Nuclear reorientation and Coulomb excitation in a magnetic field

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We study the scattering of a charged structureless particle by a nucleus, described by means of its electric multipole moments, projectile and target being immersed in a homogeneous and constant magnetic field. We present an analytical expression and a recurrence rule for the matrix elements of any multipole operator. The multipoles higher than zero induce reorientations of the target. The Coulomb excitation rates we obtained coincide in the zero-field limit with the well-known field-free result.

I. INTRODUCTION

In the recent past, astrophysics stimulated interest in studying how physical processes are altered when occurring in the presence of strong magnetic fields. White dwarfs and neutron stars seem to be endowed with intense magnetic fields up to 10^{13} G. The original proof that large magnetic fields must exist in collapsed objects comes from the flux conservation law[1] and from observation of Larmor lines in the spectrum of pulsars.[2] A thorough investigation of scattering processes of charged particles in strong magnetic fields is therefore of interest to astrophysics. Moreover, general properties of these scattering phenomena are of interest from a fundamental point of view.

Charged particles embedded in a uniform constant magnetic field can move freely along the field direction, while their transversal motion is confined in quantum Landau states. This characteristic of the charged particles motion dominates the dynamics of any collision event in intense magnetic fields, which impose their cylindrical symmetry on the scattering problem in contrast to the case of field-free scattering where the symmetry is spherical.

In the present work the motion of a charged particle moving in a homogeneous and constant magnetic field, scattered by a fixed system of charges like a nucleus, is formulated as a first-order perturbation problem. The magnetic field is taken into account exactly, whereas the Coulomb interaction is a perturbation field. The Coulomb excitation amplitudes of any multipolarity are obtained whatever the intensity of the field is. We give the appropriate definition of the rate of inelastic scattering and show that it agrees with the field-free rate in the zero-field limit. Until now, the studies of scattering of charged particles in a uniform constant magnetic field were restricted to elastic Coulomb scattering (the perturbation being the external monopole $1/r$) and to the particular case of electrons confined to low-lying Landau orbitals.[3–5]

The definition of Landau states has been treated exhaustively in the literature. In Sec. II we, however, recall some useful formulas concerning wave functions, energies, and density of states.

In Sec. III we formulate the scattering problem following two parallel approaches using either the S-matrix or Green's-function method. We warn the reader that the collision problem in the presence of a magnetic field has to be described, keeping in mind that the appropriate geometry is the cylindrical one. In this geometry we build a general expression for the rate of Coulomb excitation of any multipolarity. In the zero-field limit the spherical geometry is restored; the Coulomb scattering rates at this limit agree with the well-known expression of the cross section for the electromagnetic excitation of a nucleus by a charged particle.[6]

In Sec. IV we outline the evaluation of the matrix elements required to calculate the Coulomb scattering rates. We show in Sec. V that these matrix elements agree with the field-free elements in the zero-field limit.

II. BASIC DEFINITIONS

A. Energy states and wave functions

Let us consider the Schrödinger equation describing a structureless particle of charge q scattered by a central potential $V(\mathbf{r})$ in the presence of a uniform magnetic field $\mathbf{B}$:

$$H\psi=[H_0+V(\mathbf{r})]\psi=E\psi ,$$

with

$$H_0=\frac{1}{2m}\left[\mathbf{p}-\frac{q}{c}\mathbf{A}\right]^2 ,$$

where $\mathbf{A}$ is the vector potential of Coulomb gauge $\mathbf{A}=\frac{1}{2}\mathbf{B}\times\mathbf{r}$ and $\mathbf{r}$ is the position vector of the charged particle relative to the center of the potential.

We first call to mind eigenstates of H_0; these are known to be separable into a plane wave propagating parallel to the field and a two-dimensional harmonic-oscillator-type solution of the transversal motion. Taking the z axis along the magnetic field direction the Hamiltonian H_0 can be separated in the following way:

$$H_0=H_{\parallel}+H_{\perp} ,$$

with

$$H_{\parallel}=\frac{p_z^2}{2m}$$

and

$$H_z=\frac{p_x^2}{2m}+\frac{p_y^2}{2m}+\frac{m\Omega^2}{2}(x^2+y^2)-\Omega L ,$$

where $\Omega=\omega/2=|q|B/2mc$ is the Larmor frequency, ω being the cyclotron frequency.

Changing to cylindrical coordinates (ρ,ϕ,z), wave functions and energy states of H_0 are given by

$$\psi\equiv\phi_{n_\rho,\Lambda,k_z}=\exp(ik_z z)\phi_{n_\rho,\Lambda}(\rho,\phi) ,$$

$E=(k_z^2\hbar^2/2m)+\varepsilon_n$. The functions $\phi_{n_\rho,\Lambda}(\rho,\phi)$ and the energies $\varepsilon_n=\hbar\omega(n+\frac{1}{2})$ define the Landau states;

$$n=n_\rho+\frac{[|\Lambda|-(\mathrm{sgn}q)\Lambda]}{2}$$

is the principal quantum number and Λ the magnetic quantum number.

Explicitly, the Landau orbitals are harmonic-oscillator wave functions:

$$\phi_{n_\rho,\Lambda}(\rho,\phi)=\exp(i\Lambda\phi)R_{n_\rho,|\Lambda|}(\rho) ,$$

with radial functions

$$R_{n_\rho,|\Lambda|}(\rho)=\exp\left[-\frac{\rho^2}{2b^2}\right]\left[\frac{\rho}{b}\right]^{|\Lambda|}L_{n_\rho}^{|\Lambda|}\left[\frac{\rho^2}{b^2}\right] .$$

The parameter b defines the oscillator length

$$b=\gamma^{-1/2}=\left[\frac{2\hbar}{m\omega}\right]^{1/2} ;$$

$L_{n_\rho}^{|\Lambda|}$ are Laguerre polynomial defined by

$$L_n^\alpha(x)=(-1)^\alpha\frac{n!}{(n-\alpha)!}\exp(x)x^{-\alpha}\frac{d^{n-\alpha}}{dx^{n-\alpha}}[\exp(-x)x^n] .$$

A Landau state of principal quantum number n is infinitely degenerate.

Classically, the particle is spiralling around the direction of the field, the width of the spiral being smaller for a larger magnetic field and inversely

$$\langle r\rangle^2=\gamma^{-1}(n+\tfrac{1}{2}),\quad n=0,1,2,\ldots ,$$

where

$$\gamma^{-1}=\frac{2\hbar c}{|q|B} .$$

For a principal quantum number n, there is an infinity of such "classical orbits," their guiding centers being located on a cylindrical shell whose radius is

$$\langle r_0\rangle^2=\gamma^{-1}(s+\tfrac{1}{2}),\quad s=0,1,2,\ldots .$$

This relation gives a geometrical interpretation to the quantum number s and to the degeneracy of the nth Landau state. The quantum numbers n, Λ, and s are linearly dependent according to $\Lambda=(\mathrm{sgn}q)(s-n)$. The choice of the quantum numbers (n,s) leads to the Sokolov expression of the wave function ψ:

$$\psi\equiv\Phi_{n,s,k_z}=\exp(ik_z z)\phi_{n,s}(\rho,\phi) ,$$

where

$$\phi_{n,s}(\rho,\phi)=\exp[i(n-s)\phi]I_{n,s}(\gamma\rho^2)$$

and

$$I_{n,s}(\gamma\rho^2)=(n!s!)^{-1/2}\exp\left[-\gamma\frac{\rho^2}{2}\right](\gamma\rho^2)^{|n-s|/2}\times Q_s^{n-s}(\gamma\rho^2) .$$

The functions $Q_s^{n-s}(\gamma\rho^2)$ are associated Laguerre polynomials

$$Q_s^{n-s}(\gamma\rho^2)=\begin{cases}s!L_s^{n-s}(\gamma\rho^2) & \text{for } s<n\\ n!L_n^{s-n}(\gamma\rho^2) & \text{for } n<s .\end{cases}$$

The quantum numbers n,s are non-negative integers, the correspondence with the quantum numbers n_ρ,Λ being obviously

$$n=n_\rho+\frac{|\Lambda|-(\mathrm{sgn}q)\Lambda}{2},\quad \Lambda=(\mathrm{sgn}q)(s-n)$$

so that the correspondence is

$$(n_\rho,(\mathrm{sgn}q)\Lambda>0)\rightarrow(n=n_\rho,\ s=n+(\mathrm{sgn}q)\Lambda) ,$$

$$(n_\rho,(\mathrm{sgn}q)\Lambda<0)\rightarrow(n=n_\rho-(\mathrm{sgn}q)\Lambda,\ s=n_\rho) .$$

The energy of a state being defined by the principal quantum number n, there exists an infinity of degenerated states according to the "geometrical" quantum number s. This implies that a particular linear combination of pure states Φ_{n,s,k_z} has to be used in order to describe a scattering problem. Later on we will come back on this point (Sec. III).

B. Boundary conditions and density of states

We now give the definitions of the density of states and of the boundary conditions appropriate to the magnetic field case, in a way similar to that used for a three-dimensional plane wave (i.e., the field-free case).

