PHYSICAL REVIEW A
GENERAL PHYSICS

THIRD SERIES, VOLUME 38, NUMBER 4 AUGUST 15, 1988

Higher-order WKB approximations in supersymmetric quantum mechanics

R. Adhikari and R. Dutt
Department of Physics, Visva-Bharati University, Santiniketan 73125, West Bengal, India

A. Khare* and U. P. Sukhatme
Department of Physics, University of Illinois at Chicago, Chicago, Illinois 60680
(Received 28 December 1987)

In the framework of the recently proposed supersymmetric WKB (SWKB) approximation scheme, we obtain an explicit expression for the quantization condition which contains all terms up to order $\hbar^6$. For spherically symmetric potentials, we find that the SWKB approach automatically yields wave functions with the correct threshold behavior. This is in contrast to the usual WKB scheme, where proper $r \to 0$ behavior necessitates the use of cumbersome "Langer corrections." Previous authors have shown that the leading-order ($\hbar^0$) SWKB quantization integral gives exact bound-state spectra for analytically solvable shape-invariant potentials. For these cases, we show that the higher-order correction terms vanish identically. Finally, for nonanalytically solvable potentials, a comparison of our results (comprising of higher-order corrections) with numerically determined eigenvalues reveals very good accuracy.

I. INTRODUCTION

The WKB method[1–8] is one of the most useful techniques for computing approximate eigenvalues of the one-dimensional Schrödinger equation. In the lowest-order approximation, the WKB quantization condition is

$$\int_{x_1}^{x_2} \sqrt{2m[E-V(x)]}\,dx = (n+\tfrac{1}{2})\pi\hbar, \quad n=0,1,2,\ldots . \tag{1.1}$$

In general, Eq. (1.1) yields moderately accurate eigenvalues as analytic functions of the parameters contained in the potential. However, for additional accuracy, it is necessary to consider second- and higher-order corrections in $\hbar$. Initial work along these lines was done by Dunham,[9] while subsequently the technique was applied to various physical problems by Krieger and coworkers,[10–13] Bender *et al.*,[14] Kesarwani and Varshni,[15–18] and others.[19–26] Recently, Seetharaman *et al.*[27] have expressed Eq. (1.1) in terms of complete elliptic integrals and obtained an explicit expression for the energy eigenvalues of the anharmonic-oscillator potential.

A natural way of using the WKB approximation for three-dimensional problems with spherical symmetry is to apply the one-dimensional WKB formalism to the radial Schrödinger equation

$$\frac{d^2\chi}{dr^2} + \frac{2m}{\hbar^2}[E - V_{\rm eff}(r)]\chi(r) = 0 , \tag{1.2}$$

where the effective potential $V_{\rm eff}(r)$ is

$$V_{\rm eff}(r) = V(r) + \frac{l(l+1)\hbar^2}{2mr^2} . \tag{1.3}$$

Such a straightforward application leads to an important difficulty. It was observed[10,28] that the WKB reduced radial wave function at the origin has a behavior which is different from that of the true wave function

$$\chi(r) \underset{r\to 0}{\sim} r^{l+1} . \tag{1.4}$$

It was suggested that this defect could be remedied by treating the strength of the angular momentum barrier term $l(l+1)$ not as a fixed quantity but as an adjustable parameter K. Langer[29] pointed out that K should take the value $(l+\frac{1}{2})^2$ in the lowest-order quantization formula. This replacement is commonly known as the "Langer correction." When one goes beyond the lowest order and includes higher-order corrections, the difficulty associated with the threshold behavior of the wave function reappears again indicating that the Langer correction needs modification at each order of approximation. However, the algebraic procedure for adjusting this correction for higher-order approximations is quite difficult and cumbersome.[10,28]

WKB calculations also suffer from another serious limitation. Except for the harmonic-oscillator potential, they fail to reproduce exact analytic results for other solvable potentials.

The purpose of this work is to discuss in some detail a new formulation of the WKB approximation which is free of the above-mentioned drawbacks. This new supersymmetric WKB (SWKB) approach was pioneered by Comtet *et al.*[30] using the framework of supersymmetric quantum mechanics.[31,32] They obtained a lowest-order SWKB formula

$$\int_a^b [2m(E-\phi^2)]^{1/2}dx = n\pi\hbar \,, \tag{1.5}$$

where a and b are the turning points defined by

$$\phi^2(a)=\phi^2(b)=E_n^{(-)} \tag{1.6}$$

and $E=E_n^{(-)}$ is an eigenenergy of H_-. Here, the Hamiltonians $H_\pm$ correspond to the supersymmetric partner potentials $V_\pm$:

$$H_\pm = \frac{p^2}{2m} + V_\pm(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \phi^2(x) \pm \frac{\hbar}{\sqrt{2m}}\phi'(x) \,, \tag{1.7}$$

