Brief Reports

Brief Reports are short papers which report on completed research which, while meeting the usual **Physical Review** *standards of scientific quality, does not warrant a regular article. (Addenda to papers previously published in the* **Physical Review** *by the same authors are included in Brief Reports.) A Brief Report may be no longer than $3\frac{1}{2}$ printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Berry phases for partial cycles

Thomas F. Jordan
Physics Department, University of Minnesota, Duluth, Minnesota 55812
(Received 7 March 1988)

A simple definition of Berry phases based on the Pancharatnam phase convention is stated in the language of ordinary quantum mechanics. It is shown to be equivalent to previous definitions for a cycle of transformations built continuously from infinitesimal transformations. It also applies to discrete transformations and partial cycles. The phase for a partial cycle can be the same as for the complete cycle. This suggests possibilities for simpler experiments. In an example considered here, each part of a split beam of light is put through two half-wave plates. The light is not brought back to the original polarization state, but interference produced by the Berry phase can be observed because the different parts of the beam are brought to the same polarization state.

A new manifestation of Berry phases has been found[1] in Pancharatnam's work[2] with polarized light. Transformations of photon polarization states give the same Berry phases as rotations generated by Pauli matrices.[3] This has led to new measurements of these phases.[4] There is clear physical motivation for the phase convention used by Pancharatnam for polarization states. It provides the basis for a new definition of Berry phases that extends to a more general setting. This has been elaborated in the language of differential geometry.[5]

Here the language of ordinary quantum mechanics is used to give a simple definition of Berry phases based on the Pancharatnam phase convention. It is shown to be equivalent to previous definitions[6,7] for a cycle of transformations built continuously from infinitesimal transformations. It also applies to discrete transformations and partial cycles. The phase for a partial cycle can be the same as for the complete cycle. This suggests possibilities for simpler experiments. In an example considered here, each part of a split beam of light is put through two half-wave plates. The light is not brought back to the original polarization state, but interference produced by the Berry phase can be observed because the different parts of the beam are brought to the same final polarization state.

Consider a state vector $|m\rangle$ that is transformed by a unitary operator U. Suppose $\langle m|U|m\rangle$ is not zero. Let

$$\langle m|U|m\rangle = e^{i\gamma_m(U)}|\langle m|U|m\rangle| \ . \tag{1}$$

Following Pancharatnam[2] and Berry,[3] we call $\gamma_m(U)$ the phase of $U|m\rangle$ relative to $|m\rangle$. We say $U|m\rangle$ and $|m\rangle$ have the same phase if $\gamma_m(U)$ is zero. To see the motivation for this, imagine a superposition of $|m\rangle$ and $U|m\rangle$ and consider

$$\|\,|m\rangle + U|m\rangle\,\|^2 = 2+2|\langle m|U|m\rangle|\cos[\gamma_m(U)] \ . \tag{2}$$

The interference term is proportional to $\cos[\gamma_m(U)]$ and is maximum when $\gamma_m(U)$ is zero.

The phase can remain unchanged. The rotation operators for spin $\frac{1}{2}$ provide an important example. Let $\mathbf{S}$ be the Pauli spin operators. Consider the unitary operator

$$S = e^{-i\boldsymbol{\theta}\cdot\mathbf{S}} = \cos(\tfrac{1}{2}\theta) - i\sin(\tfrac{1}{2}\theta)(\hat{\boldsymbol{\theta}}\cdot 2\mathbf{S}) \tag{3}$$

for a rotation through the angle θ around the axis in the direction $\hat{\boldsymbol{\theta}}$. Let $|m\rangle$ be an eigenvector of $\hat{\mathbf{k}}\cdot\mathbf{S}$ with $\hat{\mathbf{k}}$ perpendicular to $\hat{\boldsymbol{\theta}}$. Then $\langle m|\hat{\boldsymbol{\theta}}\cdot\mathbf{S}|m\rangle$ is zero, so $\langle m|S|m\rangle$ is just $\cos(\frac{1}{2}\theta)$, and $\gamma_m(S)$ is zero.

A Berry phase is a change of phase produced by a sequence of transformations in which each transformation by itself does not change the phase. Let $S_1, S_2, \ldots, S_Q$ be unitary operators, let $U(0)$ be 1,

$$U(q) = S_q \cdots S_2 S_1 \tag{4}$$

for $q = 1, 2, \ldots, Q$, let U be $U(Q)$ and $|m\rangle$ a state vector. Suppose

$$\langle m|U^{-1}(q)S_{q+1}U(q)|m\rangle \tag{5}$$

for $q = 0, 1, 2, \ldots, Q$ are all real and positive. Then we can call $\gamma_m(U)$ a Berry phase.