Along the field direction **z**, the usual boundary condition for the plane wave $\exp(ik_z z)$ on a quantization length L_z defines the density of states $dn_z=L_z/2\pi$ and the plane wave normalization constant $(L_z)^{-1/2}$. Within the plane perpendicular to the field, the energy $\varepsilon_\perp=\hbar^2k_\perp^2/2m$ yields the quantized energy $\varepsilon_n=(n+\frac{1}{2})\hbar\omega$ and the quantized value $k_\perp^2=4\gamma(n+\frac{1}{2})$. Going to the classical limit, the following correspondence holds:

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}dk_x\int_{-\infty}^{+\infty}dk_y=\frac{1}{(2\pi)^2}\int_0^{2\pi}d\varphi_\perp\int_0^\infty k_\perp dk_\perp \rightarrow\sum_{n=0}^{\infty}\omega_n ,$$

where ω_n is the degeneracy of the nth Landau level. The continuum in $k_\perp^2$ coalesces into a discrete spectrum of levels such that the degeneracy between two successive Landau levels is

$$\omega_n = \frac{1}{(2\pi)^2}\int_{K_n}^{K_{n+1}} k_\perp dk_\perp \int_0^{2\pi} d\varphi_\perp ,$$

where $K_n^2 = 4\gamma(n+\frac{1}{2})$, i.e., $\omega_n = \gamma/\pi$. Moreover, the normalization of the wave function $\phi_{n,s}(\rho,\phi)$ on an elementary area (πL^2) in the plane perpendicular to the field is given by

$$N^2 \sum_{\substack{s \\ s\in(\pi L^2)}} \int_0^{2\pi} d\phi \int_0^{\infty} \rho\, d\rho\, |I_{n,s}(\gamma\rho^2)|^2 = 1$$

or

$$N^2 \sum_{\substack{s \\ s\in(\pi L^2)}} \frac{\pi}{\gamma} = 1 ,$$

owing to the normalization of the functions $I_{n,s}$,

$$\int_0^{\infty} dt\, |I_{n,s}(t)|^2 = 1 .$$

According to the relation $\langle r_0\rangle^2 = \gamma^{-1}(s+\frac{1}{2})$, the number of s quantum states in the basic area (πL^2) is given by

$$\int_0^{2\pi} d\varphi_0 \int_0^{L} r_0 dr_0 = 2\pi \sum_{\substack{s \\ s\in(\pi L^2)}} \frac{ds}{2\gamma} = \pi L^2 ,$$

i.e.,

$$\sum_{\substack{s \\ s\in(\pi L^2)}} ds = \gamma L^2 .$$

In conclusion, the normalization constant N of the Landau states is $N = (\pi L^2)^{-1/2}$.

In the basic cylindrical volume, $V = \pi L^2$. L_z, the general normalized wave function, is

$$\Phi_{n,s,k_z} = \frac{1}{L_z^{1/2}} \exp(ik_z z) \frac{1}{(\pi L^2)^{1/2}} \phi_{n,s}(\rho,\phi) , \qquad (2.1)$$

and the density of states is defined such that

$$\int \rho(n,s,k_z) d\mathbf{k} = \int_{-\infty}^{+\infty} \rho(k_z) dk_z \sum_{n=0}^{\infty} \rho(n) \sum_{s=0}^{\infty}$$

$$\equiv \frac{L_z}{2\pi} \int_{-\infty}^{+\infty} dk_z (\pi L^2) \frac{\gamma}{\pi} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} . \qquad (2.2)$$

C. The Coulomb interaction

The scattering of a structureless charged particle (q,m) by a nucleus of charge (ze), the system being embedded in a constant magnetic field $\mathbf{B} = B\mathbf{1}_z$, is described by the total Hamiltonian

$$H = H_0(\mathbf{r}) + H_N(\xi) + V(\mathbf{r},\xi) ,$$

where $H_0(\mathbf{r})$ is the Hamiltonian operator for the scattered particle in the constant magnetic field (Sec. II A), $H_N(\xi)$ is the intrinsic Hamiltonian describing the states of the nuclear target, and $V(\mathbf{r},\xi)$ is the interaction potential. We choose the Coulomb interaction

$$V(\mathbf{r},\xi) = q\frac{\rho(\xi)}{|\mathbf{r}-\xi|}$$

between the scattered particle whose $\mathbf{r}$ is the vector coordinate and the target nucleus whose ξ and $\rho(\xi)$ are, respectively, the internal coordinates and the nuclear charge density.

We expand the Coulomb potential in multipole components and retain the external terms only:

$$V(\mathbf{r},\xi) = q \sum_{l\geq 0} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} r^{-l-1} Y_{lm}(\omega) m^*_{E_{l,m}} , \qquad (2.3)$$

where $m_{E_{l,m}}$ defines the electric multipole operator of the nucleus,

$$m_{E_{l,m}} = \xi^l Y_{lm}(\omega_\xi)\rho(\xi) .$$

This expression of the Coulomb interaction factorizes the operators in the $\mathbf{r}$ and ξ coordinates. Moreover, the unperturbed states, solutions of the Schrödinger equation for $H_0(\mathbf{r}) + H_N(\xi)$, are also factorized in $\mathbf{r}$ and ξ,

$$\psi^0_\alpha(\mathbf{r},\xi) = \Phi_{n,s,k_z}(\mathbf{r})\,|I_\alpha, M_\alpha\rangle , \qquad (2.4)$$

where $\Phi_{n,s,k_z}(\mathbf{r})$ describes the motion of the scattered particle (2.1) and where $|I_\alpha, M_\alpha\rangle$ describes a nuclear state of angular momentum I_α and magnetic component M_α. Finally, the energy E^0_α is defined by the sum

$$E^0_\alpha = \frac{k_z^2\hbar^2}{2m} + \hbar\omega(n+\tfrac{1}{2}) + E_{I_\alpha} . \qquad (2.5)$$

III. COULOMB SCATTERING RATES OF ANY MULTIPOLARITY

The scattering amplitudes can be derived following two parallel approaches, the S-matrix formulation and the Green's-function method. To make things clear about the problem of the field-free limit, we give both formulations.

A. The Green's-function method

The scattering state $\Phi^+_\alpha(\mathbf{r})$ associated with the incoming wave $\Phi_\alpha(\mathbf{r})$ of the particle in the magnetic field is defined by an integral equation of the type

$$\Phi^+_\alpha(\mathbf{r}) = \Phi_\alpha(\mathbf{r}) + \int G^+_E(\mathbf{r},\mathbf{r}')v(\mathbf{r}')\Phi^+_\alpha(\mathbf{r}')d\mathbf{r}' \qquad (3.1)$$

where we use the shortened notation $\alpha \equiv (n,s,k_z)$. In (3.1), the perturbation $v(\mathbf{r})$ is one of the operators $[r^{-l-1}Y_{lm}(\omega)]$, a function of the projectile coordinates only, associated with the electric multipole operator $m_{E_{l,m}}$ of the target nucleus (cf. Sec. II C).

The function $G^+_E(\mathbf{r},\mathbf{r}')$ is the Green's function associated with the Hamiltonian H_0 of a particle in the presence of a magnetic field for a defined energy E. The integral representation of this Green's function is

$$G_E^+(\mathbf{r},\mathbf{r}')=\sum_{n',s'}\phi_{n',s'}(\rho,\phi)\phi^*_{n',s'}(\rho',\phi')\times\int_{-\infty}^{+\infty}\frac{dk_z'}{2\pi}\frac{\exp[ik_z'(z-z')]}{E-E_{n',s',k_z'}}, \tag{3.2}$$

where

$$E_{n',s',k_z'}\equiv\frac{k_z'^2\hbar^2}{2m}+\hbar\omega(n'+\tfrac{1}{2}) .$$

Starting from this general expression of the Green's function, some authors[3–5] perform the integration over dk_z' and obtain a new formula,

$$G_E^+(\mathbf{r},\mathbf{r}')=\frac{-im}{\hbar^2}\sum_{n',s'}\phi_{n',s'}(\rho,\phi)\phi^*_{n',s'}(\rho',\phi')\times\frac{\exp(ik_z'\,|\,z-z'\,|\,)}{|\,k_z'\,|}, \tag{3.3}$$

where

$$k_z'^2\equiv\frac{2m}{\hbar^2}E-4\gamma(n'+\tfrac{1}{2}) .$$

Taking the asymptotic limit $|\,z\,|\to\infty$, the scattering wave (3.1) transforms into

$$\Phi_\alpha^+(\mathbf{r})\equiv\Phi^+_{n,s,k_z}(\mathbf{r})\to\Phi_{n,s,k_z}(\mathbf{r})+\sum_{n',s'}\phi_{n',s'}(\rho,\phi)\frac{\exp(\pm ik_z'z)}{|\,k_z'\,|}\times A_{ns,n's'}^{(k_z,k_z')}\quad\text{as } z\to\infty , \tag{3.4}$$

where

$$A_{ns,n's'}(k_z,k_z')\equiv\frac{-im}{\hbar^2}\int\phi^*_{n',s'}(\rho,\phi)\exp(\mp ik_z'z)\times v(\mathbf{r})\Phi^+_{n,s,k_z}(\mathbf{r})d\mathbf{r} \tag{3.5}$$

appear as transmission and reflection coefficients of a one-dimensional scattering process.