and $\phi(x)$ is the superpotential. A remarkable feature of the SWKB quantization condition, Eq. (1.5), is that it is not only accurate for large n (as any WKB approximation should be in the classical limit), but is also exact by construction for $n=0$ since $E_0^{(-)}=0$ satisfies Eq. (1.5). It was quickly discovered that the lowest-order SWKB quantization relation reproduces the exact bound-state spectra[30] for several analytically solvable potentials. Subsequently, it was shown by us that the reason behind obtaining the exact analytic results for these potentials is that they satisfy the "shape-invariance" condition[33–36] and for shape-invariant potentials, lowest-order SWKB is necessarily exact.[34,35] We also explicitly computed the $O(\hbar^2)$ correction term and showed that it vanishes for all known shape-invariant potentials. Recently, using a complex integration technique, Raghunathan *et al.*[37] have explicitly shown that *all* the higher-order corrections are zero for the Rosen-Morse potential and have claimed that a similar result holds for other shape-invariant potentials too. Thus there is no doubt that for shape-invariant potentials SWKB is superior to the usual WKB approximation. What about non-shape-invariant potentials for which lowest-order SWKB is not exact? In this connection one might raise the following questions: (i) As for the WKB case, can one also evaluate the higher-order correction terms in the case of SWKB quantization? (ii) If yes, then are the higher-order correction terms reasonably small for potentials which are not shape-invariant? (iii) Does one obtain the correct threshold ($r\to 0$) behavior of the SWKB radial wave function for spherically symmetric potentials?

The purpose of this paper is to answer these questions in some detail. In Sec. II, we quickly review the procedure for obtaining higher-order corrections for the WKB method. Starting with the WKB result, we explicitly obtain the SWKB quantization condition to $O(\hbar^6)$ in Sec. III. Our result is Eq. (3.15). We also show that the SWKB wave function for spherically symmetric potentials has the correct r^{l+1} threshold behavior. For shape-invariant potentials, we confirm that higher-order correction terms vanish identically. Finally, as an example of nonanalytically solvable potentials, we consider in Sec. IV the class of anharmonic oscillators $V(x) = A^2x^{4d+2} + A\hbar(2d+1)x^{2d}$, $A>0$, $d\geq 0$. The leading $O(\hbar^0)$ result is quite good, and is corrected in the right direction by the $O(\hbar^2)$ terms. We find that the $O(\hbar^4)$ and $O(\hbar^6)$ corrections are very small. A short discussion and conclusions are given in Sec. V.

II. HIGHER-ORDER TERMS IN THE WKB APPROXIMATION

Let us briefly indicate the main steps in the derivation of higher-order correction terms within the usual WKB framework. For any one-dimensional potential $V(x)$, the energy eigenvalue problem is given by the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E\right]\psi(x) = 0 \,. \tag{2.1}$$

We consider the case of two classical turning points x_1 and x_2 ($\geq x_1$) given by

$$E - V(x) = 0 \,. \tag{2.2}$$

Inserting the wave function

$$\psi(x) = \exp\left[+i\hbar\int^x S(x)dx\right] , \tag{2.3}$$

where

$$S(x) = \sum_{n=0}^{\infty} (\hbar/i)^n S_n(x) \tag{2.4}$$

in Eq. (2.1) and equating the coefficients of successive powers of $\hbar$ to zero, we obtain,

$$S_0(x) = \sqrt{2m[E-V(x)]} \,, \tag{2.5}$$

$$S'_{n-1} = -\sum_{m=0}^{n} S_{n-m}S_m, \quad n=1,2,\ldots \,. \tag{2.6}$$

Writing the recurrence formula (2.6) explicitly, one finds

$$S_0S_1 = -\tfrac{1}{2}S'_0 \,, \tag{2.7a}$$

$$S_0S_2 = -\tfrac{1}{2}S'_1 - \tfrac{1}{2}S_1^2 \,, \tag{2.7b}$$

$$S_0S_{2k} = -\tfrac{1}{2}S'_{2k-1} - \sum_{m=1}^{k} S_m S_{2k-m} - \tfrac{1}{2}S_k^2 \quad (k=2,3,4,\ldots) \,, \tag{2.7c}$$

$$S_0S_{2k+1} = -\tfrac{1}{2}S'_{2k} - \sum_{m=1}^{k} S_m S_{2k-m+1} \quad (k=1,2,3,\ldots) \,. \tag{2.7d}$$

These recurrence relations may be simplified and solved successively to obtain various $S_i, i=1,2,\ldots$. Following the procedure of extending the domain of x to the complex plane, Dunham obtained the energy quantization condition[38]

$$\oint S_0 dx + \sum_{n=2}^{\infty} (\hbar/i)^n \oint S_n(x)dx = (n+\tfrac{1}{2})h \ . \tag{2.8}$$

It has been shown explicitly by Bender *et al.*,[14] that S_{2n+1}, $n \geq 1$ is itself a total derivative, and hence all odd terms S_3, S_5, etc., when integrated along the closed contour in (2.8) vanish. Following the method of Krieger *et al.*,[10,11] the terms involving S_2, S_4, S_6, etc., are simplified by repeated integration by parts. Furthermore, converting the nonintegrable singularities in these terms to integrable singularities, the quantization condition up to order $\hbar^6$ may be obtained as

$$\begin{aligned}(2m)^{1/2}\int_{x_1}^{x_2}(E-V)^{1/2}dx &- \frac{\hbar^2}{24(2m)^{1/2}}\frac{d}{dE}\int_{x_1}^{x_2}V''(E-V)^{-1/2}dx \\ &+ \frac{\hbar^4}{2880(2m)^{3/2}}\frac{d^3}{dE^3}\int_{x_1}^{x_2}[7(V'')^2-5V'V'''](E-V)^{-1/2}dx \\ &- \frac{\hbar^6}{725\,760(2m)^{5/2}}\left\{\frac{d^4}{dE^4}\int_{x_1}^{x_2}216(V''')^2(E-V)^{-1/2}dx\right. \\ &\left.+\frac{d^5}{dE^5}\int_{x_1}^{x_2}[93(V'')^3-224V'V''V'''+35(V')^2V''''](E-V)^{-1/2}dx\right\} = (n+\tfrac{1}{2})\pi\hbar \ ,\end{aligned} \tag{2.9}$$

