# Radiation pressure from the vacuum: Physical interpretation of the Casimir force

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We calculate the vacuum-field radiation pressure on two parallel, perfectly conducting plates. The modes outside the plates push the plates together, those confined between the plates push them apart, and the net effect is the well-known Casimir force. Implications of this result for the form of the Poynting vector in quantum electrodynamics are discussed.

The Casimir force between two perfectly conducting plates is one of the most frequently cited examples of physical effects attributable to the vacuum electromagnetic field.[1] This force may be regarded as a macroscopic manifestation of the retarded van der Waals force between two neutral polarizable particles. Although different viewpoints are possible,[2] it is most often derived by the consideration of the vacuum electromagnetic energy $\frac{1}{2}\hbar\omega$ per mode of frequency $\omega$: the difference between the vacuum electromagnetic energy for infinite plate separation and a finite plate separation $d$ is the interaction energy between the plates, from which the force is calculated.

The fact that the free energy depends on the plate separation accounts for the force due to the vacuum field. However, beyond this observation it does not appear that any more direct physical interpretation of the force has been proposed, nor has any intuitive explanation been given for why the force is attractive. In this paper we present an extremely simple explanation of the Casimir force using the classically familiar concept of radiation pressure. This approach makes it obvious why there should be a force due to the vacuum field, and furthermore has more general implications for the form of the Poynting vector in quantum electrodynamics (QED).

Consider the radiation pressure exerted by a plane wave incident normally on one of the plates. This pressure is equal to twice the energy per unit volume $E$ of the incident field; the factor of 2 is due to the perfect reflectivity assumed for the plates. If the wave has an angle of incidence $\theta$, however, the radiation pressure is

$$P=F/A=2E\cos^2\theta \ . \tag{1}$$

Two factors of $\cos\theta$ appear because (1) the normal component of the linear momentum imparted to the plate is proportional to $\cos\theta$, and (2) the element of area $A$ is increased by $(\cos\theta)^{-1}$ compared with the case of normal incidence.

Consider now the vacuum field between the plates. The modes formed by reflections off the plates obviously act to push the plates apart. A mode of frequency $\omega$ contributes a pressure

$$P=2(\tfrac{1}{2})(\tfrac{1}{2}\hbar\omega)V^{-1}\cos^2\theta=(\hbar\omega/2V)k_z^2/k^2 \ , \tag{2}$$

where $k=\omega/c$ and $V$ is the quantization volume. A factor of $\frac{1}{2}$ has been inserted because the zero-point energy of a mode, $\frac{1}{2}\hbar\omega$, is divided equally between waves propagating toward and away from each of the plates. For large plates $k_x$ and $k_y$ take on a continuum of values, whereas $k_z=n\pi/d$, with $n$ a positive integer. Summing up the contributions from all the modes of the space between the plates, we have the total outward pressure

$$P_{\text{out}}=(\hbar c/\pi^2 d)\times\sum_{n=1}^{\infty}\int_0^\infty dk_x\int_0^\infty dk_y\frac{(n\pi/d)^2}{[k_x^2+k_y^2+(n\pi/d)^2]^{1/2}} \tag{3}$$

on each plate. In writing this expression a factor of two has been inserted to allow for the two independent polarizations.

The vacuum-field modes outside the "resonator" formed by the plates have a continuum of allowed frequencies. These modes obviously act to push the plates together by reflection off the plates. The total inward pressure exerted by these modes may be obtained from (3) by replacing $\sum_n$ by $(d/\pi)\int dk_z$,

$$P_{\text{in}}=(\hbar c/\pi^3)\int_0^\infty dk_x\int_0^\infty dk_y\int_0^\infty dk_z\frac{k_z^2}{(k_x^2+k_y^2+k_z^2)^{1/2}} \ . \tag{4}$$

Both $P_{\text{out}}$ and $P_{\text{in}}$ are infinite, but it is only the difference that is physically meaningful. After some simple algebra we can cast this difference in the form

