Simultaneous diffusion and reaction processes in plasma dynamics

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A particular solution, which describes the shape-preserving evolution of an explosive localized state (ELS) in one, two, and three spatial dimensions, is obtained for the reaction-diffusion equation with a quadratic creative reaction term. This state, the ELS, is discovered to be of fundamental significance, since states which are "nearby" an ELS are attracted to it. A new technique of analysis is introduced, which enables one to describe the dynamical evolution of the amplitude and width of density profiles. These variables are found to be governed by two coupled nonlinear differential equations. The dynamic theory covers, in a unified form, the cases of one, two, and three spatial dimensions.

I. INTRODUCTION

Reaction-diffusion equations have recently attracted considerable attention, partly due to their occurrence in many fields of science, in physics as well as in chemistry or biology, partly due to the interesting features and rich variety of properties of their solutions.[1,2] The processes of diffusion and reaction each play essential roles in the dynamics of many systems, e.g., in plasma, or semiconductor physics. The simultaneous occurrence of these processes may lead to solutions which can only exist if both kinds of processes do contribute to the dynamics.[3,4] Exact solutions are rare in this connection.

Particularly interesting phenomena occur when the tendency of the nonlinear reaction term is such as to create large solutions. Such situations may lead to explosive instabilities such as have recently been studied in connection with fusion energy research on mirror machines,[5] where a certain mode, a so-called density-independent flute mode becomes explosively unstable in the presence of ion-cyclotron oscillations. Whereas the presence of such nonlinear reaction terms of the creation type causes the amplitude to grow drastically, the contrary is true for a steep gradient diffusive process. As a result threshold phenomena may occur and interesting situations develop where the balance between the different processes becomes crucial.

Even if situations of the type described above may be well suited for computer studies, analytical results are fundamental and useful for making comparisons with the results of computer simulations, in order to provide check points and to offer means of interpretation of computer experiments and of real experiments in the laboratory. The analytic results are also of principle interest in connection with the fundamental question of finding more general solutions to the reaction-diffusion equations.[3,4,6] This last issue is indeed a challenging one, as has been demonstrated by previous examples, such as solitons,[7] or shock-wave solutions of the so-called Burger's equation.[8] The reaction-diffusion equations also offer a variety of new questions of fundamental nature.[2,6]

The analytic solution as well as the dynamic equations, which we here study, are for certain given dependences of the diffusion coefficients on density. For various practical situations, several different choices are motivated. Often the diffusion coefficient is taken as constant, for simplicity or proportional to the density. The only reason we have here made such choices is that those are the only ones for which explicit exact solutions have been obtained for the forms of the equations we consider.[3] In addition, however, computer simulation studies for other choices have been made to confirm a more general principal validity of the analytic results obtained.[9]

II. REACTION-DIFFUSION EQUATIONS WITH AN ANNIHILATION TYPE OF NONLINEAR REACTION TERM

In plasma physics, reaction-diffusion equations, where the reaction term describes annihilation of particles, are frequently used. In cold plasmas electron-ion recombination plays an essential role. Applications are found in various laboratory experiments as well as in space research. In hot burning plasmas, fusion reactions are expected to be responsible for the energy production of future thermonuclear devices.

As concerns the theoretical description of such plasmas, a quadratic nonlinear loss term, accounting for the reaction processes, is a typical ingredient in the equations describing the dynamics of the system. In addition, diffusion terms are generally to be considered. Simultaneous diffusion and reaction processes introduce new interesting solutions.[3,4] The purpose of this section is to elucidate the properties and the role of such solutions, which are of a particular kind but of physical significance.

A. Basic equations and exact particular solutions in one, two, and three dimensions

Consider the equation

$$\frac{\partial n}{\partial t}=D\nabla^2 n-\alpha n^2 , \tag{1}$$

where D and α represent diffusion and reaction coefficients, and n refers to, for example, an electron-plasma density. If we introduce the convenient normalization of space and time coordinates, namely,

$$(\alpha/D)^{1/2}x\to x \quad \text{and} \quad \alpha t\to t , \tag{2}$$

Eq. (1) becomes

$$\frac{\partial n}{\partial t}=\nabla^2 n-n^2 . \tag{3}$$

For symmetric configurations the diffusion operator is

$$\nabla^2=\frac{1}{x^\gamma}\frac{\partial}{\partial x}\left[x^\gamma\frac{\partial}{\partial x}\right] , \tag{4}$$

where $\gamma=0$, 1, and 2, corresponding to the dimensions d ($d=\gamma+1$). The simplest form of similarity-type solution can be written

$$n(x,t)=t^\mu\phi(\xi) , \tag{5}$$

$$\xi=x/t^\nu , \tag{6}$$

where ξ is the similarity variable and μ and ν are constants to be determined.