For obtaining corresponding correction terms for spherically symmetric three-dimensional problems described by the radial Schrödinger equation (2.1), Krieger *et al.*[11] invoked the Langer transformation

$$r=e^x, \quad \chi(r)=e^{x/2}\psi(x) \tag{2.10}$$

and showed that the leading-order term of the series expansion (2.4) is of the form

$$S_0 = \left[E - V(r) - \frac{(l+\frac{1}{2})^2\hbar^2}{2mr^2}\right]^{1/2} , \tag{2.11}$$

which clearly indicates that in the centrifugal term $l(l+1)$ has been replaced by $(l+\frac{1}{2})^2$. One may then easily check from (2.3), (2.10), and (2.11) that the correct threshold behavior (for potentials less singular than $1/r^2$) is

$$\chi_l(r) \underset{r\to 0}{\sim} r^{1/2}e^{(l+1/2)\ln r} ,$$

which is same as indicated in (1.4). However, it was pointed out by several authors[10,28] that the replacement of $l(l+1)$ by $(l+\frac{1}{2})^2$ is not valid if second- and higher-order correction terms are included. For example, if one includes terms up to S_2 in (2.3) and (2.4), it can be shown that for obtaining the correct threshold behavior of $\chi(r)$ at the origin, $l(l+1)\equiv K$ has to be replaced by the solution of the equation

$$K + \frac{1}{64K} = l(l+1) \ . \tag{2.12}$$

It emerges that Langer-type corrections vary from order to order, and the calculation becomes cumbersome and tedious for higher-order WKB approximations.

III. HIGHER-ORDER SWKB APPROXIMATION

Equation (2.9), being quite general for any potential $V(x)$ which can hold bound states, will be our starting point for obtaining the new expansion series in supersymmetric WKB approximation. Let us rewrite (2.9) as

$$I_1 + I_2 + I_3 + I_4 = (n+\tfrac{1}{2})\pi\hbar \ , \tag{3.1}$$

where I_1, I_2, I_3, and I_4 denote, respectively, the first, second, third, and fourth terms on the left-hand side of (2.9). As mentioned before, Comtet *et al.*[30] first replaced $V(x)$ by $V_-(x)$ as given by Eq. (1.7) and keeping the leading term ϕ^2, obtained the modified quantization condition (1.5). It is necessary to emphasize here that there are two distinguishing features between WKB and SWKB quantization relations (1.1) and (1.5), respectively: Firstly, the integrand in the supersymmetric case contains not the full potential

$$V_-(x) = \phi^2(x) - \frac{\hbar}{\sqrt{2m}}\phi'(x) \ ,$$

but only the leading term, ϕ^2. Secondly, the turning points $x_{1,2}$ in (1.1), are the solutions of

$$E_n^{(-)} - \phi^2(x) + \frac{\hbar}{\sqrt{2m}}\phi'(x) = 0 \ , \tag{3.2}$$

whereas the turning points a and b in (1.5) are the solutions of Eq. (1.6). Clearly,

$$\lim_{\hbar\to 0} x_{1,2} = a,b \ . \tag{3.3}$$

This indicates that while obtaining higher-order SWKB integrals similar to those in (2.9), one may expect contributions to different orders in $\hbar$ due to the change of the limits $x_{1,2}\to a,b$ as well as due to the terms contributed by the expansion of the integrands which also contain the parameter $\hbar$. Now, our job is to do the expansions of the integrals in (2.9) in $\hbar$ systematically and to collect terms of the same order in $\hbar$.

In obtaining the rearranged series in $\hbar$, let us first examine whether any contribution at all comes from the change of limits. For this purpose we concentrate, say, on the first integral I_1 in (3.1),

$$I_1=\int_{x_1(\hbar)}^{x_2(\hbar)} f(x,\hbar)dx \ , \tag{3.4}$$

where

$$f(x,\hbar)\equiv \left[2m\left[E-\phi^2+\frac{\hbar}{\sqrt{2m}}\phi'\right]\right]^{1/2} . \tag{3.5}$$

One may expand I_1 in powers of $\hbar$ as

$$I_1=\sum_{n=0}^{\infty}\hbar^n I_{1n} \ , \tag{3.6}$$

where

$$I_{1n}=\frac{1}{n!}\lim_{\hbar\to 0}\frac{d^n I_1}{d\hbar^n}, \quad n=0,1,2,\ldots . \tag{3.7}$$