In the Coulomb excitation, the energy conservation is guaranteed by $E_\alpha^0=E_\beta^0$ [Eq. (2.5)]. It means that $|\,k_z'\,|$ has to be replaced in the Green's function (3.3) by

$$|\,k_z'\,|=\left[k_z^2+4\gamma(n-n')-\frac{2m}{\hbar^2}\Delta E_{\alpha\beta}\right]^{1/2},$$

with $\Delta E_{\alpha\beta}\equiv E_{I_\beta}-E_{I_\alpha}$. The elastic scattering ($\Delta E_{\alpha\beta}=0$, $\Delta n=0$, $k_z'=\pm k_z$) is similar to the collision in one dimension: observable quantities being transmission coefficient containing the contribution from the transmitted wave without scattering. The presence of the magnetic field allows for transitions between Landau states, $\Delta n\neq0$ without nuclear excitation, $\Delta E_{\alpha\beta}=0$. These define a pseudoelastic scattering.

If one assumes that the nuclear Zeeman states remain degenerated in energy, our definition of elastic and pseudoelastic scattering includes the nuclear reorientation process (cf. Sec. III C). The inelastic scattering rates ($\Delta E_{\alpha\beta}\neq0$) are proportional to the square of the transition amplitudes (3.5) from an initial state Φ_{n,s,k_z} to a final one $\Phi_{n',s',k_z'}$. Although the state Φ_{n,s,k_z} does not reduce to a plane wave at the zero-field limit, the Green's function $G_E^+(\mathbf{r},\mathbf{r}')$ as defined in (3.2) reduces to the plane wave Green's function.[7] However, the scattering wave function defined in (3.4) is an asymptotic form ($|\,z\,|\to\infty$), from which it is no more possible to come back to a spherical scattering wave by the extinction of the magnetic field. That kind of treatment implicitly assumes the occurrence of a very intense magnetic field leading to an extremely thin confinement around the field direction. The scattering rates $|\,A_{ns,n's'}(k_z,k_z')\,|^2$, defined from (3.5) with Landau states as initial and final states, take the correct field-free form (cf. Sec. V); however to obtain the correct field-free scattering cross section, one has to perform both the summations over (n',s',k_z') and the average over (n,s,k_z) according to the precise definitions of the density of states for a charged particle in a magnetic field (cf. Sec. II). Moreover, in a beam scattering experiment, the distribution of the initial states of the charged particles inside the field has to be defined owing to the physical situation involved (cf. Sec. V).

Starting from the general expression (3.2) of the Green's function, another point of view has been adopted by one author[8] who performs the integration and both summations, obtaining a closed expression. The scattering wave function built by means of that Green's function is a regular function of the magnetic field. Consequently, the extinction of the magnetic field is exactly obtained, leading to the well-known results of reference.[6]

B. The *S*-matrix formulation

In the *S*-matrix formulation, the required *S*-matrix element is

$$\langle\beta\,|\,S\,|\,\alpha\rangle=\langle\beta\,|\,\alpha\rangle-\frac{i}{\hbar}\int dt\int d\mathbf{r}\,d\xi\,\psi_\beta^{0*}(\mathbf{r},\xi,t)V(\mathbf{r},\xi)\times\psi_\alpha^+(\mathbf{r},\xi,t) ,$$

where

$$\psi_\alpha^0(\mathbf{r},\xi,t)=\psi_\alpha^0(\mathbf{r},\xi)\exp\left[\frac{i}{\hbar}E_\alpha^0t\right]$$

is a solution of $[H_0(\mathbf{r})+H_N(\xi)]\psi=i\hbar\delta_t\psi$ and $\psi_\alpha^+(\mathbf{r},\xi,t)$ is the scattering wave associated with $\psi_\alpha^0(\mathbf{r},\xi,t)$ at $t\to-\infty$. The average transition probability per unit time and per energy interval dE_β^0 in the final state is

$$w_{\beta\alpha}=|\,\langle\beta\,|\,S-1\,|\,\alpha\rangle\,|^2\rho(E_\beta^0)dE_\beta^0 ,$$

where $\rho(E_\beta^0)$ defines the density in energy of the final states and

$$|\,\langle\beta\,|\,S-1\,|\,\alpha\rangle\,|^2=\frac{2\pi}{\hbar}\delta(E_\beta^0-E_\alpha^0)\times\left|\int\psi_\beta^{0*}(\mathbf{r},\xi)V(\mathbf{r},\xi)\psi_\alpha^+(\mathbf{r},\xi)d\mathbf{r}\,d\xi\right|^2.$$

In the first-order approximation, one replaces the scattering wave $\psi_\alpha^+(\mathbf{r},\xi)$ by the unperturbed wave (2.4),

$$\psi_\alpha^0(\mathbf{r},\xi)\equiv\Phi_{n,s,k_z}(\mathbf{r})\,|\,I_\alpha M_\alpha\rangle ,$$

so that the partial rate $w_{\beta\alpha}$ becomes

$$w_{\beta\alpha}=\frac{2\pi}{\hbar}\delta(E_\beta^0-E_\alpha^0)\times|\langle n',s',k_z'|\langle I_\beta M_\beta|V(\mathbf{r},\xi)|I_\alpha M_\alpha\rangle|n,s,k_z\rangle|^2\times\rho(E_\beta^0)dE_\beta^0\ , \tag{3.6}$$

in which the matrix elements of the interaction are identical to the scattering amplitudes $A_{ns,n's'}(k_z,k_z')$ [Eq. (3.5)], defined in the Green's-function approach.

Using the explicit form of $V(\mathbf{r},\xi)$ as given in (2.3), the partial rate $w_{\beta\alpha}$ becomes

$$w_{\beta\alpha}=\frac{2\pi}{\hbar}\delta(E_\beta^0-E_\alpha^0)\left|\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi q}{2l+1}\langle n',s',k_z'|r^{-l-1}Y_{lm}(\hat{\mathbf{r}})|n,s,k_z\rangle\langle I_\beta M_\beta|m_{E_{lm}}^*|I_\alpha M_\alpha\rangle\right|^2\rho(E_\beta^0)dE_\beta^0\ . \tag{3.7}$$

This defines the Coulomb excitation probability of the final nuclear substate $|I_\beta M_\beta\rangle$ from the initial one ($|I_\alpha M_\alpha\rangle$) connected with the transition from the state $|n,s,k_z\rangle$ to the state $|n',s',k_z'\rangle$ of the scattered particle embedded in the magnetic field. The Coulomb scattering rate describing the nuclear transition $I_\alpha\to I_\beta$ is obtained by summing the partial rates (3.7) over the final states and by averaging them over the initial ones,

$$W_{\beta\alpha}=\sum_\alpha{}^*\sum_\beta w_{\beta\alpha}\ ,$$

where, explicitly, $\sum_\alpha^*$ and $\sum_\beta$ means, respectively, $1/2I_\alpha+1\sum_\alpha\sum_{n,s,k_z}^*$ and $\sum_\beta\sum_{n',s',k_z'}$.

The nuclear matrix elements are well known from the traditional Coulomb excitation; they are related, after summation over the nuclear magnetic quantum numbers to the reduced probabilities $B_{E_l}^{I_\alpha\to I_\beta}$,

$$W_{\beta\alpha}=\frac{2\pi}{\hbar}\sum_{n,s,k_z}{}^*\sum_{n',s',k_z'}\int\delta(E_\beta^0-E_\alpha^0)\rho(E_\beta^0)\sum_{l\geq 0}\frac{(4\pi)^2q^2}{(2l+1)^3}\delta(l,I_\alpha,I_\beta)B_{E_l}^{I_\alpha\to I_\beta}\times\sum_m|\langle n',s',k_z'|r^{-l-1}Y_{lm}(\hat{\mathbf{r}})|n,s,k_z\rangle|^2dE_\beta^0\ . \tag{3.8}$$

Let us now define what we mean by the summation $\sum_{n',s',k_z'}$ over the final states of the scattered particle in the magnetic field. The density $\rho(E_\beta^0)$ stands for the density of final states in the energy range $(E_\beta^0,\,E_\beta^0+dE_\beta^0)$, taking the following relation into account:

$$E_\beta^0=\frac{k_z'^2\hbar^2}{2m}+\hbar\omega(n'+\tfrac{1}{2})+E_{I_\beta}\ .$$

From this relation, it appears that the summations $\sum_{n',s',k_z'}$ and the integration $\int\rho(E_\beta^0)dE_\beta^0$ are overlapping. As defined in Sec. II B the density of states $|n',s',k_z'\rangle$ normalized on the basic volume $V=(\pi L^2)L_z$ is such that

$$\sum_{n',s',k_z'}\equiv\sum_{n',s'}\int_{-\infty}^{+\infty}dk_z'\rho(k_z')\rho(n',s')=\frac{L_z}{2\pi}\int_{-\infty}^{+\infty}dk_z'(\pi L^2)\frac{\gamma}{\pi}\sum_{n'=0}^{\infty}\sum_{s'=0}^{\infty}\ . \tag{3.9}$$

For an energy E, n' and k_z' are correlated according to

$$E=\frac{k_z^2\hbar^2}{2m}-\Delta E_{\alpha\beta}=\frac{k_z'^2\hbar^2}{2m}+\hbar\omega\Delta n\ ,$$

with $\Delta n\equiv n'-n$, so that

$$dk_z'\equiv\left[\frac{m}{2\hbar^2}\right]^{1/2}\frac{dE}{(E-\hbar\omega\Delta n)^{1/2}}\ . \tag{3.10}$$

Finally, $\sum_{n',s',k_z'}\int\rho(E_\beta^0)dE_\beta^0$ can be replaced by

$$\frac{L_z}{2\pi}(\pi L^2)\frac{\gamma}{\pi}\sum_{n'=0}^{N'}\sum_{s'=0}^{\infty}\left[\frac{m}{2\hbar^2}\right]^{1/2}\int\frac{dE_\beta^0\rho(E_\beta^0)}{(E_\beta^0-\hbar\omega\Delta n)^{1/2}}\ , \tag{3.11}$$

where N' is the maximum integer value of n' associated with $k_z'=0$.

