hat{C}\rangle^2)} . \quad (3.7c)$$

We first calculate the necessary expectation values with respect to the SU(1,1) CS $|\xi,k\rangle$ for arbitrary index k. Denoting these expectation values as $\langle\ \rangle$, it is straightforward to show that

$$\langle\hat{C}\rangle=|\xi|\cos\varphi(1-|\xi|^2)^{2k}S_1 , \quad (3.8a)$$

$$\langle\hat{S}\rangle=-|\xi|\sin\varphi(1-|\xi|^2)^{2k}S_1 , \quad (3.8b)$$

$$\langle\hat{C}^2\rangle=\tfrac{1}{2}-\tfrac{1}{4}(1-|\xi|^2)^{2k} +\tfrac{1}{2}|\xi|^2(1-|\xi|^2)^{2k}\cos(2\varphi)S_2 , \quad (3.8c)$$

$$\langle\hat{S}^2\rangle=\tfrac{1}{2}-\tfrac{1}{4}(1-|\xi|^2)^{2k} -\tfrac{1}{2}|\xi|^2(1-|\xi|^2)^{2k}\cos(2\varphi)S_2 , \quad (3.8d)$$

where

$$S_1=\sum_{m=0}^{\infty}(2k)_m\left[\frac{m+2k}{m+1}\right]^{1/2}\frac{|\xi|^{2m}}{m!} , \quad (3.9a)$$

$$S_2=\sum_{m=0}^{\infty}(2k)_m\left[\frac{(m+2k+1)(m+2k)}{(m+2)(m+1)}\right]^{1/2}\frac{|\xi|^{2m}}{m!} , \quad (3.9b)$$

and where $(\ )_m$ is the usual Pochammer symbol. The minus sign in Eq. (3.8b) is due to the manner of defining the phase of the variable ξ. Also we need

$$\langle K_0\rangle=k\cosh\theta,\quad \langle K_0^2\rangle=\tfrac{1}{2}k(k+1)(\cosh^2\theta-1)+k^2 , \quad (3.10)$$

from which we obtain

$$(\Delta K_0)^2=\tfrac{1}{2}k\sinh^2\theta . \quad (3.11)$$

For the case of oscillator realization it is possible to show that $(\Delta K_0)^2=(\Delta N)^2/4$. In turn, it can be shown that for the SU(1,1) CS with $k=\frac{1}{4}$ that $(\Delta N)^2=2(n^2+n)$, where

n is given by Eq. (2.13).

Note that

$$\langle \hat{C}^2\rangle+\langle \hat{S}^2\rangle=1-\tfrac{1}{2}(1-|\xi|^2)^{2k}\ , \tag{3.12}$$

which has the limits $\frac{1}{2}$ and 1 for $|\xi|$ small and large ($|\xi|\to 1$), respectively, just as in the case of the usual phase operators.[2-5] Note also that for $|\xi|\to 0$,

$$\langle \hat{C}\rangle=(2k)^{1/2}|\xi|\cos\varphi\ , \tag{3.13a}$$

$$\langle \hat{S}\rangle=-(2k)^{1/2}|\xi|\sin\varphi\ , \tag{3.13b}$$

$$\langle \hat{C}^2\rangle=\tfrac{1}{4}+\tfrac{1}{2}|\xi|^2\sqrt{k(2k+1)}\cos(2\varphi)\ , \tag{3.13c}$$

$$\langle \hat{S}^2\rangle=\tfrac{1}{4}-\tfrac{1}{2}|\xi|^2\sqrt{k(2k+1)}\cos(2\varphi)\ . \tag{3.13d}$$