This is the same as previous definitions for transformations that are built continuously as products of infinitesimal transformations. Suppose

$$U(q+\Delta q)=e^{-iG(q)\Delta q}U(q)\ , \tag{6}$$

with the $G(q)$ Hermitian operators. Then the matrix elements (5) are

$$\begin{aligned}\langle m \mid U^{-1}(q)U(q+\Delta q)\mid m\rangle \\ =\langle m \mid U^{-1}(q)U(q)\mid m\rangle \\ -i\langle m \mid U^{-1}(q)G(q)U(q)\mid m\rangle\Delta q\end{aligned} \tag{7}$$

to first order in Δq. The condition that they are real and positive is

$$\langle m \mid U^{-1}(q)G(q)U(q)\mid m\rangle=0\ . \tag{8}$$

This is the basis of previous definitions of Berry phases[6,7] which are equivalent to the more traditional[8] and geometric[9] definitions.

For example, consider Berry phases for spin.[7-13] As the unit vector $\hat{\mathbf{k}}$ changes from $\hat{\mathbf{k}}(q)$ to $\hat{\mathbf{k}}(q+\Delta q)$, the operator $\hat{\mathbf{k}}\cdot\mathbf{S}$ for the projection of the spin in the direction of $\hat{\mathbf{k}}$ can be changed from $\hat{\mathbf{k}}(q)\cdot\mathbf{S}$ to $\hat{\mathbf{k}}(q+\Delta q)\cdot\mathbf{S}$ by using the unitary operator for a rotation that takes $\hat{\mathbf{k}}(q)$ to $\hat{\mathbf{k}}(q+\Delta q)$. There are many rotations that do this. The Berry phases are obtained[7,12] from the product of the unitary rotation operators if the rotation from $\hat{\mathbf{k}}(q)$ to $\hat{\mathbf{k}}(q+\Delta q)$ is always around the axis perpendicular to $\hat{\mathbf{k}}(q)$ and $\hat{\mathbf{k}}(q+\Delta q)$. This choice of rotations is equivalent to the condition (8) for the rotation operators.[7]

Whenever it can be used, the definition given here is equivalent to other definitions of Berry phases. It cannot be used when $\langle m \mid U \mid m\rangle$ is zero. Other definitions are not so restricted.

Suppose $\mid m\rangle$ is an eigenvector of U. Then

$$U\mid m\rangle=e^{i\gamma_m(U)}\mid m\rangle\ . \tag{9}$$

This is how Berry phases normally appear. The sequence of transformations with $S_1,S_2,\ldots,S_Q$ takes the state represented by $\mid m\rangle$ around a closed loop back to the same state. The state represented by $\mid m\rangle$ is not changed by U. The state vector is changed only by a phase factor. That is the Berry phase.

The definition given here also applies to a partial cycle, a sequence of transformations that does not form a closed loop. The phase for a partial cycle can actually be the same as the phase obtained when the cycle is completed. Suppose the cycle is completed by one more unitary operator S_{Q+1}. That means $\langle m \mid U^{-1}S_{Q+1}U\mid m\rangle$ is real and positive and $\mid m\rangle$ is an eigenvector of $S_{Q+1}U$. The sequence of transformations with $S_1,S_2,\ldots,S_Q,S_{Q+1}$ forms a closed loop. We have

$$S_{Q+1}U\mid m\rangle=e^{-i\Omega_m}\mid m\rangle\ . \tag{10}$$

We write $-\Omega_m$ for the phase $\gamma_m(S_{Q+1}U)$. It is the Berry phase calculated for the closed loop. For example, for rotations it is obtained from the solid angle enclosed by the loop. We get

$$\begin{aligned}\langle m \mid U\mid m\rangle&=e^{-i\Omega_m}\langle m \mid S_{Q+1}^{-1}\mid m\rangle\\ &=e^{-i\Omega_m}\langle m \mid S_{Q+1}\mid m\rangle^*\ .\end{aligned} \tag{11}$$

If $\langle m \mid S_{Q+1}\mid m\rangle$ is not zero, the Berry phase $\gamma_m(U)$ for the partial cycle can be obtained from Ω_m and the phase of $\langle m \mid S_{Q+1}\mid m\rangle$. In particular, if $\langle m \mid S_{Q+1}\mid m\rangle$ is real and positive, $\gamma_m(U)$ is just $-\Omega_m$; the Berry phase for the partial cycle is the same as for the completed cycle. For example, S_{Q+1} could be the rotation operator (3).

Measuring a Berry phase may not require a complete cycle of transformations. Using partial cycles may make experiments simpler.