Let us now conclude by the definition of the averaging $\sum_{n,s,k_z}^*$ over the initial states of the scattered particle. An initial momentum k_z being given, the initial state of the scattering process is defined as an incident beam of uniform density which is represented by a mixture of Landau states where the different s values have the same weight. Classically, it means a uniform beam of particles representing all possible impact parameters $\langle r_0\rangle$, $[\langle r_0\rangle^2=\gamma^{-1}(s+\frac{1}{2})]$. For an initial momentum k_z, the averaging $\sum_{n,s}^*$ is thus defined by

$$\sum_{n,s}{}^*=\frac{1}{j_z}\frac{\gamma}{\pi}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\ , \tag{3.12}$$

where the total initial current j_z is given by

$$j_z=\frac{\gamma}{\pi}\frac{k_z\hbar}{m}\sum_s|I_{n,s}(\gamma\rho^2)|^2=\frac{\gamma}{\pi}\frac{k_z\hbar}{m}\ . \tag{3.13}$$

Introducing these explicit definitions, we may now present the scattering rate (3.8) as follows:

$$W_{\beta\alpha}=\frac{m}{\hbar k_z}\frac{2\pi}{\hbar}\frac{L_z}{2\pi}(\pi L^2)\frac{\gamma}{\pi}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n'=0}^{N'}\sum_{s'=0}^{\infty}\left[\frac{m}{2\hbar^2}\right]^{1/2}\int\frac{\rho(E_\beta^0)dE_\beta^0}{(E_\beta^0-\hbar\omega\Delta n)^{1/2}}$$
$$\times\sum_{l\geq 0}\frac{(4\pi)^2q^2}{(2l+1)^3}\delta(l,I_\alpha,I_\beta)\delta(E_\beta^0-E_\alpha^0)B_{E_l}^{I_\alpha\to I_\beta}$$
$$\times\sum_{m=-l}^{+l}|\langle n',s',k_z'|r^{-l-1}Y_{lm}(\hat{\mathbf{r}})|n,s,k_z\rangle|^2 . \quad (3.14)$$

This scattering rate describes the process in which a uniform beam of charged particles of wave vector k_z in the field direction scatters into all possible Landau states according to the Coulomb excitation $(I_\alpha,E_{I_\alpha})\to(I_\beta,E_{I_\beta})$ of the target nucleus. In Sec. V we show that in the zero-field limit, formula (3.14) agrees with the field-free Coulomb excitation rate.

C. Angular momentum and energy dependence

In the scattering of a charged particle by a target nucleus, embedded in a magnetic field, the Coulomb excitation of a nuclear state I_β starting from an initial state I_α is defined by the electric multipole momenta $m_{E_{lm}}$, where l is fixed by the nuclear spins,

$$|I_\alpha-I_\beta|\leq l\leq I_\alpha+I_\beta , \quad (3.15)$$

and where m is fixed by the magnetic quantum numbers of the initial and final states of the scattered particle,

$$m=\Lambda'-\Lambda . \quad (3.16)$$

Energy conservation implies

$$\frac{k_z^2\hbar^2}{2m}=\frac{k_z'^2\hbar^2}{2m}+\hbar\omega\Delta n+\Delta E_{\alpha\beta} \quad (3.17)$$

so that the nuclear transition occurs only if

$$\frac{k_z^2\hbar^2}{2m}\geq\hbar\omega\Delta n+\Delta E_{\alpha\beta} ,$$

where $\Delta n=n'-n$ and $\Delta E_{\alpha\beta}=E_{I_\beta}-E_{I_\alpha}$. The presence of the magnetic field allows for transition between Landau states $\Delta n\neq 0$ without nuclear excitation $\Delta E_{\alpha\beta}=0$. The elastic and pseudoelastic scatterings, associated with the monopole component of the Coulomb interaction, have previously been dealt with in the literature.[3-5,8]

For a nuclear transition $I_\alpha\to I_\beta$, the left-hand side of (3.17) takes continuous values in k_z. On the contrary, on the right-hand side of this relation, one term is a continuous varying term whereas the other varies by discrete jumps. So the momentum k_z' is a multivalued function of E (or k_z), depending on the values of Δn, as shown in Fig. 1.

This pattern is strongly dependent on the value $\hbar\omega$, i.e., on the field strength and on the mass of the particle. For a scattered e^- (or e^+), the $\hbar\omega$ value is about 2000 times smaller than for a proton. This bundle of curves will be narrower for an electron than for a proton in the same field conditions. The qualitative dependence of the scattering rate (3.14) as a function of the initial energy or momentum k_z shall reflect this behavior of the phase space factor.

D. The nuclear reorientation process in elastic scattering

If one assumes that the mass of the nucleon is large enough in regard to that of the scattered particle so that the nuclear Zeeman effect can be neglected, the $(2I_{\alpha(\beta)}+1)$ magnetic nuclear substates $(I_{\alpha(\beta)},M_{\alpha(\beta)})$ have the same intrinsic energy $E_{I_\alpha(I_\beta)}$. Let us rewrite the Coulomb excitation amplitude for a given multipole order,

$$a_{\alpha\beta}^{lm}=\frac{4\pi q}{2l+1}\langle n',s',k_z'|r^{-l-1}Y_{lm}(\hat{\mathbf{r}})|n,s,k_z\rangle$$
$$\times(-1)^{I_\beta-M_\beta}\begin{pmatrix}I_\beta & l & I_\alpha\\ -M_\beta & -m & M_\alpha\end{pmatrix}\langle I_\beta\|m_{E_l}^*\|I_\alpha\rangle .$$

Given the nuclear states (I_α,M_α) and (I_β,M_β), the contributing multipole-order l values are $|I_\alpha-I_\beta|\leq l\leq I_\alpha+I_\beta$. Selection rules in the magnetic quantum numbers are present within each matrix element in the excitation amplitude $a_{\alpha\beta}^{lm}$,

$$m=M_\alpha-M_\beta \text{ and } m=(n'-s')-(n-s)=\Lambda'-\Lambda .$$

These relations are exactly the expression of the conservation law for the magnetic component of the whole system,

$$M_\alpha+\Lambda=M_\beta+\Lambda' \text{ or } M_\alpha-M_\beta=\Lambda'-\Lambda .$$

In what we previously defined as elastic and pseudoelastic scattering, both being associated with $\Delta E_{\alpha\beta}=0$ (Sec. III A), a nuclear reorientation effect is possible. Owing to the transverse quantification of the states of the scattered particle in the magnetic field, transitions of any

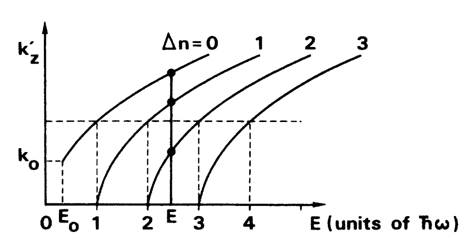

FIG. 1. Final momentum k_z' as a function of the initial energy $E=k_z^2\hbar^2/2m-\Delta E_{\alpha\beta}$ according to the energy conservation relation [(3.17) in the text].

$m = M_\alpha - M_\beta = \Lambda' - \Lambda$ with $\Delta n = \Delta(k_z^2) = 0$ or $\Delta n = (4\gamma)^{-1}\Delta(k_z^2) \neq 0$ are included in what we called the elastic or the pseudoelastic scattering, respectively. These transitions correspond to $m = M_\alpha - M_\beta = s - s'$ taking all integers values with any sign; they are associated with transitions between classical orbits of the scattered particle corresponding to a different geometrical radius r_0 $[\langle r_0 \rangle^2 = \gamma^{-1}(s+\frac{1}{2})]$, as shown in Fig. 2. For a given nuclear spin I, all nuclear transitions between magnetic substates (I,M) and (I,M') are included in the elastic and pseudoelastic scattering processes ($\Delta E = 0$); they are associated with the transitions $s \rightarrow s'$ ($\Delta s \equiv s - s' = \Lambda' - \Lambda = M' - M \equiv \Delta M$) between two Landau orbitals $\phi_{n,s}$ (see Sec. II A) of the particle scattering in the magnetic field.

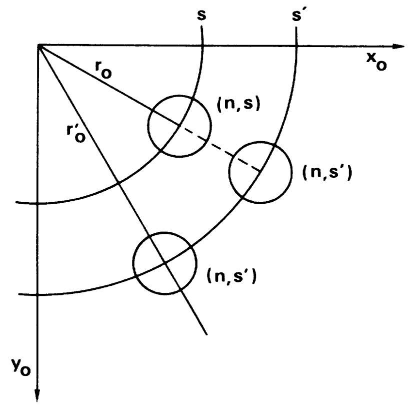

FIG. 2. Classical orbits of the scattered particle corresponding to different geometrical radii r_0 or quantum numbers s, $\langle r_0 \rangle^2 = \gamma^{-1}(s+\frac{1}{2})$.

