Approximate analytical approaches to nonlinear pulse propagation in optical fibers: A comparison

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A comparison is made of two recently suggested approximate analytical methods for the investigation of nonlinear pulse propagation in optical fibers, as described by the nonlinear Schrödinger (NLS) equation. One approach is based on a variational procedure involving Ritz's optimization while the other makes use of invariants of the NLS equation. Despite their different character the approximation schemes are shown to lead to identical results for the evolution of the pulse amplitude and pulse width.

The analysis of nonlinear pulse propagation in optical fibers has been based primarily on numerical investigations of the nonlinear Schrödinger (NLS) equation, which determines the evolution of the slowly varying envelope of the optical wave pulse. Additional information has been provided by various analytical methods, the most important being the inverse scattering theory, which provides exact analytical results. However, the set of problems which can be solved to give explicit analytical solutions is disconcertingly small. For example, for nonsoliton initial conditions the evolution of the pulse is characterized by a complicated nonperiodic oscillatory behavior involving a soliton part and a radiation part which is difficult to describe analytically. Consequently, there is a need for approximate analytical descriptions of the pulse evolution. Several different approximation approaches have been suggested, e.g., variational methods,[1,2] methods making use of the invariants of the NLS,[3] and moment methods.[4]

The characteristic equation determining the evolution of the slowly varying pulse envelope ψ is the nonlinear Schrödinger equation (NLS), which reads

$$i\frac{\partial\Psi}{\partial x}=\alpha\frac{\partial^2\Psi}{\partial\tau^2}+\kappa|\Psi|^2\Psi , \tag{1}$$

where x denotes the distance of propagation, τ the retarded time, and α and κ are coefficients determined by dispersion and nonlinearity, respectively. In the variational approach[1,2] the NLS equation is restated as a variational problem in terms of the Lagrangian L, given by

$$L=\frac{i}{2}\left[\psi\frac{\partial\psi^*}{\partial x}-\psi^*\frac{\partial\psi}{\partial x}\right]-\alpha\left|\frac{\partial\psi}{\partial\tau}\right|^2+\frac{\kappa}{2}|\psi|^4 . \tag{2}$$

The trial function to be used in the optimization procedure is

$$\psi(x,\tau)=A(x)\exp\left[-\frac{\tau^2}{2a^2(x)}+ib(x)\tau^2\right] . \tag{3}$$

The form of the ansatz function has been chosen so as to incorporate features which are known to be inherent parts of the pulse evolution;[1-3,5,6] changing amplitude $|A|$, pulse width $a(x)$, frequency chirping effects $b(x)$, and a variable wave-number shift $\arg A$.

The variational problem can now be reduced to the form

$$\delta\int\langle L\rangle dx=0 , \tag{4}$$

where

$$\langle L\rangle=\int_{-\infty}^{\infty}L_G\,d\tau \tag{5}$$

and L_G denote the result of inserting the ansatz, Eq. (3), into the Lagrangian given by Eq. (2). The reduced variational problem, expressed by Eq. (4), results in a set of coupled ordinary differential equations for the Gaussian parameter functions $A(x)$, $A^*(x)$, $a(x)$, and $b(x)$, which together determine the pulse evolution.[1] The set of differential equations can be reduced to a single equation for the pulse width parameter, viz.,

$$\frac{1}{2}\left[\frac{da}{dx}\right]^2+\pi(a)=0 , \tag{6}$$

where the potential function $\pi(a)$ is given by

$$\pi(a)=2\alpha^2\left[\frac{1}{a^2}-\frac{1}{a_0^2}\right]-\alpha\kappa\sqrt{2}E_0\left[\frac{1}{a}-\frac{1}{a_0}\right] . \tag{7}$$