From Eqs. (3) and (4) with (5) and (6) we obtain by matching powers of t,

$$\mu=-1, \quad \nu=\tfrac{1}{2} , \tag{7}$$

which corresponds to a similarity solution of the form

$$n_s(x,t)=t^{-1}\phi(x/t^{1/2}) . \tag{8}$$

A class of solutions $n(x,t)=n_s(x,t+t_0)=(t+t_0)^{-1}\phi(x/(t+t_0)^{1/2})$ can also be generated with t_0 arbitrary, due to the time translational invariance of Eq. (1).

The function $\phi(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2\phi}{d\xi^2}+\left[\frac{\gamma}{\xi}+\frac{\xi}{2}\right]\frac{d\phi}{d\xi}+\phi-\phi^2=0 . \tag{9}$$

The Eq. (9) has the following interesting particular solution,[3,4] namely,

$$\phi=\frac{a}{p+\xi^2}+\frac{b}{(p+\xi^2)^2} . \tag{10}$$

There is a stationary solution for $b=0$, $p=0$, and $a=6-2\gamma$, i.e., $\phi=(6-\gamma)/\xi^2$, corresponding to $n(x,t)=(6-2\gamma)/x^2$.

The more interesting solution, where

$$p=2(15+\gamma+5\sqrt{6+2\gamma}) , \tag{11}$$

$$a=12(4+\sqrt{6+2\gamma}) , \tag{12}$$

$$b=-24p \tag{13}$$

may, for instance, describe how a plasma density profile with a bell-shaped structure evolves in time and space. The solution can, furthermore, be used to construct solutions of coupled reaction-diffusion rate equations.[3] Such equations are encountered in the description of media in which reactions between species of different kinds occur, for example a burning fusion plasma, where deuterium and tritium nuclei undergo reactions as well as diffusion.

B. Evolution of narrow local perturbations of the particular solution for the plasma density

The purpose of this section is to study the evolution of a narrow perturbation which we may represent initially by the simple expression

$$\Delta n(x,0)=\varepsilon n_0\cos[(x-x_1)/L_1] , \tag{14}$$

where $\varepsilon\ll 1$, $|x-x_1|/L_1\ll\pi/2$ and where we consider L_1 as constant, much less than the width of the main profile. The perturbed density may then be described by

$$n_p(x,t)=n(x,t)+\Delta n(x,t) , \tag{15}$$

where $n(x,t)$ denotes the unperturbed plasma density and $\Delta n(x,t)$ the perturbation at a certain time.

Introducing expression (15) into Eq. (3) with the diffusion operator (4) and neglecting $(\Delta n)^2$ terms we obtain

$$\frac{\partial(\Delta n)}{\partial t}=\frac{\partial^2(\Delta n)}{\partial x^2}+\frac{\gamma}{x}\frac{\partial(\Delta n)}{\partial x}-2(\Delta n)n . \tag{16}$$

For $x\approx x_1$, we have $[\partial(\Delta n)/\partial x]_{x=x_1}\approx 0$, if $x_1\neq 0$, whereas if the perturbation occurs in the center, i.e., if $x_1=0$, we have $[\gamma x^{-1}\partial(\Delta n)/\partial x]_{x=x_1}\simeq-\gamma L^{-2}\varepsilon n_0$. We may, accordingly, use the notation

$$[\gamma x^{-1}\partial(\Delta n)/\partial x]_{x=x_1}=-\gamma L_1^{-2}\varepsilon n_0\delta_{0x_1} ,$$

where δ_{0x_1} denotes a Kronecker symbol, $\delta_{0x_1}=1$ if $x_1=0$, but $\delta_{0x_1}=0$ if $x_1\neq 0$. We find

$$\frac{\Delta n(x,t)}{\Delta n(x,0)}=\exp\left[-L_1^{-2}(1+\gamma\delta_{0x_1})t-2\int_0^t n(x,t')dt'\right] , \tag{17}$$

with

$$n(x,t)=\frac{1}{t+t_0}\left[\frac{a}{p+\dfrac{x^2}{t+t_0}}+\frac{b}{\left[p+\dfrac{x^2}{t+t_0}\right]^2}\right] , \tag{18}$$

where the constants a, b, and p are given by expressions (11)–(13).