 

$$P_{\rm out}-P_{\rm in}=(\pi^2\hbar c/4d^4)\times\left[\sum_{n=1}^{\infty}n^2\int_0^\infty\frac{dx}{(x+n^2)^{1/2}}-\int_0^\infty du\,u^2\int_0^\infty\frac{dx}{(x+u^2)^{1/2}}\right]. \quad (5)$$

Various techniques may be employed to "regularize" (5) and extract the finite, physical result. For instance, a formal application of the Euler-Maclaurin summation formula yields

$$P_{\rm out}-P_{\rm in}=-\pi hc/480d^4\,, \quad (6)$$

which is the well-known expression for the Casimir force per unit area.

We conclude, therefore, that the Casimir force between the plates is simply a consequence of the radiation pressure associated with the QED vacuum field with zero-point energy $\frac{1}{2}\hbar\omega$ per mode of the field.[3] From this point of view it may be no surprise that the Casimir force in this case is attractive: since the modes in the space outside the plate form a continuum, whereas those inside are restricted to discrete values of $k_z$, there are "more" modes outside to push the plates together by radiation pressure than there are modes between the plates to push them apart. However, this intuitive argument is superficial in that both the inward and outward radiation pressures on the plates are infinite. In the case of a spherically conducting shell, for instance, the effect of the vacuum field is to produce a radially *outward* force, as first shown by Boyer.[4] The fact that the net radiation pressure is repulsive in this case emerges only after taking into account certain properties of the zeros of spherical Bessel functions,[5] based again on the work of Boyer.[6]

It is obvious, of course, that there is a radiation pressure on the plates in the presence of real photons. The simple analysis leading to Eq. (6) shows that the Casimir effect is just the zero-temperature limit of this *classical* radiation force. This is analogous to regarding the nonrelativistic portion of the Lamb shift as a quadratic Stark shift for an atom in a broadband field: the Lamb shift is just the Stark shift in the limit where the field has energy $\frac{1}{2}\hbar\omega$ per mode.[7] Various other vacuum QED effects may be interpreted similarly as zero-temperature limits of purely classical or well-known quantum effects. Although the derivation above makes this obvious in the case of the Casimir force between conducting plates, there does not appear, in the vast literature on the subject, to be any calculation based on so simple a classical concept as radiation pressure.[3]

The notion that Casimir forces result from the radiation pressure of the vacuum has a general mathematical formulation in terms of the Maxwell stress tensor for the quantized field. It follows from the (operator) Maxwell equations that the momentum density of the free field,

$$\hat{\mathbf{g}}=(1/8\pi c)(\hat{\mathbf{E}}\times\hat{\mathbf{B}}-\hat{\mathbf{B}}\times\hat{\mathbf{E}})\,, \quad (7)$$

obeys the equation of continuity

$$\partial\hat{g}^{\,i}/\partial t-\partial\hat{\sigma}^{\,ij}/\partial x^j=0\,, \quad (8)$$

where $\hat{\sigma}^{\,ij}$ is the Maxwell stress tensor

$$\hat{\sigma}^{\,ij}=(1/8\pi)[\hat{E}^{\,i}\hat{E}^{\,j}+\hat{E}^{\,j}\hat{E}^{\,i}+\hat{B}^{\,i}\hat{B}^{\,j}+\hat{B}^{\,j}\hat{B}^{\,i}-(\hat{E}^{\,2}+\hat{B}^{\,2})\delta^{ij}]\,. \quad (9)$$

[We use the caret symbol (^) to denote quantum-mechanical operators.] Equation (8) describes the local conservation of field momentum. By integrating (8) over a volume $V$ and using the divergence theorem, we find that the rate of change of the momentum in $V$ (i.e., the force on that volume) equals the integral of $\hat{\sigma}^{\,ij}$ over the surface $\Sigma$ bounding $V$. The expectation value of this force is found to be

$$F^i\equiv\int_\Sigma\langle\hat{\sigma}^{\,ij}\rangle n^j da\,, \quad (10)$$

where $n^j$ is the unit outward normal to the surface and $da$ is the element of surface area. If $\Sigma$ is the surface of a conductor, then (10) gives the radiation force on the conductor for any state of the field. In particular, when the field is in the vacuum state, Eq. (10) gives the Casimir force (6) on parallel, perfectly conducting plates when the field modes are chosen to satisfy the electromagnetic boundary conditions on the surfaces of the plates.