We now consider the asymptotic case for $|\xi|\to 1$. For finite k the higher-order terms in the series S_1 must be very close to those of the series

$$S_0=\sum_{m=0}^{\infty}(2k)_m\frac{|\xi|^{2m}}{m!}$$
$$=\frac{1}{(2k-1)!}\sum_{m=0}^{\infty}\frac{(|\xi|^2)^m}{m!}(2k+m-1)!\ , \tag{3.14}$$

which is, in fact, a sort of hypergeometric function. Using standard procedures,[12] this series can be written in integral form as

$$S_0=\frac{1}{(2k-1)!}\int_0^{\infty}du\ e^{-u(1-x)}u^{2k-1}\ , \tag{3.15}$$

where we have set $|\xi|^2=x$. Now make the change of variable $y=u(1-x)$. Since $0\le x<1$, the limits are unchanged and we obtain

$$S_0=\frac{1}{(2k-1)!}\frac{1}{(1-x)^{2k}}\int_0^{\infty}e^{-y}y^{2k-1}dy\ . \tag{3.16}$$

But

$$\int_0^{\infty}e^{-y}y^{2k-1}dy=(2k-1)!\ , \tag{3.17}$$

so we have

$$S_0=\frac{1}{(1-|\xi|^2)^{2k}}\ . \tag{3.18}$$

Now suppose we write

$$S_0=\sum_{m=0}^{\infty}A_m x^m \tag{3.19}$$

and

$$S_1=\sum_{m=0}^{\infty}B_m x^m\ , \tag{3.20}$$

where

$$A_m=(2k)_m/m!$$

and

$$B_m=A_m[(m+2k)/(m+1)]^{1/2}\ .$$

For sufficiently large M we can write

$$S_1=\sum_{m=0}^{M}B_m x^m+\sum_{m=M}^{\infty}A_m x^m\ . \tag{3.21}$$

On the other hand, we have

$$S_0=\sum_{m=0}^{M}A_m x^m+\sum_{m=M}^{\infty}A_m x^m\ , \tag{3.22}$$

so that

$$S_1=S_0+\sum_{m=0}^{M}A_m\left[1-\left[\frac{m+2k}{m+1}\right]^{1/2}\right]x^m \tag{3.23}$$

$$=\frac{1}{(1-|\xi|^2)^{2k}}+\sum_{m=0}^{M}\frac{(2k)_m}{m!}\left[1-\left[\frac{m+2k}{m+1}\right]^{1/2}\right]|\xi|^{2m}\ . \tag{3.24}$$

Thus we have

$$(1-|\xi|^2)^{2k}S_1=1+(1-|\xi|^2)^{2k}\sum_{m=0}^{M}\frac{(2k)_m}{m!}\left[1-\left[\frac{m+2k}{m+1}\right]^{1/2}\right]|\xi|^{2m}\ . \tag{3.25}$$

Apparently, as $|\xi|\to 1$, the last term becomes vanishingly small. Thus in this limit

$$\langle \hat{C}\rangle\approx|\xi|\cos\varphi\approx\cos\varphi\ , \tag{3.26a}$$

$$\langle \hat{S}\rangle\approx-|\xi|\sin\varphi\approx-\sin\varphi\ . \tag{3.26b}$$

A similar analysis for S_2 leads to

$$(1-|\xi|^2)^{2k}S_2=1+(1-|\xi|^2)^{2k}\sum_{m=0}^{M}\frac{(2k)_m}{m!}\left[1-\left[\frac{(m+2k)(m+2k+1)}{(m+1)(m+2)}\right]^{1/2}\right]|\xi|^{2m}\ , \tag{3.27}$$

from which follows that as $|\xi|\to 1$,

$$\langle \hat{C}^2\rangle\approx\cos^2(\varphi)\ , \tag{3.28a}$$

$$\langle \hat{S}^2\rangle\approx\sin^2(\varphi)\ . \tag{3.28b}$$

Apparently then the preceding analysis shows that in the limit of high-field excitation ($|\xi|\to 1$) we recover the classical results for the expectation values $\hat{C}$, $\hat{S}$, $\hat{C}^2$ and $\hat{S}^2$, just as is the case for the ordinary coherent states.

Next we examine the uncertainty products in these limits.