E_0 denotes the pulse energy $[E_0=|A(x)|^2a(x)=|A_0|^2a_0]$, which is a constant of motion. The variational equations also determine the frequency-chirp parameter $b(x)$ in terms of the pulse-width parameter as follows:

$$b(x)=-\frac{1}{4\alpha}\frac{1}{a}\frac{da}{dx} . \tag{8}$$

Finally, the phase $\phi(x)$ of the complex amplitude $A(x)=|A(x)|\exp[i\phi(x)]$ is determined by

$$\frac{d\phi}{dx}=\frac{\alpha}{a^2}-\frac{5\sqrt{2}}{8}\kappa|A|^2 . \tag{9}$$

Recently a new alternative analytical approach was suggested which is based on the invariants of the NLS equation.[3] This approach could provide a shortcut to the determination of the evolution of the pulse envelope, but involves the use of an ansatz function for the evolving complex pulse envelope. Obviously, the quality of the result depends on how well the ansatz models the actual pulse evolution. In this respect the invariant approach is similar to the variational approach. In order to compare the invariant and the variational methods we will therefore use the same trial function [Eq. (3)].

The NLS equation possesses an infinite set of invariants. However, some additional information is required concerning the functional form of the frequency chirp if one is to base the invariant method on the three leading invariants of the NLS equation only. These invariants are given by[7]

$$I_1=\int_{-\infty}^{\infty} |\Psi(x,\tau)|^2 d\tau \;, \tag{10a}$$

$$I_2=\int_{-\infty}^{\infty} \left[\psi^*(x,\tau)\frac{\partial\psi}{\partial\tau}(x,\tau)-\Psi(x,\tau)\frac{\partial\psi^*}{\partial\tau}(x,\tau)\right]d\tau \;, \tag{10b}$$

$$I_3=\int_{-\infty}^{\infty}\left[2\alpha\left|\frac{\partial\psi}{\partial\tau}\right|^2-\kappa|\Psi|^4\right]d\tau \;. \tag{10c}$$

The first invariant I_1 implies energy conservation and is given by

$$I_1=|A_1(x)|^2 a(x)\sqrt{\pi}$$

if the trial function of Eq. (3) is inserted into Eq. (10a). The second invariant I_2 is trivially satisfied by the trial function given by Eq. (3).

It is crucial that the ansatz used in the third invariant I_3 to obtain a characteristic equation for the pulse-width variation does not violate the lower-order invariants. In the work of Sodha and Kumar[3] the ansatz is deficient by violating the first invariant I_1 and their analysis consequently leads to unphysical results, e.g., predicts pulse-width collapse at a finite distance of propagation.

If the amplitude $A(x)$ is eliminated by means of the first invariant I_1 and the chirp parameter $b(x)$ is expressed in terms of the pulse width $a(x)$ according to Eq. (8), the third invariant I_3 will yield an evolution equation for $a(x)$ after substitution of the ansatz given by Eq. (3) into I_3 [Eq. (10c)]. One finds

$$I_3=I_1\left[\frac{\alpha}{a^2}+\frac{1}{4\alpha}\left[\frac{da}{dx}\right]^2-\frac{\kappa I_1}{a\sqrt{2\pi}}\right] \;, \tag{11}$$

which can be recast into the potential form of Eqs. (6) and (7) when I_1 and I_3 are calculated from the initial conditions.

The comparison can be extended to include the effect of a damping term $-i\gamma\psi$, in which case the NLS equation becomes

$$i\frac{\partial\psi}{\partial x}=\alpha\frac{\partial^2\Psi}{\partial\tau^2}+\kappa|\psi|^2-i\gamma\psi \;. \tag{12}$$

A variational reformulation of the damped NLS equation, Eq. (12), can be accomplished as follows:[2] introduce the transformation

$$\psi(x,\tau)=\varphi(x,\tau)\exp(-\gamma x) \;.$$

Then Eq. (12) becomes

$$i\frac{\partial\varphi}{\partial x}=\alpha\frac{\partial^2\varphi}{\partial\tau}+\kappa\exp(-2\gamma x)|\varphi|^2\varphi \;, \tag{13}$$