IV. EVALUATION OF THE COULOMB MATRIX ELEMENT

We summarize the evaluation of the Coulomb matrix elements whatever the multipole order l is. The matrix elements $\langle n',s',k_z' \,|\, r^{-l-1}Y_{lm}(\hat{\mathbf{r}}) \,|\, n,s,k_z \rangle$ are not independent from each other. Those matrix elements corresponding to negative angular momentum values $\Lambda = n - s < 0$ (and/or $\Lambda' = n' - s' < 0$) are related to the ones with positive values $\Lambda = n - s > 0$ (and/or $\Lambda' = n' - s' > 0$) simply by interchanging n with s (and/or n' with s'). Furthermore, a three-term recurrence formula links the matrix elements of adjoining indices. First, we show how to obtain this recurrence formula; second, we calculate exactly the matrix element which initiates that recurrence.

A. The recurrence formula

The matrix element $\langle n',s',k_z' \,|\, r^{-l-1}Y_{lm}(\hat{\mathbf{r}}) \,|\, n,s,k_z \rangle$ is explicitly defined in cylindrical coordinates (ρ,ϕ,z) using the following: (a) for the wave functions, the expression (cf. Sec. II A)

$$| n,s,k_z \rangle = \exp(ik_z z)\exp[i(n-s)\phi] I_{n,s}(\gamma\rho^2) \; ;$$

(b) for the spherical operator $[r^{-l-1}Y_{lm}(\hat{\mathbf{r}})]$, an integral representation factorized in the cylindrical variables [Appendix A, Eq. (A5)],

$$r^{-l-1}Y_{lm}(\hat{\mathbf{r}}) = \frac{2\exp(im\phi)}{\Gamma\left[\dfrac{l+m+\sigma+1}{2}\right]} \times \int_0^\infty du\, u^{l+m+\sigma} z^\sigma \times \exp(-u^2z^2)\rho^m \exp(-u^2\rho^2) \times {}_1F_1(-\kappa, m+1; u^2\rho^2) \; , \quad (4.1)$$

where $\kappa \equiv (l-m-\sigma)/2$ and $\sigma = 0$ or 1 according to the parity of $(l-m)$. One can write the matrix element $\langle n',s',k_z' \,|\, r^{-l-1}Y_{lm}(\hat{\mathbf{r}}) \,|\, n,s,k_z \rangle$ as a multiple integration,

$$\frac{2\Gamma(\kappa+1)\Gamma(m+1)}{\Gamma(\kappa+m+1)\Gamma(l-\kappa+\frac{1}{2})} \int_0^\infty d\rho\,\rho \int_{-\infty}^{+\infty} dz \int_0^{2\pi} d\phi \exp(iqz)\exp[i(n-s-n'+s')\phi]\exp(im\phi) \times I_{n,s}(\gamma\rho^2) I_{n',s'}(\gamma\rho^2) \int_0^\infty [du\, u^{l+m+\sigma} z^\sigma \exp(-u^2z^2)\rho^m \exp(-u^2\rho^2) L_\kappa^m(u^2\rho^2)] \; , \quad (4.2)$$

where q is the momentum transfer in the field direction ($q = k_z - k_z'$); the Laguerre polynomial L_κ^m, proportional to the hypergeometric function in (4.1), is used in place of this latter.

Let us assume positive values for the quantum numbers $\Lambda = n - s$ and $\Lambda' = n' - s'$; so each Landau function $I_{n,s}(\gamma\rho^2)$ is defined by

$$I_{n,s}(t) = (n!s!)^{-1/2} \exp\left[-\frac{t}{2}\right] t^{(n-s)/2} s! L_s^{n-s}(t) \; . \quad (4.3)$$

Performing the integration over the azimuthal angle ϕ leads immediately to the selection rule (3.16), $m = \Lambda' - \Lambda$:

$$\int_0^{2\pi} d\phi \exp[i(n-s-n'+s')\phi]\exp(im\phi) = 2\pi\delta(n-s, n'-s'-m) \; . \quad (4.4)$$

The integration over z is obvious; it leads to

$$\int_{-\infty}^{+\infty} dz \exp(iqz) z^\sigma \exp(-u^2z^2) = \left[\frac{iq}{2}\right]^\sigma \frac{\sqrt{\pi}}{u^{2\sigma+1}} \exp\left[\frac{-q^2}{4u^2}\right] .$$

Using the new variable $t = \gamma\rho^2$ and the explicit form (4.3) of the Landau functions, the integration over ρ reduces to

$$\frac{(n!s!n'!s'!)^{-1/2}}{2\gamma^{(m/2)+1}}K_{ss'}^{nn'}\left[\frac{u^2}{\gamma},l\right], \tag{4.5}$$

where

$$K_{ss'}^{nn'}\left[\frac{u^2}{\gamma},l\right]\equiv\int_0^\infty dt\, t^{n-s+m}\exp(-t)\exp\left[-\frac{u^2t}{\gamma}\right]L_\kappa^m\left[\frac{u^2t}{\gamma}\right]s!L_s^{n-s}(t)s'!L_{s'}^{n'-s'}(t) \tag{4.6}$$

is defined for each l value; the quantum number m being connected to $\{n,s;n',s'\}$ by the selection rule (4.4). Finally, the matrix element reduces to

$$\langle n',s',k_z'\,|\,r^{-l-1}Y_{lm}(\hat{\mathbf{r}})\,|\,n,s,k_z\rangle=\frac{2\pi\sqrt{\pi}\Gamma(\kappa+1)\Gamma(m+1)\delta(n-s,n'-s'-m)}{\gamma^{(m/2)+1}\Gamma(\kappa+m+1)\Gamma(l-\kappa+\frac{1}{2})(n!n'!s!s'!)^{1/2}}\left[\frac{iq}{2}\right]^\sigma V_{ss'}^{nn'}\left[\frac{q^2}{\gamma},l\right], \tag{4.7}$$

where

$$V_{ss'}^{nn'}\left[\frac{q^2}{\gamma},l\right]\equiv\int_0^\infty du\, u^{l+m-\sigma-1}\exp\left[\frac{-q^2}{4u^2}\right]\times K_{ss'}^{nn'}\left[\frac{u^2}{\gamma},l\right]. \tag{4.8}$$

The explicit calculations to evaluate the integrals (4.6) and to define subsequently the recurrence formula between the matrix elements $V_{ss'}^{nn'}(q^2/\gamma,l)$ are derived in Appendix B. Given n, n', $\Lambda=n-s>0$, and $\Lambda'=n'-s'>0$, the following recurrence formula holds:

$$V_{ss'}^{nn'}\left[\frac{q^2}{\gamma},l\right]=\left[s+s'-1+\frac{q^2}{4\gamma}\right]V_{s-1,s'-1}^{nn'}\left[\frac{q^2}{\gamma},l\right]-(s-1)(s'-1)V_{s-2,s'-2}^{nn'}\left[\frac{q^2}{\gamma},l\right]. \tag{4.9}$$

In this relation, the four quantum numbers $(n,s;n',s')$ are connected by the selection rule on the magnetic quantum numbers:

$$m\equiv(n'-s')-(n-s).$$

Each matrix element is l dependent and this recurrence formula holds whatever is the l value defined by the nuclear spins selection rule (3.15), $|I_\alpha-I_\beta|\le l\le I_\alpha+I_\beta$.

B. The explicit expressions of $V_{s0}^{nn'}$

It is possible to define an explicit analytic expression for the first term $V_{s0}^{nn'}$ of the recurrence formula (4.9), l and m being fixed. For $s'=0$ and n,s given, the selection rule $m\equiv\Lambda'-\Lambda$ implies that n' is fixed: $n'\equiv n-s+m$ or, equivalently, $n-n'\equiv s-m$.