Using Leibnitz's formula

$$\frac{d}{d\hbar}\int_{x_1(\hbar)}^{x_2(\hbar)} f(x,\hbar)dx=\int_{x_1}^{x_2}\frac{df}{d\hbar}dx+f(x_2,\hbar)\frac{dx_2(\hbar)}{d\hbar} -f(x_1,\hbar)\frac{dx_1(\hbar)}{d\hbar} \ , \tag{3.8}$$

we get from (3.7)

$$I_{10}=\int_a^b [2m(E-\phi^2)]^{1/2}dx \ , \tag{3.9}$$

$$I_{11}=\lim_{\hbar\to 0}\left[\int_{x_1}^{x_2}\frac{df}{d\hbar}dx+f(x_2,\hbar)\frac{dx_2}{d\hbar}-f(x_1,\hbar)\frac{dx_1}{d\hbar}\right] , \tag{3.10}$$

and so on. The last two terms in (3.10) are the contributions from the limits as those are functions of $\hbar$. However, by virtue of (3.2)

$$f(x_2,\hbar)=f(x_1,\hbar)=0 \ , \tag{3.11}$$

and hence the contributions due to the change of limits from $x_{1,2}$ to a,b are zero in the $\hbar$ expansion. Thus, Eqs. (3.5) and (3.10) lead to

$$I_{11}=\int_a^b \frac{\phi' dx}{2(E-\phi^2)^{1/2}}=\frac{\pi}{2} . \tag{3.12}$$

Similarly,

$$I_{12}=\frac{-1}{8(2m)^{1/2}}\int_a^b \frac{(\phi')^2 dx}{(E-\phi^2)^{3/2}} . \tag{3.13}$$

Proceeding in this way one gets

$$\begin{aligned} I_1=&(2m)^{1/2}\int_a^b (E-\phi^2)^{1/2}dx+\frac{\pi\hbar}{2}-\frac{\hbar^2}{8(2m)^{1/2}}\int_a^b (\phi')^2(E-\phi^2)^{-3/2}dx \\ &+\frac{\hbar^3}{16(2m)}\int_a^b(\phi')^3(E-\phi^2)^{-5/2}dx-\frac{5\hbar^4}{128(2m)^{3/2}}\int_a^b(\phi')^4(E-\phi^2)^{-7/2}dx \\ &+\frac{7\hbar^5}{256(2m)^2}\int_a^b(\phi')^5(E-\phi^2)^{-9/2}dx-\frac{21\hbar^6}{1024(2m)^{5/2}}\int_a^b(\phi')^6(E-\phi^2)^{-11/2}dx+O(\hbar^7) . \end{aligned} \tag{3.14}$$

It is interesting to note that I_1, which is of order ($\hbar^0$) in the normal WKB scheme [see (2.9)], now contains all orders of $\hbar$ due to the change of the potential $V(x)$ by the superpotential $\phi(x)$. Similarly, I_2, I_3, and I_4 which are the integrals of orders ($\hbar^2$), ($\hbar^4$), and ($\hbar^6$), respectively, in the usual WKB formalism, also involve all powers of $\hbar$ beyond their leading terms. Explicit expressions for I_2, I_3, and I_4 are given in the Appendix.

As first sight, one may be surprised to see the appearance of odd powers of $\hbar$ such as $\hbar^3$, $\hbar^5$, etc., in the quantization integrals I_1–I_4. However, interestingly as expected, one finds complete cancelation among such terms when one adds up I_1, I_2, I_3, and I_4 and only the even powers of $\hbar$ make nonzero contributions. After some algebraic simplification, one gets

$$\begin{aligned} &(2m)^{1/2}\int_a^b(E-\phi^2)^{1/2}dx-\frac{\hbar^2 E}{6(2m)^{1/2}}\frac{d^2}{dE^2}\int_a^b(\phi')^2(E-\phi^2)^{-1/2}dx \\ &+\frac{\hbar^4}{720(2m)^{3/2}}\left[\frac{d^2}{dE^2}\int_a^b 30\phi'\phi'''(E-\phi^2)^{-1/2}dx+\frac{d^3}{dE^3}\int_a^b(-8(\phi')^4-31\phi(\phi')^2\phi'' \right. \\ &\qquad\left. +7\phi^2(\phi'')^2-5\phi^2\phi'\phi''')(E-\phi^2)^{-1/2}dx\right] \\ &+\frac{\hbar^6}{90\,720(2m)^{5/2}}\left[\frac{d^3}{dE^3}\int_a^b 378(\phi''')^2(E-\phi^2)^{-1/2}dx \right. \\ &\qquad+\frac{d^4}{dE^4}\int_a^b[-2160\phi\phi'\phi''\phi'''+1674(\phi')^2(\phi'')^2-108\phi^2(\phi''')^2](E-\phi^2)^{-1/2}dx \\ &\qquad+\frac{d^5}{dE^5}\int_a^b[96(\phi')^6-1119\phi(\phi')^4\phi''+729\phi^2(\phi')^2(\phi'')^2+399\phi^2(\phi')^3\phi''' \\ &\qquad\left. -93\phi^3(\phi'')^3+224\phi^3\phi'\phi''\phi'''-35\phi^3(\phi')^2\phi''''](E-\phi^2)^{-1/2}dx\right]=n\pi\hbar . \end{aligned} \tag{3.15}$$