i.e., a NLS equation with a decreasing nonlinear coupling coefficient

$$\kappa(x)\equiv\kappa\exp(-2\gamma x) \;.$$

Equation (13) can be expressed as a variational problem in terms of the Lagrangian L of Eq. (2), although with $\kappa\to\kappa(x)$. The subsequent analysis is analogous to the case of the undamped NLS equation, Eq. (1). Reducing the variational ordinary differential equations for $A(x)$, $A^*(x)$, $a(x)$, and $b(x)$, a single nonlinear oscillator equation for the pulse width $a(x)$ emerges, viz.,

$$\frac{d^2a}{dx^2}=\frac{4\alpha^2}{a^3}-\alpha\kappa(x)\frac{E_0\sqrt{2}}{a^2} \;. \tag{14}$$

We emphasize that no assumptions regarding the length scale for the variation of $\kappa(x)$ has been made in the calculations leading to Eq. (14).

In the presence of damping, the integrals of Eqs. (10) which define I_1, I_2, and I_3 are no longer invariant but vary with x. Denote these integrals by $I_1(x)$, $I_2(x)$ and $I_3(x)$, respectively. Their change with distance is given by

$$I_1(x)=I_1(0)\exp(-2\gamma x) \;, \tag{15a}$$

$$I_2(x)=I_2(0)\exp(-2\gamma x) \;, \tag{15b}$$

$$\frac{dI_3}{dx}=(-2\gamma)I_3+2\gamma\kappa\int_{-\infty}^{\infty}|\psi|^4 d\tau \;. \tag{15c}$$

Using the ansatz given by Eqs. (3) and (8) yields

$$\frac{d^2a}{dx^2}=\frac{4\alpha^2}{a^3}-\alpha\kappa(x)\frac{E_0\sqrt{2}}{a^2} \;. \tag{16}$$

Equation (16) is identical to the corresponding approximate solution Eq. (14) for the damped NLS equation obtained using the variational approach above.

Equations (6), (7), and (16) correctly reproduce the characteristic qualitative features of nonlinear pulse propagation, including the nonlinear oscillatory behavior of the pulse width in strongly nonlinear situations, cf. Refs. 1 and 2. On the other hand, an ansatz of the form given by Eq. (3), irrespective of whether it is used in connection with a variational or an invariant approach, is not flexible enough to describe *quantitatively* all the characteristic features of the nonlinear pulse evolution, e.g., the damped oscillatory behavior of the pulse amplitude and pulse width in the case of nonsoliton initial conditions. In such situations the solution consists of a soliton part and a dispersive decaying radiative part. By combining the advantages of the variational and the invariant approaches, we have recently been able to determine the parameters of the emerging soliton as well as the asymptotic properties of the dispersive tail.[8]

A comparison between the variational and invariant methods reveals the following features: Both are based on an ansatz function in terms of four parameter functions. In the variational approach the variational equations give rise to the proper relations between the parameters, including the constant of motion $a(x)|A(x)|^2=\text{const}$. In the invariant approach the first invariant gives the information corresponding to this constant of motion. However, the dynamic relation between the chirp parameter $b(x)$ and the pulse width $a(x)$ has to be introduced *a priori* and does not follow naturally as in the variational approach. In addition, the variational approach consistently determines the variation of the phase of the amplitude $A(x)$, an information which is lost in the invariant approach.

The variational method can be modified to apply to related pulse propagation problems, e.g., mutual pulse interactions.[9] It can also be extended to higher accuracy by using a more flexible ansatz in terms of additional parameter functions, although this usually means that the corresponding system of variational equations has to be investigated numerically.

It is not clear how to extend the invariant method to higher accuracy or to provide further dynamic relations between the parameters. For example, using the ansatz in the fifth invariant only gives rise to an algebraic relation between $b(x)$ and the other parameters and does not lead to a dynamic consistent relation between $b(x)$ and $a(x)$ as in Eq. (8).

Thus we conclude that the variational and invariant methods give the same characteristic equation for the variation of the pulse width. The variational approach is algebraically more complicated than the invariant approach but is also more flexible and in particular consistently determines the relation between the parameter functions where the invariant approach has to introduce independent *a priori* information.

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