For (l,m) given, one has explicitly

$$V_{s0}^{n(n')}\left[\frac{q^2}{\gamma},l\right]=\int_0^\infty du u^{l+m-\sigma-1}\exp\left[\frac{-q^2}{4u^2}\right]\times K_{s0}^{n(n')}\left[\frac{u^2}{\gamma},l\right], \tag{4.10}$$

where $K_{s0}^{n(n')}$ defined in (4.6) reduces to

$$K_{s0}^{n(n')}\left[\frac{u^2}{\gamma},l\right]=\int_0^\infty dt\, t^m d_t^s[t^n\exp(-t)]\times\exp\left[-\frac{u^2t}{\gamma}\right]L_\kappa^m\left[\frac{u^2t}{\gamma}\right]. \tag{4.11}$$

Performing the integration over u with the help of the identity[9]

$$L_\kappa^{2\mu}(z)(-1)^\kappa z^{\mu+(1/2)}\exp\left[-\frac{z}{2}\right]\equiv W_{\kappa+\mu+(1/2),\mu}(z),$$

we obtain

$$\int_0^\infty du\, u^{l+m-\sigma-1}\exp\left[-\frac{q^2}{4u^2}\right]\exp\left[-\frac{u^2t}{\gamma}\right]L_\kappa^m\left[\frac{u^2t}{\gamma}\right]\equiv(-1)^\kappa\left[\frac{\gamma}{t}\right]^{m/2}\left[\frac{q}{2}\right]^{m+2\kappa}K_m(q\sqrt{t/\gamma}). \tag{4.12}$$

By use of the definitions (4.10),(4.11), use of the relation (4.12), and performing s partial integrations, the matrix element (4.10) reduces to

$$V_{s0}^{n(n')}\left[\frac{q^2}{\gamma},l\right]=(-1)^{\kappa+s}\int_0^\infty\left\{dt\exp(-t)t^n d_t^s\left[t^m\left[\frac{t}{\gamma}\right]^{-m/2}\left[\frac{q}{2}\right]^{m+2\kappa}K_m(q\sqrt{t/\gamma})\right]\right\}.$$

This last integral is evaluated analytically[10] so that

$$V_{s0}^{n(n')}\left[\frac{q^2}{\gamma},l\right]=(-1)^{(l-m-\sigma)/2}\left[\frac{q^2}{4}\right]^{(l+m-\sigma)/2}\Gamma(n+1)\Gamma(n'+1)\Psi\left[n'+1,n'-n+1,\frac{q^2}{4\gamma}\right], \tag{4.13}$$

where Ψ is the confluent hypergeometric function of the second kind and q defines the momentum transfer in the magnetic field direction.

V. RESTORATION OF FIELD-FREE CROSS SECTIONS

Recently, several theoretical articles contributed to the description of charged particles scattering in the presence of external electromagnetic fields. The difficulty sometimes encountered in these calculations is that the scattering waves do not take the field-free forms if Landau states are used as initial and final states (Sec. III A). Some authors present solutions to this problem of the zero magnetic field limit by constructing a superposition of Landau states which satisfies free-wave boundary conditions when the field is switched off. However, the way to reproduce the features of the scattering phenomena in the zero magnetic field limit does not seem to be unique.

A. Laboratory experimental situation

Until now, in the literature, the situation which has been considered is the laboratory one. A beam of charged particles of fixed momentum $\mathbf{k}$ represented in the field-free region by a plane wave is assumed to enter a region in which there is a homogeneous magnetic field. So one needs to know what are the occupation amplitudes of the Landau states when a free particle enters the magnetic field.

Faisal[11] and Ohsaki[7] give two alternative adiabatic expansions for the wave function to be used. The constructed state is a coherent superposition of the degenerate Landau eigenstates inside the field which are occupied by the free particle as it adiabatically enters the field. Zarcone *et al.*[12] also describe the very different situation of a sudden switching on of the magnetic field, to define how the plane wave splits into a combination of Landau states. This sudden picture is only valid for weak fields or for very fast particles at the highest laboratory fields. In laboratory conditions the correct situation is always an adiabatic one. More recently, coherent states of charged particles in a homogeneous magnetic field have been thoroughly studied by Varró[13] and Varró and Ehlotzsky,[14] the method they use to build such states is based on the one-dimensional harmonic-oscillator algebra. The coherent states defined by Varró[13] correspond to a Gaussian probability distribution of the electron position in the transverse plane, gyrating along circles with the cyclotron frequency; the center of these distributions moves along possible classical trajectories.

B. Stellar environments

When the scattering of charged particles takes place in a stellar environment like white dwarfs or neutron stars, new properties relative to the behavior of the scattered particles have to be taken into account. In order to describe the free field scattering of a wave packet with initial momentum $\mathbf{k}$ distributed according to an amplitude weight function $g(\mathbf{k})$, we use a wave function like

$$\Psi_E(\mathbf{r},t)=g(\mathbf{k})\frac{\exp(i\mathbf{k}\cdot\mathbf{r})}{L^{3/2}}\exp\left[-i\frac{k^2\hbar^2}{2m}t\right] \tag{5.1}$$

and assume that $\int d\mathbf{k}\,|g(\mathbf{k})|^2=1$. The wave-packet solution for a charged particle in the presence of a magnetic field would be described similarly by the wave function

$$\phi_E(\mathbf{r},t)=\sum_{n,s,k_z} C_{n,s}g_n(k_z)\frac{\exp(ik_z z)}{L_z^{3/2}}\frac{\phi_{n,s}(\rho,\phi)}{(\pi L^2)^{1/2}} \times\exp\left[-\frac{i}{\hbar}\left[\hbar\omega(n+\tfrac{1}{2})+\frac{k_z^2\hbar^2}{2m}\right]t\right] . \tag{5.2}$$

In a stellar environment, we have to show how the magnetic field changes the thermodynamic properties of the electron gas, assuming the charged particle to be an electron, for example. Canuto and Ventura[1] have discussed this problem. The statistical averaging is performed using a Fermi distribution in (5.2),

$$g_n(k_z)\equiv g_n(E)=\left[1+\exp\left[\frac{E_n-\mu}{k_BT}\right]\right]^{-1} , \tag{5.3}$$

where k_B is the Boltzmann factor, T is the temperature of the Fermi gas, μ is the chemical potential, and E_n is the energy

$$E_n\equiv\frac{k_z^2\hbar^2}{2m}+\hbar\omega(n+\tfrac{1}{2}) .$$

As previously seen (Sec. III), the scattering in the presence of a magnetic field is similar to a scattering between systems having internal structures: the one of the nuclear target and the one bound to the Landau states of the spiraling motion of the scattered particle due to the presence of the magnetic field. Only the motion in the direction of the field (z direction) is a free motion, whose energies and moments vary continuously. In such a scattering whose geometry is defined by a cylindrical symmetry, the direction of the magnetic field is the only one to be considered at infinity[3] (Sec. III A). But in the definition of the scattering rates one has to use a scattering state like that defined in (2.1) and to solve the averaging $\sum^*_{n,s,k_z}$ and the summation $\sum_{n',s',k_z'}\int dE^0_\beta\rho(E^0_\beta)$ according to the rules defined in (Sec. III B). In (3.14) we gave an appropriate definition of the Coulomb scattering rate and we show now that this rate agrees with the field-free rate[6] to the zero magnetic field limit.

C. The zero-field limit

We can decrease the field strength in two ways. One is the adiabatic process with n kept constant. In this case, one cannot obtain the plane-wave scattering result because the Landau orbital $\phi_{n,s}(\rho,\phi)$ $(\Lambda=n-s\neq0)$ becomes zero in the zero-field limit. The other approach is a process keeping the transverse energy $k_\perp^2=4\gamma(n+\frac{1}{2})$ constant. The quantum number n diverges according to

$1/\gamma$ when γ goes to zero. To show that the scattering rate $W_{\beta\alpha}$ [Eq. (3.14)] reduces exactly to the field-free one in the zero magnetic field limit, we proceed in two steps. First, we find the zero-field limit of the matrix element (4.7),

$$U_{ss'}^{nn'} \equiv \langle n',s',k_z' \mid r^{-l-1} Y_{lm}(\hat{\mathbf{r}}) \mid n,s,k_z \rangle \ .$$

Secondly, we use the zero-field limits of the summation $\sum_{n',s',k_z'}$ and averaging $\sum^*_{n,s,k_z}$ according to their explicit definitions (3.9)–(3.13).

Let us show how to obtain the zero-field limit of a matrix element $U_{ss'}^{nn'}$. We use the expression (A2) of the spherical operator $r^l Y_{lm}(\hat{\mathbf{r}})$ and the integral representation (A3) of r^{-2l-1}. Solving the integration over dz

$$\int_{-\infty}^{+\infty} dz \exp(iq_z z) z^{l-m-2k} \exp(-u^2 z^2) = \frac{2\sqrt{\pi} i^\sigma (-1)^{\kappa-k}}{(2u)^{l-m-2k+1}} \exp\left[-\frac{q_z^2}{4u^2}\right] H_{l-m-2k}\left[\frac{q_z}{2u}\right]$$

with the Hermite polynomials H_{2n} or H_{2n+1}, the matrix element $U_{ss'}^{nn'}$ transforms itself into

$$U_{ss'}^{nn'} = \sum_{k=0}^{\kappa} C_k^{lm} 2\pi \delta(n-s, n'-s'-m) \frac{2\sqrt{\pi} i^\sigma (-1)^{\kappa-k}}{\Gamma(l+\frac{1}{2}) 2^{l-m-2k}} \int_0^\infty du\, u^{l+m+2k-1} \exp\left[\frac{-q_z^2}{4u^2}\right] H_{l-m-2k}\left[\frac{q_z}{2u}\right] \times \int_0^\infty d\rho\, I_{ns}(\gamma\rho^2) I_{n's'}(\gamma\rho^2) \rho^{m+2k+1} \exp(-u^2\rho^2) \ , \tag{5.4}$$

where the coefficients C_k^{lm} are explicitly given by

$$C_k^{lm} = \frac{(-1)^k}{k!4^k} \frac{m!(l-m)!}{(m+k)!(l-m-2k)!} \ .$$

We keep n,s fixed and define the transverse momentum or energy $q_\perp^2 = 4\gamma(n-n')$: the quantum number n' diverges and the zero-field limits of $I_{n's'}(\gamma\rho^2)$ and $I_{ns}(\gamma\rho^2)$ are, respectively,

$$\lim_{\gamma\to 0} I_{n's'}(\gamma\rho^2) = J_{n-s+m}(q_\perp \rho) \ ,$$

where $n'-s'$ has been replaced by $n'-s' = n-s+m$ and J is the Bessel function of integer order, and

$$\lim_{\gamma\to 0} I_{ns}(\gamma\rho^2) = \left[\frac{\Gamma(n+1)}{\Gamma(s+1)\Gamma(n-s+1)}\right]^{1/2} (\gamma\rho^2)^{(n-s)/2} \ ,$$

where the zero order in γ is defined when $n=s$.