In the limit of low excitation, $|\xi|\to 0$, the uncertainty product $U(|\xi|^2,\varphi)$ becomes, from Eqs. (3.11) and (3.13),

$$U_1(|\xi|^2_1\varphi)\approx\tfrac{1}{2}k\sinh^2\theta\frac{\frac{1}{4}}{(2k)|\xi|^2\sin^2\varphi}, \quad (3.29)$$

where $(\Delta C)^2=\frac{1}{4}$ in this limit. Using the identity

$$|\xi|=\tanh(\theta/2)=\sinh\theta/(\cosh\theta+1), \quad (3.30)$$

then, as $|\xi|\to 0$, we obtain

$$\lim_{|\xi|\to 0} U_1(|\xi|^2,\varphi)=\frac{1}{4\sin^2\varphi}\geq\tfrac{1}{4}. \quad (3.31)$$

Similarly, for $U_2(|\xi|^2,\varphi)$,

$$\lim_{|\xi|\to 0} U_2(|\xi|^2,\varphi)=\frac{1}{4\cos^2\varphi}\geq\tfrac{1}{4}, \quad (3.32)$$

and for $U_3(|\xi|^2)$, which is independent of φ,

$$\lim_{|\xi|\to 0} U_3(|\xi|^2)=\tfrac{1}{2}. \quad (3.33)$$

These low-excitation limits are the same as those obtained in the corresponding uncertainty products for the ordinary coherent states for low-field excitation.

However, in the case of high excitation, we find, contrary to the case of the ordinary coherent states, that the uncertainty products tend to increase without bound rather than to minimize. It is easiest to demonstrate this numerically. We consider only the case of the $k=\frac{1}{4}$ states corresponding to the squeezed states of the single-mode electromagnetic field. In Fig. 1 we exhibit U_1 for various phase angles as a function of average photon number n. As can quite clearly be seen, for higher n, fluctuations become increasingly enhanced. Similar behavior is also evident for U_3, which is illustrated in Fig. 2. These enhanced fluctuations are expected since the squeezed state has no classical counterpart. We also recall that for higher n, the greater the squeezing and from Figs. 1 and 2, the greater the fluctuations in the number-phase uncertainty products.

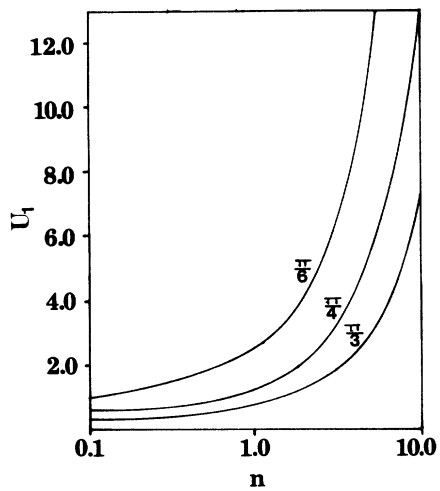

FIG. 1. U_1 vs n for $\varphi=\pi/6$, $\pi/4$, and $\pi/5$.

IV. PHASE OPERATORS FOR su(1,1) AND THE QUADRATURE COMPONENTS OF THE SQUARED FIELD AMPLITUDE

The formalism that we have developed is valid for any $\mathcal{D}^+(k)$ representation of SU(1,1). In this section we consider the particular realization of the su(1,1) Lie algebra given in terms of the boson operators, Eq. (2.4). This is the appropriate realization for applications in quantum optics if we restrict ourselves to single-mode fields. The question arises as to the physical interpretation of the SU(1,1) phase operators in this case.

Now the usual phase operators associated with the Heisenberg-Weyl algebra of the bose operators a and $a^\dagger$ may be written as

$$\hat{e}=(aa^\dagger)^{1/2}a,\quad \hat{e}^\dagger=a^\dagger(aa^\dagger)^{1/2}. \quad (4.1)$$