We wish to point out that the stress-tensor formalism carries with it certain implications concerning the form of the Poynting vector in quantum electrodynamics. For any relativistic theory the energy flux density $\hat{\mathbf{S}}$ (in our case the Poynting vector) is related to the momentum density $\hat{\mathbf{g}}$ by a factor $c^2$,

$$\hat{\mathbf{S}}=c^2\hat{\mathbf{g}}\,. \quad (11)$$

Several different expressions for the Poynting vector of the quantized field can be found in the literature. Among these are the vector

$$\hat{\mathbf{S}}=(c/8\pi)(\hat{\mathbf{E}}\times\hat{\mathbf{B}}-\hat{\mathbf{B}}\times\hat{\mathbf{E}})\,, \quad (12)$$

which follows from the momentum density (7), the normally ordered form

$$\hat{\mathbf{S}}_N=(c/4\pi):\hat{\mathbf{E}}\times\hat{\mathbf{B}}: \quad (13)$$

which is $\hat{\mathbf{S}}$ less the vacuum expectation value of $\hat{\mathbf{S}}$, and the "quantum-optical form"[8]

$$\hat{\mathbf{S}}_Q=(c/4\pi)(\hat{\mathbf{E}}^{(-)}\times\hat{\mathbf{B}}^{(+)}-\hat{\mathbf{B}}^{(-)}\times\hat{\mathbf{E}}^{(+)})\,, \quad (14)$$

where $(+)$ and $(-)$ indicate the positive-frequency (photon-annihilation) and negative-frequency (photon-creation) parts of the field, respectively. With $\hat{\mathbf{S}}_N$ and $\hat{\mathbf{S}}_Q$ are associated momentum densities $\hat{\mathbf{g}}_N$ and $\hat{\mathbf{g}}_Q$, and from these and the Maxwell equations one readily derives equations of continuity, analogous to (8), which involve stress tensors $\hat{\sigma}^{\,ij}_N$ and $\hat{\sigma}^{\,ij}_Q$, respectively. In this way the Poynting vector determines the stress tensor. Conversely, the existence of Casimir forces places constraints on the form of the Poynting vector. The expressions (13) and (14) for the Poynting vector lead to stress tensors which are in normal form and therefore have vanishing expectation values. The existence of Casimir forces thus rules out these forms as completely *general* expressions for the Poynting vector of quantum electrodynamics.

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[2]See, for instance, A. O. Barut and J. P. Dowling, Phys. Rev. A **36**, 2550 (1987) for a treatment based on self-energy; and P. W. Milonni, Phys. Rev. A **25**, 1315 (1982) for a discussion based on radiation reaction.

[3]It should be noted that Debye referred briefly to such an interpretation in his Baker lectures on molecular forces. ["The (Casimir) force may be interpreted as a zero-point pressure of electromagnetic waves."] See B. Chu, *Molecular Forces* (Wiley Interscience, New York 1967), p. 75.

[4]T. H. Boyer, Phys. Rev. **174**, 1764 (1968).

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[6]T. H. Boyer, J. Math. Phys. **10**, 1729 (1969).

[7]See P. W. Milonni, Phys. Scr. **T21**, 102 (1988).

[8]R. Loudon, *The Quantum Theory of Light,* 2nd ed. (Clarendon, Oxford, 1983), pp. 184–187.