According to these limits we have

$$\lim_{\gamma\to 0} \int d\rho\, I_{ns}(\gamma\rho^2) I_{n's'}(\gamma\rho^2) \rho^{m+2k+1} \exp(-u^2\rho^2) = \int d\rho\, \rho^{m+2k+1} \exp(-u^2\rho^2) J_m(q_\perp \rho) = \frac{k!u^{-m-2k-2}}{2} \exp\left[-\frac{q_\perp^2}{4u^2}\right] \frac{q_\perp^m}{(2u)^m} L_k^m \left[\frac{q_\perp^2}{4u^2}\right] \ .$$

The expression (5.4) at the zero-field limit finally reads

$$\lim_{\gamma\to 0} U_{ss'}^{nn'} = \sum_{k=0}^{\kappa} C_k^{lm} \frac{2\pi\sqrt{\pi} i^\sigma (-1)^{\kappa-k} k!}{2^{l-2k}\Gamma(l+\frac{1}{2})} q_\perp^m \times \int_0^\infty \left[du\, u^{l-m-3} \exp\left[-\frac{q_z^2+q_\perp^2}{4u^2}\right] \times H_{l-m-2k}\left[\frac{q_z}{2u}\right] L_k^m \left[\frac{q_\perp^2}{4u^2}\right]\right] \ ,$$

which is exactly the expression we obtained for the field-free matrix element $\int d\mathbf{r} \exp(i\mathbf{q}\cdot\mathbf{r}) r^{-l-1} Y_{lm}(\hat{\mathbf{r}})$ when we used the development of the plane wave in cylindrical coordinates,

$$\exp(i\mathbf{q}\cdot\mathbf{r}) = \sum_\lambda i^\lambda \exp[i\lambda(\varphi-\varphi_0)] \exp(iq_z z) J_\lambda(q_\perp \rho) \ .$$

One obtains the final expression

$$\lim_{\gamma\to 0} U_{ss'}^{nn'} = \frac{4\pi i^l}{(2l-1)!!} \sum_{k=0}^{\kappa} C_k^{lm} \frac{q_\perp^{m+2k} q_z^{l-m-2k}}{(q_z^2+q_\perp^2)} = \frac{4\pi i^l}{(2l-1)!!} q^{l-2} Y_{lm}(\hat{\mathbf{q}}) \ . \tag{5.5}$$

In Sec. II B we showed how the continuum of free particle states coalesces into equally spaced harmonic-oscillator energy states in the presence of a magnetic field. Their degeneracy is defined by

$$\omega_n = \frac{1}{(2\pi)^2} \int_{K_n}^{K_{n+1}} k_\perp^2 dk_\perp \int_0^{2\pi} d\varphi_\perp$$

with $K_n^2 = 4\gamma(n+\frac{1}{2})$. If n becomes very large, the following correspondences hold.

(a) When $\gamma \to 0$, the summation over the final quantum states

$$\sum_{n's'k_z'} \equiv \frac{L_z}{2\pi} \int_{-\infty}^{+\infty} dk_z' \left[\frac{\gamma}{\pi}\right] (\pi L^2) \sum_{n'} \sum_{s'}$$

reduces to

$$\frac{L^3}{(2\pi)^3} \int d\mathbf{k}' \equiv \frac{m^2 L^3}{(2\pi\hbar)^3} \int v' dE' \int d\Omega' \tag{5.6}$$

with the same normalized length L in each direction.

(b) The value of k_z being given, when $\gamma \to 0$, the expression

$$\sum_{n,s}^* \mid \langle n',s',k_z' \mid r^{-l-1} Y_{lm}(\hat{\mathbf{r}}) \mid n,s,k_z \rangle \mid^2 \equiv \frac{\frac{\gamma}{\pi} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \mid \langle n',s',k_z' \mid r^{-l-1} Y_{lm}(\hat{\mathbf{r}}) \mid n,s,k_z \rangle \mid^2}{\frac{\gamma}{\pi} \frac{k_i \hbar}{m}}$$

reduces to

$$\frac{1}{v_i} \lim_{\gamma\to 0} \mid \langle n',s',k_z' \mid r^{-l-1} Y_{lm}(\hat{\mathbf{r}}) \mid n,s,k_z \rangle \mid^2 \ . \tag{5.7}$$

Coming back to the scattering rate $W_{\beta\alpha}$ defined in (3.14) and using the field-free limits obtained in (5.5)–(5.7), we find

$$\lim_{\gamma\to 0} W_{\beta\alpha}=\frac{2\pi}{\hbar}\frac{m^2}{(2\pi\hbar)^3 v_\alpha}\int dE^0_\beta d\Omega_\beta v_\beta \delta(E^0_\beta-E^0_\alpha)\sum_{l\geq 0}\frac{(4\pi)^2 q^2}{(2l+1)^3}\delta(l,I_\alpha,I_\beta)B_{E_l}^{I_\alpha\to I_\beta}$$
$$\times\frac{(4\pi)^2}{[(2l-1)!!]^2}|\mathbf{k}_\alpha-\mathbf{k}_\beta|^{2l-4}\sum_m Y_{lm}(\hat{\mathbf{q}})Y^*_{lm}(\hat{\mathbf{q}})$$
$$=\sum_{l\geq 0}\delta(l,I_\alpha,I_\beta)\frac{16\pi q^2}{\hbar^2 v_\alpha^2}\frac{k_\alpha k_\beta}{[(2l+1)!!]^2}B_{E_l}^{I_\alpha\to I_\beta}|\mathbf{k}_\alpha-\mathbf{k}_\beta|^{2l-4}d\Omega_\beta\ , \quad (5.8)$$

where the momentum transfer $\mathbf{q}$ is replaced by $\mathbf{q}=\mathbf{k}_\alpha-\mathbf{k}_\beta$. This last expression (5.8) is exactly the differential cross section for Coulomb excitation calculated in the plane-wave Born approximation.[6]

VI. CONCLUSIONS

In this paper we studied the scattering of a charged structureless particle by a nucleus described by means of its electric multipole moments, projectile, and target being immersed in a homogeneous and constant magnetic field. The effects of the magnetic field are taken into account exactly, the electric interaction being described through first-order perturbation. Physical situations corresponding to such conditions are of astrophysical interest.

We present an analytical expression and a recurrence rule for the matrix elements of any multipole operator. The Coulomb excitation rates we derived coincide in the zero-field limit with the well-known results of a direct field-free evaluation.

To consider multipoles higher than the zero one (Rutherford scattering) is not a mere generalization; these multipoles are able to induce reorientations of the target, processes which are included in our treatment. We also consider the physics of Coulomb scattering in a magnetic field either in a laboratory situation or in stellar environments.

APPENDIX A

The cylindrical symmetry of our physical problem suggests transformation of the expression

$$r^{-l-1}Y_{lm}(\theta,\phi) \quad (A1)$$

into a formula where the cylindrical coordinates (ρ,ϕ,z) are factorized, as far as possible.

We present two useful expressions of (A1), starting with a formula[8] we obtained earlier:

$$r^l Y_{lm}=\exp(im\phi)\rho^m z^{l-m}\times{}_2F_1\left[-\frac{l-m}{2},-\frac{l-m-1}{2};m+1;-\frac{\rho^2}{z^2}\right] . \quad (A2)$$

A first way consists in completing formula (A2) by an integral representation of r^{-2l-1}, namely,

$$r^{-2l-1}=\frac{2}{\Gamma(l+\frac{1}{2})}\int_0^\infty du\ u^{2l}\exp(-u^2r^2) . \quad (A3)$$

A second way consists in transforming formula (A2) by means of the well-known property

$${}_2F_1(a,b;c;t)=(1-t)^{-a}{}_2F_1\left[a,c-b;c;\frac{t}{t-1}\right] .$$

The expression we are interested in reads finally

$$r^{-l-1}Y_{lm}=\exp(im\phi)\rho^m z^\sigma r^{-2\tau}{}_2F_1\left[-\kappa,\tau;m+1;\frac{\rho^2}{r^2}\right] , \quad (A4)$$

where we used the following notation. (a) $l-m=2\kappa+\sigma$, where σ is equal to 0 or 1 according to the even or odd character of $(l-m)$; consequently 2κ is the largest even integer included in $(l-m)$. (b) $\tau=l-\kappa+\frac{1}{2}=\kappa+\sigma+m+\frac{1}{2}$ is a positive half integer.