Here E corresponds to $E_n^{(-)}$. Equation (3.15) is our new SWKB quantization relation up to order $\hbar^6$ and should be treated as the analogue of Eq. (2.9) obtained in the usual WKB approximation. The first two terms of Eq. (3.15), i.e., up to order $\hbar^2$ were obtained earlier by us.[34]

Equation (3.15) preserves the basic properties of unbroken supersymmetry. In particular, for $n=0$, the turning points a and b are coincident when $E_0 \equiv E_0^{(-)}=0$ and SWKB is exact by construction. Furthermore, it is important to point out that if instead of $E_n^{(-)}$ one is interested in obtaining a quantization formula for $E_n^{(+)}$ [i.e., energy eigenvalues associated with the sypersymmetric partner potential $V_+(x)$] one needs to replace $\hbar\phi'(x)$ by $-\hbar\phi'(x)$, which is equivalent to the replacement $\hbar \to -\hbar$ in Eq. (3.15). It is easy to see that the SWKB quantization condition for $E_n^{(+)}$ is again given by Eq. (3.15) except that the right-hand side will be $(n+1)\pi\hbar$. Thus one sees that all higher-order SWKB calculations retain the basic supersymmetric relation

$$E_{n+1}^{(-)}=E_n^{(+)}, \quad n=0,1,2,\ldots . \tag{3.16}$$

Before exploring the applications of our higher-order SWKB quantization relation, we would like to emphasize that unlike the conventional WKB approach, one obtains the correct threshold behavior of the radial wave function for spherically symmetric problems without making any Langer-type modifications. From the lowest-order term, one obtains

$$S_0 \underset{r\to 0}{\sim} [2m(E-\phi^2)]^{1/2} \sim -i\hbar(l+1)/r$$

and consequently

$$\psi(r) \underset{r\to 0}{\sim} \exp\left[\frac{i}{\hbar}\int^r S_0 dr\right] \sim r^{l+1} . \tag{3.17}$$

Including higher-order terms such as S_1, S_2, etc., in (2.3), we have checked that the SWKB wave function behaves like r^{l+1} as $r\to 0$ to all orders in $\hbar$, i.e., the SWKB formalism contains the correct threshold behavior in a natural way.

As mentioned in Sec. I, it has been shown[34] that for shape invariant potentials, the leading term in Eq. (3.15) gives the exact bound-state spectra and the $O(\hbar^2)$ contribution vanishes. Now that we have obtained a more accurate expression containing $O(\hbar^4)$ and $O(\hbar^6)$ terms, it is possible to see what their contributions are. We have explicitly computed these contributions for all known shape-invariant potentials (see Table I in Ref. 35) and found them to be zero. (The calculations are tedious but straightforward.) This provides an independent check on the proof[37] regarding the vanishing of higher-order corrections for shape-invariant potentials.

IV. EIGENVALUES OF NON-SHAPE-INVARIANT POTENTIALS

in the previous section, we have discussed shape-invariant potentials, for which all higher-order SWKB integrals are explicitly zero. In this section, we deal with potentials for which at least some of the higher-order integrals are nonzero. One such problem is the anharmonic potential,[33,34]

$$V(x)=A^2x^{4d+2}+(2d+1)\hbar A x^{2d}, \quad A>0, \quad d\geq 0 , \tag{4.1}$$

for which the superpotential is

$$\phi(x)=Ax^{2d+1} . \tag{4.2}$$

Treating V as V_+ and using the relation (3.19), we obtain from (5.2) and (3.18) the energy quantization relation

$$\frac{Q\sqrt{\pi}\,\Gamma\left[\frac{4d+3}{4d+2}\right]}{\Gamma\left[\frac{3d+2}{2d+1}\right]}+\frac{\hbar^2\sqrt{\pi}(d+1)\Gamma\left[\frac{4d+1}{4d+2}\right]}{6Q\Gamma\left[\frac{d}{2d+1}\right]}$$
$$-\frac{\hbar^4\sqrt{\pi}(d+2)\Gamma\left[\frac{4d-1}{4d+2}\right]}{720Q^3\Gamma\left[\frac{d-1}{2d+1}\right]}\left[60d(2d-1)+\frac{(4d-1)}{(2d+1)}[-4(2d+1)^2-31d(2d+1)+14d^2-5d(2d-1)]\right]$$
$$+\frac{\hbar^6\sqrt{\pi}(d+3)(3d+4)\Gamma\left[\frac{4d-3}{4d+2}\right]}{181440Q^5(2d+1)^2\Gamma\left[\frac{d-2}{2d+1}\right]}$$
$$\times\{3024d^2(2d-1)^2(2d+1)+216d^2(4d-3)[-40(2d+1)(2d-1)+31(2d+1)^2-2(2d-1)^2]$$
$$+(8d-1)(4d-3)[48(2d+1)^3-1119d(2d+1)^2+1458d^2(2d+1)+399d(2d+1)(2d-1)$$
$$-372d^3+448d^2(2d-1)-70d(d-1)(2d-1)]\}=(n+1)\pi\hbar , \tag{4.3}$$

TABLE I. Comparison of energy eigenvalues computed up to order ($\hbar^6$) with the numerical results for the potential $V(x)=x^2+x^6/9$ in units of $2m=\hbar=1$.