In expression (17) we find after integration

$$\int_0^t n\,dt'=\frac{1}{p}\left[a+\frac{b}{p}\right]\ln\left[\frac{p(t+t_0)+x^2}{pt_0+x^2}\right]-\frac{b}{p}\frac{tx^2}{(pt_0+x^2)[p(t+t_0)+x^2]} . \tag{19}$$

For $x_1=0$ we obtain

$$\frac{\Delta n(0,t)}{\Delta n(0,0)}=\left[\frac{t}{t_0}+1\right]^{-q}\exp\left[-\frac{\gamma+1}{L_1^2}t\right], \tag{20}$$

where from (19) and (11)–(13)

$$q=\frac{2}{p}\left[a+\frac{b}{p}\right]=\frac{12(2+\sqrt{6+2\gamma})}{15+\gamma+5\sqrt{6+2\gamma}},$$

or $q=1.96$ for $\gamma=0$, (1D); $q=1.92$ for $\gamma=1$, (2D); and $q=1.89$ for $\gamma=2$, (3D), where (1D), (2D), and (3D) denote one, two, and three dimensions, respectively.

For the unperturbed solution we have from expression (18),

$$\frac{n(0,t)}{n(0,0)}=\left[\frac{t}{t_0}+1\right]^{-1},\quad t_0=\frac{q}{2}[n(0,0)]^{-1}. \tag{21}$$

The conclusion from expressions (20) and (21) is that the narrow perturbation always decays faster than the original unperturbed plasma density, and more so the smaller the width L_1 is. The decay of the perturbation is slightly faster, for the same width L_1, the higher the dimension $d=\gamma+1$. The validity of the results obtained is limited by the fact that we regard L_1 as a constant, which is, however, a good approximation for an early stage of the development of the perturbation.

C. Evolution of the plasma density for large scale deviations from the form of the particular solutions

For given values of the parameters a, b, p, and t_0 the exact particular solution (18) expresses the evolution in space and time (for 1D, 2D, and 3D) of a certain initial plasma profile $n(0,t)$. In the solution expressed by relation (18) the parameter p is a measure of the width of the profile.

It is interesting to consider a more general form of solution, namely,

$$n(x,t)=A(t)\left[\frac{a}{P(t)+\frac{x^2}{t+t_0}}+\frac{bP(t)/p}{\left[P(t)+\frac{x^2}{t+t_0}\right]^2}\right], \tag{22}$$

where A and P depend on time, whereas a, b, and p are the constant parameters defined by expressions (11)–(13). For $x=0$ the relation (22) takes the form

$$n(0,t)=\frac{A}{p}\left[a+\frac{b}{p}\right], \tag{23}$$

which has the same relative contributions from the two main terms in the bracket of (22) as has, for $x=0$, the relation (18).

Let us introduce the expression (22) into the original equation (3) and consider the development in time of A and P in the central region, i.e., for x near zero. Expanding each term of the equation in x^2, we obtain two coupled equations, one from the constant terms and the other from the x^2 terms, in terms if A and P, namely,

$$P\frac{\partial A}{\partial t}-A\frac{1}{t+t_0}\left[(t+t_0)\frac{\partial P}{\partial t}-\Gamma\right]+A^2K=0, \tag{24}$$

$$P\frac{\partial A}{\partial t}-A\frac{1}{t+t_0}\left[P+2(t+t_0)\frac{\partial P}{\partial t}-M\right]+2A^2K=0, \tag{25}$$

where

$$\Gamma=2(1+\gamma)\frac{a+2b/p}{a+b/p}=2(3+\gamma-\sqrt{6+2\gamma}),$$

$$M=4(3+\gamma)\frac{a+3b/p}{a+2b/p}=2\Gamma,$$

and $K=12(2+\sqrt{6+2\gamma})$. It follows from (24) and (25) that

$$A=\frac{1}{t+t_0},\quad P=p+\frac{\Delta}{t+t_0}, \tag{26}$$

where Δ is an arbitrary constant. It should be stressed that the solution (26) has been obtained by considering only the situation in the central region, near $x=0$.