The hypergeometric function ${}_2F_1(-\kappa,\tau;m+1;\rho^2/r^2)$ is a polynomial of degree κ in ρ^2/r^2; formula (A4) thus contains a finite number of negative powers of r. Using the integral representation (A3) one easily obtains our final factorized expression

$$\frac{2}{\Gamma(\tau)}\exp(im\phi)\int_0^\infty du\ u^{l+m+\sigma}z^\sigma\exp(-u^2z^2)\rho^m\times\exp(-u^2\rho^2)\,{}_1F_1(-\kappa;m+1;u^2\rho^2) . \quad (A5)$$

We assumed a positive value of m; Laplace functions with a negative value of m are easily derived from the former ones by means of a well-known formula.

APPENDIX B: RECURRENCE FORMULA FOR THE MATRIX ELEMENTS

In (4.6) we defined

$$K^{nn'}_{ss'}\left[\frac{u^2}{\gamma},l\right]=\int_0^\infty dt\ t^{n-s+m}\exp(-t)\exp\left[-\frac{u^2t}{\gamma}\right]\times s!L_s^{n-s}(t)s'!L_{s'}^{n'-s'}(t)\times L_\kappa^m\left[\frac{u^2t}{\gamma}\right] , \quad (B1)$$

where $\kappa=(l-m-\sigma)/2$ and $\sigma=0$ or 1 according to the parity of $(l-m)$. We perform in (B1) an integration by parts and we use the Rodrigues formula

$$s!L_s^{n-s}(t)=\exp(t)t^{s-n}\frac{\delta^s}{\delta_t}[\exp(-t)t^n] \quad (B2)$$

and the relation

$$\delta_t L_{s'}^{n'-s'}(t)=-L_{s'-1}^{n'-s'+1}(t) . \quad (B3)$$

We first obtain the expression

$$K_{ss'}^{nn'} = s' K_{s-1,s'-1}^{nn'} - \int_0^\infty dt\,(s-1)! L_{s-1}^{n-s+1}(t) t^{-m+1} \delta_t^{s'} [\exp(-t) t^{n'}] \delta_t \left[t^m L_\kappa^m \left[\frac{u^2 t}{\gamma} \right] \exp\left[-\frac{u^2 t}{\gamma} \right] \right] . \tag{B4}$$

After one more integration by parts and making use of all the relations previously defined [(B2), (B3), and (B4) itself with (s,s') replaced by $(s-1,s'-1)$], we establish the formula

$$\begin{aligned} K_{ss'}^{nn'} &= (s'+s-1) K_{s-1,s'-1}^{nn'} - (s-1)(s'-1) K_{s-2,s'-2}^{nn'} \\ &\quad + (s'-1)! \int_0^\infty dt\, \delta^{s-1} [\exp(-t) t^n] L_{s'-1}^{n'-s'+1}(t) t^m \\ &\quad \times \delta_t \left\{ t^{-m+1} \delta_t \left[t^m L_\kappa^m \left[\frac{u^2 t}{\gamma} \right] \exp\left[\frac{-u^2 t}{\gamma} \right] \right] \right\} . \end{aligned} \tag{B5}$$

Using twice the relation

$$\delta_t L_\kappa^m \left[\frac{u^2 t}{\gamma} \right] = -\frac{u^2}{\gamma} L_{\kappa-1}^{m+1} \left[\frac{u^2 t}{\gamma} \right]$$

and writing explicitly the successive derivatives over t, the last integral in (B5) takes the explicit form

$$\begin{aligned} &(s'-1)! \int_0^\infty dt\, \delta^{s-1} [\exp(-t) t^n] L_{s'-1}^{n'-s'+1}(t) t^m \exp\left[-\frac{u^2 t}{\gamma} \right] \frac{u^2}{\gamma} \\ &\quad \times \left[\left[\frac{u^2 t}{\gamma} - m - 1 \right] L_\kappa^m \left[\frac{u^2 t}{\gamma} \right] + \left[2 \frac{u^2 t}{\gamma} - m - 1 \right] L_{\kappa-1}^{m+1} \left[\frac{u^2 t}{\gamma} \right] + \frac{u^2 t}{\gamma} L_{\kappa-2}^{m+2} \left[\frac{u^2 t}{\gamma} \right] \right] . \end{aligned} \tag{B6}$$

For $\kappa=0$, the second and third terms in (B6) have to be replaced by zero. For $\kappa=1$, only the third term has to be replaced by zero. For $\kappa \geq 2$, all three terms in (B6) are present.

Using (B5) and (B6), let us now write the expression of the matrix element $V_{ss'}^{nn'}(q^2/\gamma)$ defined in (4.8):

$$\begin{aligned} V_{ss'}^{nn'} &= (s'+s-1) V_{s-1,s'-1}^{nn'} - (s-1)(s'-1) V_{s-2,s'-2}^{nn'} \\ &\quad + (s'-1)! \int_0^\infty dt\, \delta^{s-1} [\exp(-t) t^m] L_{s'-1}^{n'-s'+1}(t) t^m \\ &\quad \times \int_0^\infty du\, u^{l+m-\sigma-1} \frac{u^2}{\gamma} \exp\left[-\frac{q^2}{4u^2} \right] \exp\left[-\frac{u^2 t}{\gamma} \right] \\ &\quad \times \left[\left[\frac{u^2 t}{\gamma} - m - 1 \right] L_\kappa^m \left[\frac{u^2 t}{\gamma} \right] + \left[\frac{u^2 t}{\gamma} - m - 1 \right] L_{\kappa-1}^{m+1} \left[\frac{u^2 t}{\gamma} \right] \right. \\ &\quad \left. + \frac{u^2 t}{\gamma} L_{\kappa-2}^{m+2} \left[\frac{u^2 t}{\gamma} \right] + \frac{u^2 t}{\gamma} L_{\kappa-1}^{m+1} \left[\frac{u^2 t}{\gamma} \right] \right] . \end{aligned} \tag{B7}$$

By use of relation (B3) we can replace $L_{\kappa-1}^{m+1}(u^2 t/\gamma)$ by

$$L_{\kappa-1}^{m+1} \left[\frac{u^2 t}{\gamma} \right] = \frac{-\gamma}{2ut} \delta_u L_\kappa^m \left[\frac{u^2 t}{\gamma} \right]$$

so that one integral like

$$I = \int_0^\infty du\, u^{l+m-\sigma-1} \frac{u^2}{\gamma} \exp\left[\frac{-q^2}{4u^2} \right] \exp\left[\frac{-u^2 t}{\gamma} \right] \frac{u^2 t}{\gamma} L_{\kappa-1}^{m+1} \left[\frac{u^2 t}{\gamma} \right]$$

transforms itself into

$$I = \frac{-1}{2\gamma} \int_0^\infty du\, u^{l+m-\sigma+2} \exp\left[\frac{-q^2}{4u^2} \right] \exp\left[\frac{-u^2 t}{\gamma} \right] \delta_u L_\kappa^m \left[\frac{u^2 t}{\gamma} \right] .$$

Performing an integration by parts leads to the expression

$$\begin{aligned} I &= \frac{q^2}{4\gamma} \int_0^\infty du\, u^{l+m-\sigma-1} \exp\left[\frac{-q^2}{4u^2} \right] \exp\left[\frac{-u^2 t}{\gamma} \right] L_\kappa^m \left[\frac{u^2 t}{\gamma} \right] \\ &\quad + \int_0^\infty du\, u^{l+m-\sigma-1} \frac{u^2}{\gamma} \exp\left[\frac{-q^2}{4u^2} \right] \exp\left[\frac{-u^2 t}{\gamma} \right] \left[\frac{l+m-\sigma+2}{2} - \frac{u^2 t}{\gamma} \right] L_\kappa^m \left[\frac{u^2 t}{\gamma} \right] . \end{aligned} \tag{B8}$$

By using this last form of I, we can transform the formula (B7) into the new relation

$$V_{ss'}^{nn'}=\left[s'+s-1+\frac{q^2}{4\gamma}\right]V_{s-1,s'-1}^{nn'}-(s-1)(s'-1)V_{s-2,s'-2}^{nn'}$$
$$+\frac{(s'-1)!}{\gamma}\int_0^\infty dt\,\delta^{s-1}[\exp(-t)t^m]L_{s'-1}^{n'-s'+1}(t)t^m$$
$$\times\int_0^\infty du\,u^{l+m-\sigma+1}\exp\left[\frac{-q^2}{4u^2}\right]\exp\left[\frac{-u^2t}{\gamma}\right]$$
$$\times\left[\kappa L_\kappa^m\left[\frac{u^2t}{\gamma}\right]+\left[\frac{u^2t}{\gamma}-m-1\right]L_{\kappa-1}^{m+1}\left[\frac{u^2t}{\gamma}\right]+\frac{u^2t}{\gamma}L_{\kappa-2}^{m+2}\left[\frac{u^2t}{\gamma}\right]\right]\right]. \quad \text{(B9)}$$

By use of the relations between contiguous Laguerre polynomials, the bracket of the last term of (B9) is equivalent to $\kappa(L_\kappa^m+L_{\kappa-1}^{m+1}-L_\kappa^{m+1})$, which is identical to zero. We thus obtain the recurrence formula (4.9),

$$V_{ss'}^{nn'}\left[\frac{q^2}{\gamma},l\right]=\left[s'+s-1+\frac{q^2}{4\gamma}\right]V_{s-1,s'-1}^{nn'}\left[\frac{q^2}{\gamma},l\right]-(s-1)(s'-1)V_{s-2,s'-2}^{nn'}\left[\frac{q^2}{\gamma},l\right].$$

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