n	$O(\hbar^0)$	$O(\hbar^2)$	$O(\hbar^4)$	$O(\hbar^6)$	Numerical
0	1.3077	1.2142	1.2142	1.2566	1.1175
1	3.6989	3.6346	3.6346	3.6363	3.6364
2	6.7953	6.7430	6.7430	6.7433	6.7440
3	10.462	10.417	10.417	10.417	10.417
4	14.621	14.581	14.581	14.581	14.581
5	19.220	19.183	19.183	19.183	19.183
6	24.220	24.186	24.186	24.186	24.186
7	29.591	29.559	29.599	29.559	29.559
8	35.309	35.279	35.279	35.279	35.279
9	41.355	41.326	41.326	41.326	41.326
10	47.710	47.683	47.683	47.683	47.683

where

$$Q=(2m)^{1/2}(E^{d+1}/A)^{1/(2d+1)} .$$

Since the Schrödinger equation cannot be solved analytically for the potentials under consideration, we compare the SWKB energy eigenvalues with the numerical results obtained by us using a Runge-Kutta integration program. As an illustration, we consider two cases: $A=\frac{1}{3}$, $d=1$ and $A=1$, $d=\frac{1}{3}$. Using the units $2m=\hbar=1$ in (5.3), we thus obtain for $A=\frac{1}{3}$, $d=1$

$$\frac{3^{1/3}E^{2/3}\Gamma(\frac{1}{3})}{\Gamma^2(\frac{2}{3})2^{4/3}}+\frac{2^{1/3}E^{-2/3}\Gamma(\frac{2}{3})}{\Gamma^2(\frac{1}{3})3^{4/3}}-\frac{1318E^{-10/3}2^{2/3}\Gamma(\frac{1}{3})}{109\,35\Gamma^2(\frac{2}{3})3^{5/3}}=n+1 . \quad (4.4)$$

In (5.4), successive terms corresponds to order ($\hbar^0$), ($\hbar^2$), and ($\hbar^6$), respectively. For this case, the term of order ($\hbar^4$) is identically zero. Likewise, for the choice $A=1$, $d=\frac{1}{3}$, one gets

$$\frac{3E^{4/5}\Gamma(\frac{3}{5})}{\Gamma^2(\frac{4}{5})2^{13/5}}+\frac{2^{8/5}E^{-4/5}\Gamma(\frac{2}{5})}{9\Gamma^2(\frac{1}{5})}-\frac{469E^{-12/5}2^{4/5}\Gamma(\frac{1}{5})}{30\,375\Gamma^2(\frac{3}{5})}=n+1 . \quad (4.5)$$

For this choice of A and d, it is interesting to note that one has terms of order ($\hbar^0$), ($\hbar^2$), and ($\hbar^4$), but order ($\hbar^6$) vanishes. Energy eigenvalues computed from (4.4) and (4.5) are shown, respectively, in Tables I and II along with the numerical results obtained by us. The comparison reveals that in general there is an improvement of accuracy due to $O(\hbar^2)$ terms, but subsequent higher-order correction terms are negligible. From Table I, it is seen that the result for the ground state begins to deviate from its actual value if one includes the next nonvanishing correction term beyond order ($\hbar^2$). This is perhaps due to the asymptotic nature of the SWKB series and is not so surprising when one recalls that similar behavior is also observed in case of normal WKB calculations. Bender *et al.*[14] performed higher-order WKB calculations for the anharmonic potential $V(x)=x^4$ and observed that the ground-state energy becomes worse by the inclusion of correction term of order ($\hbar^6$) [see Table I of Ref. 14]. Similar observations were also made by Kesarwani and Varshni.[17]

V. CONCLUDING REMARKS

The SWKB quantization condition can be regarded as a rearrangement of the WKB series in powers of $\hbar$ such that (i) it is exact for $n=0$ and (ii) the lowest order gives the exact eigenvalue spectrum for shape-invariant potentials. It is clearly important to check that in spite of this rearrangement the higher-order contributions are small.

In this paper we have explicitly obtained the SWKB

TABLE II. Comparison of energy eigenvalues computed up to order ($\hbar^6$) with the numerical results for the potential $V(x)=x^{10/3}+\frac{5}{3}x^{2/3}$ in units of $2m=\hbar=1$.

n	$O(\hbar^0)$	$O(\hbar^2)$	$O(\hbar^4)$	$O(\hbar^6)$	Numerical
0	2.1422	2.0897	2.1148	2.1148	1.9850
1	5.0950	5.0642	5.0678	5.0678	5.1121
2	8.4578	8.4351	8.4363	8.4363	8.4088
3	12.118	12.100	12.100	12.100	12.118
4	16.016	16.001	16.001	16.001	15.988
5	20.116	20.103	20.103	20.103	20.113
6	24.391	24.379	24.379	24.379	24.371
7	28.822	28.811	28.811	28.811	28.817
8	33.393	33.383	33.383	33.383	33.384
9	38.094	38.085	38.085	38.085	38.090
10	42.914	42.905	42.905	42.905	42.901

quantization condition to $O(\hbar^6)$ and have shown that whereas the higher-order correction terms vanish for shape invariant-potentials, they are nonzero but quite small for other potentials. Further, we have shown that unlike the WKB approach, the SWKB method reproduces the correct threshold behavior of the reduced radial eigenfunction for spherically symmetric potentials. This suggests that the SWKB method can prove to be very good for eigenvalue determinations. In order to further compare the relative merits of the WKB and SWKB approaches, it would clearly be desirable to compare the predictions of both approaches for a number of potentials. Further, one should examine if one could compute large-order terms and phase shifts within SWKB approach. It must be noted here that the usual WKB method has a definite edge over SWKB for those potentials for which the superpotential, and hence the ground-state wave function, is not known. For such potentials, the additional step of computing the ground-state wave function either numerically of by a variational technique is necessary.