These operators in turn may be associated with the phase of the quadrature operators X_1 and X_2 of Eq. (2.9). The electric field may be expressed in terms of X_1 and X_2 as

$$E=X_1\cos(\omega t)+X_2\sin(\omega t). \quad (4.2)$$

Now if we square the field amplitude we obtain

$$E^2=Y_0+Y_1\cos(2\omega t)+Y_2\sin(2\omega t), \quad (4.3)$$

where

$$Y_0=\tfrac{1}{4}(a^\dagger a+aa^\dagger),$$
$$Y_1=\tfrac{1}{4}[a^2+(a^\dagger)^2], \quad (4.4)$$
$$Y_2=\frac{1}{4i}[a^2-(a^\dagger)^2].$$

The operators Y_1 and Y_2 are the quadratures of the squared field amplitude and satisfy the commutation relations

$$[Y_1,Y_2]=\tfrac{1}{4}i(2N+1), \quad (4.5)$$

from which one obtains the uncertainty relation

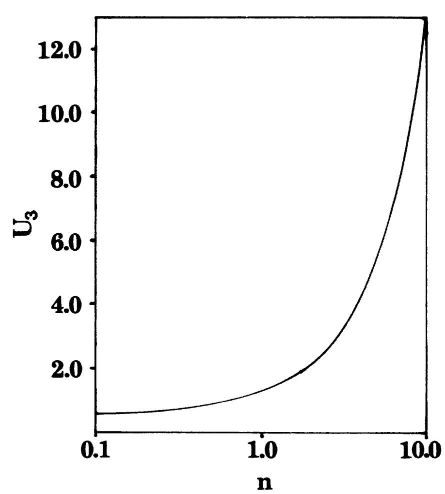

FIG. 2. U_3 vs n.

$$(\Delta Y_1)(\Delta Y_2) \geq \tfrac{1}{4} | \langle N+\tfrac{1}{2} \rangle | . \quad (4.6)$$

Hillery[13] has shown that the squeezing of the squared field amplitude is a nonclassical effect. This amounts to the requirement that $(\Delta Y_1)^2 < \frac{1}{2} | \langle N+\frac{1}{2} \rangle |$ or $(\Delta Y_2)^2 < \frac{1}{2} | \langle N+\frac{1}{2} \rangle |$.

However, Gerry and Vrscay[14] have pointed out that the algebra of the squared field amplitude is isomorphic to the su(1,1) Lie algebra. This is easy to see by noting, from Eq. (2.4), that

$$K_1 = \tfrac{1}{4}[a^2 + (a^\dagger)^2] = Y_1 ,$$
$$K_2 = \frac{1}{4i}[a^2 - (a^\dagger)^2] = Y_2 , \quad (4.7)$$
$$K_0 = \tfrac{1}{4}(a^\dagger a + a a^\dagger) = \tfrac{1}{2}(N + \tfrac{1}{2}) = Y_0 ,$$

such that

$$[K_1, K_2] = -iK_0 . \quad (4.8)$$

From this commutator it follows that

$$(\Delta K_1)(\Delta K_2) \geq \tfrac{1}{2} | \langle K_0 \rangle | , \quad (4.9)$$

which is equivalent to Eq. (4.6). These uncertainty relations and the generalized squeezing have been discussed by Wodkiewicz and Eberly.[5] Through the definition of Eqs. (3.1), or in terms of the bose operators, we have the new phase operators as

$$\hat{e} = \tfrac{1}{4}[a^2(a^\dagger)^2]^{-1/2} a^2, \quad \hat{e}^\dagger = \tfrac{1}{4}(a^\dagger)^2 [a^2(a^\dagger)^2]^{-1/2} . \quad (4.10)$$

It would seem reasonable, by analogy, to interpret the new phase for the single-mode bose realizations as the phase associated with the quadratures of the squared field amplitude.

V. CONCLUSIONS

In this paper we have developed a phase-operator formalism for the dynamical group SU(1,1). Uncertainty relations similar to the usual number-phase relations have been constructed. When these relations are evaluated with SU(1,1) coherent states we find enhanced fluctuations for high excitation indicating a nonclassical behavior. For the case single-mode quantized electromagnetic field such a state is a squeezed-vacuum state. We have argued that the new phase operator is, for the single-mode-field case, the phase associated with the quadratures of the squared field amplitude.

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