All this can be illustrated with an example using photon polarization states. Let $\mid +\rangle$ and $\mid -\rangle$ represent the states for photons with right and left circular polarization. Pauli spin operators $\mathbf{S}$ are defined by

$$\begin{aligned}&2S_1\mid\pm\rangle=\mid\mp\rangle,\quad 2S_2\mid\pm\rangle=\pm i\mid\mp\rangle\ ,\\ &2S_3\mid\pm\rangle=\pm\mid\pm\rangle\ .\end{aligned} \tag{12}$$

Let $\hat{\mathbf{k}}$ be the unit vector with spherical coordinates (θ,ϕ). Using the spin operators $\mathbf{S}$ to make a rotation through the angle θ around the axis with spherical coordinates $\pi/2$ and $\phi+\pi/2$, we obtain the state vector

$$\begin{aligned}&e^{-i\theta(-S_1\sin\phi+S_2\cos\phi)}\mid+\rangle\\ &\qquad=\cos(\tfrac{1}{2}\theta)\mid+\rangle+\sin(\tfrac{1}{2}\theta)e^{i\phi}\mid-\rangle\ ,\end{aligned} \tag{13}$$

which is an eigenvector of $\hat{\mathbf{k}}\cdot\mathbf{S}$ for the eigenvalue $\frac{1}{2}$. All the photon polarization states are of this form with θ between 0 and π and ϕ between 0 and 2π. The sphere with coordinates θ,ϕ that label these different polarization states is called the Poincaré sphere. In particular, $\theta=0$ and $\theta=\pi$ are for right and left circular polarization and $\theta=\pi/2$ is for linear polarization. For two orthogonal linear polarization states, the two values of ϕ differ by π.

A change of polarization state can be described by moving the point that represents the state on the Poincaré sphere. Suppose the point is moved by rotating the sphere through the angle δ around the axis with spherical coordinates $\theta=\pi/2$ and $\phi=\beta$. That describes[2] how the polarization state changes when light goes through a birefringent material that puts a phase difference δ between the components in the orthogonal states for linear polarizations represented by $\theta=\pi/2,\phi=\beta$ and $\theta=\pi/2,\phi=\beta+\pi$. The point moving on the Poincaré sphere is the tip of the vector $\hat{\mathbf{k}}$. The state is represented by an eigenvector of $\hat{\mathbf{k}}\cdot\mathbf{S}$. Each step in the change of state can be made with the unitary operator for a rotation that moves $\hat{\mathbf{k}}$. To get the Berry phase we choose the rotation around the axis perpendicular to the moving $\hat{\mathbf{k}}$. The Berry phase is obtained from the product of these rotation operators for the sequence of steps.[7] These rotations are generally not the same as the rotation of the Poincaré sphere; they are around different axes. They are the same when the point representing the state moves along a great circle; then the rotation of the Poincaré sphere is around the axis perpendicular to the moving $\hat{\mathbf{k}}$.

When light goes through a polarizing filter, the point on the Poincaré sphere that represents the polarization state is moved along the great circle connecting the two opposite points that represent the orthogonal states for the linear polarizations that are passed and stopped by

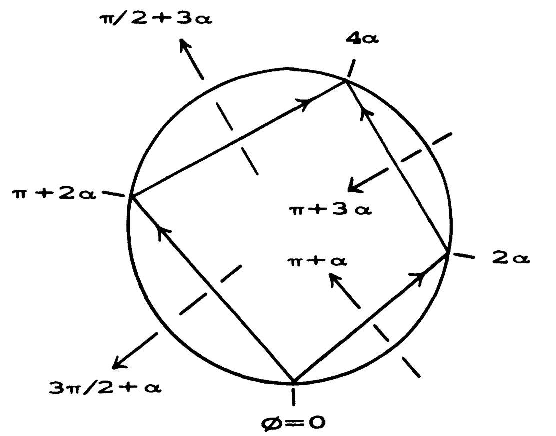

FIG. 1. View of the Poincaré sphere from the $\theta=0$ direction. The point that represents the polarization state for one part of the beam is moved from $\phi=0$ to $\phi=2\alpha$ by a rotation halfway around the axis along $\phi=\pi+\alpha$ and then to $\phi=4\alpha$ by a rotation halfway around the axis along $\phi=\pi+3\alpha$. For the other part of the beam it is moved from $\phi=0$ to $\phi=\pi+2\alpha$ by a rotation halfway around the axis along $\phi=3\pi/2+\alpha$ and then to $\phi=4\alpha$ by a rotation halfway around the axis along $\phi=\pi/2+3\alpha$. The phase difference is half the solid angle enclosed by the paths.

the filter.[2] Then the rotations we use to get the Berry phase are around the axis perpendicular to the plane of the great circle. The state is always represented by an eigenvector of $\hat{\mathbf{k}}\cdot\mathbf{S}$ with $\hat{\mathbf{k}}$ perpendicular to the axis of rotation. The rotation operators, like the operator (3), do not change the phase. If the transformation describing light going though a polarizing filter is used to complete a cycle of transformations,[4] the phase for the partial cycle will be the same as for the completed cycle.