ACKNOWLEDGMENTS

One of us (R.A.) wishes to thank the Department of Atomic Energy, Government of India for financial support. Discussions with W. Y. Keung are gratefully acknowledged. This work was supported in part by the U.S. Department of Energy and the Research Corporation.

APPENDIX

The WKB quantization condition can be expressed in the form [Eq. (3.1)]

$$\sum_{r=1}^{\infty} I_r=(n+\tfrac{1}{2})\pi\hbar\ , \tag{A1}$$

where $I_r=O(\hbar^{2r-2})$. The computation of I_1 was described in detail in Sec. III. Essentially the same procedure is used for computing higher-order terms. Defining $G\equiv[E-\phi^2(x)]^{-1/2}$, the results for I_2, I_3, and I_4 are

$$\begin{aligned}
I_2={}&\frac{-\hbar^2}{12(2m)^{1/2}}\left[\frac{d}{dE}\int_a^b[(\phi')^2+\phi\phi'']G dx\right]+\frac{\hbar^3}{24(2m)}\left[\frac{d}{dE}\int_a^b\phi'''G\,dx-\frac{d^2}{dE^2}\int_a^b 2[(\phi')^3+\phi\phi'\phi'']G\,dx\right]\\
&+\frac{\hbar^4}{24(2m)^{3/2}}\left[\frac{d^2}{dE^2}\int_a^b\phi'\phi'''G\,dx-\frac{d^3}{dE^3}\int_a^b[(\phi')^4+\phi(\phi')^2\phi'']G\,dx\right]\\
&+\frac{\hbar^5}{144(2m)^2}\left[\frac{d^3}{dE^3}\int_a^b 3(\phi')^2\phi'''G\,dx-\frac{d^4}{dE^4}\int_a^b 2[(\phi')^5+\phi(\phi')^3\phi'']G\,dx\right]\\
&+\frac{\hbar^6}{288(2m)^{5/2}}\left[\frac{d^4}{dE^4}\int_a^b 2(\phi')^3\phi'''G\,dx-\frac{d^5}{dE^5}\int_a^b[(\phi')^6+\phi(\phi')^4\phi'']G\,dx\right]+O(\hbar^7)\ ,
\end{aligned}\tag{A2}$$

$$\begin{aligned}
I_3={}&\frac{\hbar^4}{720(2m)^{3/2}}\left[\frac{d^3}{dE^3}\int_a^b[7(\phi')^4+7\phi^2(\phi'')^2-\phi(\phi')^2\phi''-5\phi^2\phi'\phi''']G\,dx\right]\\
&+\frac{\hbar^5}{1440(2m)^2}\Bigg[\frac{d^3}{dE^3}\int_a^b[-14(\phi')^2\phi'''-9\phi\phi''\phi'''+5\phi\phi'\phi''''+15\phi'(\phi'')^2]G\,dx\\
&\qquad+\frac{d^4}{dE^4}\int_a^b 2[7(\phi')^5+7\phi^2\phi'(\phi'')^2-\phi(\phi')^3\phi''-5\phi^2(\phi')^2\phi''']G\,dx\Bigg]\\
&\qquad+\frac{\hbar^6}{2880(2m)^{5/2}}\Bigg[\frac{d^3}{dE^3}\int_a^b[7(\phi''')^2-5\phi''\phi'''']Gdx\\
&\qquad\qquad+\frac{d^4}{dE^4}\int_a^b 2[-14(\phi')^3\phi'''-9\phi\phi'\phi''\phi'''+5\phi(\phi')^2\phi''''+15(\phi')^2(\phi'')^2]G\,dx\\
&\qquad\qquad+\frac{d^5}{dE^5}\int_a^b 2[7(\phi')^6+7\phi^2(\phi')^2(\phi'')^2-\phi(\phi')^4\phi''-5\phi^2(\phi')^3\phi''']G\,dx\Bigg]+O(\hbar^7)\ ,
\end{aligned}\tag{A3}$$

$$\begin{aligned}
I_4={}&\frac{-\hbar^6}{840(2m)^{5/2}}\Bigg[\frac{d^4}{dE^4}\int_a^b[9(\phi')^2(\phi'')^2+\phi^2(\phi''')^2+6\phi\phi'\phi''\phi''']G\,dx\\
&\qquad+\frac{d^5}{dE^5}\int_a^b\frac{1}{108}[93(\phi')^6+93\phi^3(\phi'')^2-393\phi(\phi')^4\phi''-84\phi^2(\phi')^3\phi'''\\
&\qquad\qquad-288\phi^2(\phi')^2(\phi'')^2-224\phi^3\phi'\phi''\phi'''+35\phi^3(\phi')^2\phi'''']G\,dx\Bigg]+O(\hbar^7)\ .
\end{aligned}\tag{A4}$$

*Permanent address: Institute of Physics, Bhubaneswar 751 005, India.

[1]G. Wentzel, Z. Phys. **38**, 518 (1982).
[2]H. A. Kramers, Z. Phys. **39**, 828 (1926).
[3]L. Brillouin, C.R. Acad. Sci. **183**, 24 (1926); J. Phys. Radium **7**, 353 (1926).
[4]J. Heading, *An Introduction to Phase-Integral Methods, Methuen Monographs on Physical Subjects* (Methuen, London/New York, 1962).
[5]B. S. Jeffreys, in *Quantum Theory I,* edited by D. R. Bates (Academic, New York/London, 1961).
[6]N. Fröman and P. O. Fröman, *JWKB Approximation, Contributions to the Theory* (North-Holland, Amsterdam, 1965).
[7]D. J. W. Geldart and D. Kiang, Am. J. Phys. **54**, 131 (1986).
[8]N. Fröman and P. O. Fröman, in *Semiclassical Descriptions of Atomic and Nuclear Collision,* edited by J. Bang and J. de Boer (Elsevier, Amsterdam, 1985).
[9]J. L. Dunham, Phys. Rev. **41**, 721 (1932).
[10]J. B. Krieger and C. Rosenzweig, Phys. Rev. **164**, 171 (1967).
[11]J. B. Krieger, M. L. Lewis, and C. Rosenzweig, J. Chem. Phys. **47**, 2942 (1967).
[12]C. Rosenzweig, J. Math. Phys. **9**, 849 (1968).
[13]J. B. Krieger, J. Math. Phys. **10**, 1455 (1969).
[14]C. M. Bender, K. Olaussen, and P. S. Wang, Phys. Rev. D **16**, 1740 (1977).
[15]R. N. Kesarwani and Y. P. Varshni, Can. J. Phys. **56**, 1488 (1978).
[16]R. N. Kesarwani and Y. P. Varshni, Can. J. Phys. **58**, 363 (1980).
[17]R. N. Kesarwani and Y. P. Varshni, J. Math. Phys. **22**, 1983 (1981).
[18]R. N. Kesarwani and Y. P. Varshni, J. Math. Phys. **23**, 92 (1982).
[19]S. C. Miller and R. H. Good, Jr., Phys. Rev. **91**, 174 (1953).
[20]P. Lu, J. Chem. Phys. **47**, 815 (1967).
[21]P. Lu, J. Chem. Phys. **53**, 845 (1970).
[22]P. Lu, S. S. Wald, and B. L. Young, Phys. Rev. D **7**, 1701 (1972).
[23]N. Fröman and P. O. Fröman, Ann. Phys. **83**, 103 (1974).
[24]P. O. Fröman, I. Karlson, and M. Lakshmanan, Phys. Rev. D **20**, 3435 (1979).
[25]F. T. Hioe, E. W. Montroll, and M. Yamawaki, *Perspectives in Statistical Physics,* edited by H. J. Rayeche (North-Holland, Amsterdam, 1981).
[26]E. A. Solovev, Pis'ma, Zh. Eksp. Teor. Fiz. **39**, 84 (1984) [JETP Lett. **39**, 100 (1984)].
[27]M. Seetharaman, S. Raghavan, and S. S. Vasan, J. Phys. A **15**, 1537 (1982); **16**, 455 (1983).
[28]M. Seetharaman and S. S. Vasan, J. Phys. **A17**, 2485 (1984); **17**, 2493 (1984); **18**, 1041 (1985); **27**, 1031 (1986).
[29]K. E. Langer, Phys. Rev. **51**, 669 (1937).
[30]A. Comtet, A. D. Bandrauk, and D. K. Campbell, Phys. Lett. **150B**, 159 (1985); A. Khare, *ibid.* **161B**, 131 (1985).
[31]E. Witten, Nucl. Phys. B **185**, 513 (1981); F. Cooper and B. Freedman, Ann. Phys. (N.Y.) **146**, 262 (1983).
[32]C. V. Sukumar, J. Phys. A **18**, 2917 (1985); **18**, 2937 (1985).
[33]L. Gendenshtein, Pis'ma, Zh. Eksp. Teor. Fiz. **38**, 299 (1983) [JETP Lett. **38**, 356 (1983)].
[34]R. Dutt, A. Khare, and U. P. Sukhatme, Phys. Lett. **181B**, 295 (1986).
[35]R. Dutt, A. Khare, and U. P. Sukhatme, Am. J. Phys. **56**, 163 (1988).
[36]It should be noted that all analytically solvable potentials are not shape invariant. See F. Cooper, J. Ginocchio, and A. Khare, Phys. Rev. D **36**, 2458 (1987).
[37]K. Raghunathan, M. Seetharaman, and S. S. Vasan, Phys. Lett. **188B**, 351 (1987).
[38]An alternative derivation using quantum action variables is due to R. Leacock and M. Padgett, Phys. Rev. Lett. **50**, 3 (1983).