When light goes through a polarizing filter, the physical process is not reversible, but the Berry phase is still obtained from unitary operators. A nonunitary operator could be used to describe the process, and the definition of the Berry phase could be extended naturally[5] to accommodate a nonunitary operator U, but that is not necessary.

A Berry phase could be measured by putting each part of a split beam of light through a partial cycle of transformations made by two half-wave plates. Each time the light goes through a plate, the point on the Poincaré sphere that represents the polarization state is moved by rotating the sphere π radians around an axis with θ coordinate $\pi/2$ and ϕ coordinate determined by the orientation of the birefringent plate. Suppose the light is initially in the state of linear polarization represented by $\theta=\pi/2$ and $\phi=0$. Each plate takes it to another state of linear polarization at $\theta=\pi/2$. Suppose the two birefringent plates for one part of the beam are oriented so the rotations of the Poincaré sphere are around axes at $\phi=\pi+\alpha$ and $\phi=\pi+3\alpha$ with α between 0 and $\pi/4$. The polarization state is changed from $\phi=0$ to $\phi=2\alpha$ by the first plate and then to $\phi=4\alpha$ by the second plate. Suppose the two birefringent plates for the other part of the beam are oriented so the rotations of the Poincaré sphere are around axes at $\phi=3\pi/2+\alpha$ and $\phi=\pi/2+3\alpha$. In that part of the beam the polarization state is changed from $\phi=0$ to $\phi=\pi+2\alpha$ by the first plate and then to $\phi=4\alpha$ by the second plate. (See Fig. 1.)

The final polarization state is the same for both parts of the beam. If both parts are treated exactly the same except for the orientations of the birefringent plates, the phase difference between the two parts will be just the difference between the Berry phases for the two partial cycles made by the transformations between the initial and final polarization states for the two parts of the beam. Both partial cycles can be completed by the transformation that takes the final state back to the initial state along the great circle on the Poincaré sphere. This transformation is made by the unitary operator for a rotation around the axis perpendicular to the plane of the great circle. During the transformation, the state is represented by an eigenvector of an operator $\hat{\mathbf{k}}\cdot\mathbf{S}$ with $\hat{\mathbf{k}}$ perpendicular to the axis of the rotation. The rotation operator, like the operator (3) again, does not change the phase. The phase of each partial cycle is the same as the phase of the completed cycle. It is the Berry phase calculated from the solid angle enclosed by the completed cycle on the Poincaré sphere. The phase difference between the two parts of the beam is the Berry phase calculated from the combination of the two solid angles, that is the solid angle Ω enclosed by the loop on the Poincaré sphere formed by the two partial cycles, one forward and one backwards, which is

$$\Omega=2\pi(\cos\alpha+\sin\alpha-1)\ . \tag{14}$$

Since the spin eigenvalue is $\frac{1}{2}$, the phase difference is $\Omega/2$.

[1]S. Ramaseshan and R. Nityanada, Current Sci. **55**, 1225 (1986).
[2]S. Pancharatnam, Proc. Ind. Acad. Sci. A **44**, 247 (1956), reprinted in *Collected Works of S. Pancharatnam* (Oxford University Press, New York, 1975).
[3]M. V. Berry (unpublished).
[4]R. Bhandari and J. Samuel, Phys. Rev. Lett. **60**, 1211 (1988).
[5]J. Samuel and R. Bhandari (unpublished).
[6]J. Anandan and L. Stodolsky, Phys. Rev. D **35**, 2597 (1987).
[7]T. F. Jordan, J. Math. Phys. (to be published).
[8]M. V. Berry, Proc. Roy. Soc. London, Ser. A **392**, 45 (1984); Nature **326**, 277 (1987).
[9]B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).
[10]R. Y. Chiao and Y. S. Wu, Phys. Rev. Lett. **57**, 933 (1986).
[11]I. Bialynicki-Birula and Z. Bialynicki-Birula, Phys. Rev. D **35**, 2383 (1987).
[12]T. F. Jordan, J. Math. Phys. **28**, 1759 (1987).
[13]R. Y. Chiao, A. Antaramian, K. M. Ganga, H. Jiao, and H. Nathel, Phys. Rev. Lett. **60**, 1214 (1988); T. F. Jordan, Phys. Rev. Lett. **60**, 1584 (1988).