We could make a more general assumption about the form of the solution than in (22), namely,

$$n(x,t)=\frac{A_1(t)}{P_1(t)+\frac{x^2}{t+t_0}}+\frac{B_1(t)}{\left[P_1(t)+\frac{x^2}{t+t_0}\right]^2}. \tag{27}$$

If we then try to determine A_1, B_1, and P_1 by inserting (27) into Eq. (3) and equate successive powers of the denominators, without any expansion, the results come out identical to (22) with (26), i.e., $A_1=aA$, $B_1=AbP/p$, $P=P_1$, which therefore gives the evolution of the density in space and time exactly over the whole space domain in one, two, and three dimensions.

It is instructive for the forthcoming analysis in the following sections to have been able, here, to confirm the results of an expansion in x^2 by comparison with an exact solution. Such solutions are in general not available. It is, furthermore, instructive to notice that the solution can be expressed in the following simple form, namely,

$$n(x,t)=\frac{1}{(t+t_0)p+\Delta+x^2}\left[a+\frac{[(t+t_0)p+\Delta]b/p}{(t+t_0)p+\Delta+x^2}\right], \tag{28}$$

from which we conclude that for different values of Δ the solution corresponds to a manifold, all of which approach, in a conformal way the profile for which $\Delta=0$. In the process of evolution this occurs from above, if $\Delta<0$; from below, if $\Delta>0$.

As a result of time translational invariance all profiles in the set are equivalent and identical for the same value of $(t+t_0)p+\Delta$. A shift in Δ is equivalent to a change in $(t+t_0)p$. No crossings in space of neighboring profiles occur with time for this set of solutions. For the specific given initial shapes of the plasma profiles (including any given value of Δ) the simultaneous effects of diffusion and annihilating reaction processes are therefore to cause a decay of the particle density in every point in space as described by relation (28). These results are intended to shed some light on the properties and the role of the exact particular solutions.

III. EXPLOSIVE INSTABILITIES OF REACTION-DIFFUSION EQUATIONS

It is the purpose of this section to consider reaction-diffusion equations for situations which are explosively unstable, i.e., where instabilities tend to grow to infinite amplitudes in a limited period of time.

A. Basic equations and assumptions

Consider the equation

$$\frac{\partial n}{\partial t}=\frac{1}{x^{\gamma}}\frac{\partial}{\partial x}\left[x^{\gamma}D\frac{\partial n}{\partial x}\right]-bn+cn^2 , \tag{29}$$

where $\gamma=0$, 1, 2, corresponding to the dimension d ($d=\gamma+1$); n is assumed to be radially symmetric for the cylindrical and spherical cases ($\gamma=1,2$) and to describe a population density; b and c are constant coefficients with $c>0$ for explosive cases; and D is the diffusion coefficient, which we assume to be of the form $D=an$, where a is a constant coefficient.

It is convenient to introduce new variables of space and time, accordingly

$$(c/a)^{1/2}x\to x \quad \text{and} \quad ct\to t .$$

The linear term $-bn$ can easily be handled by introducing the transformation

$$N=n\exp(bt),\quad \tau=b^{-1}[1-\exp(-bt)] , \tag{30}$$

where N and τ satisfy an equation which is formally identical to Eq. (29) with $b=0$. Thus, the solution to Eq. (29) with $b=0$ can directly be extended to include the effects of linear dissipation (or growth). In the following it therefore suffices that we treat the case $b=0$.

The remaining equation can be written as

$$\frac{\partial n}{\partial t}=\frac{1}{x^{\gamma}}\frac{\partial}{\partial x}\left[x^{\gamma}n\frac{\partial n}{\partial x}\right]+n^2 \tag{31}$$

or

$$\frac{\partial n}{\partial t}=\frac{1}{2}\frac{\partial^2 n^2}{\partial x^2}+\frac{1}{2}\frac{\gamma}{x}\frac{\partial n^2}{\partial x}+n^2 . \tag{32}$$

B. Equilibria and explosive localized solutions (ELS) in one, two, and three dimensions

For $[\partial n(x,t)]/(\partial t)\equiv 0$ (all x), the remaining equation for $n=n_e(x,t)$ is

$$\frac{d^2n_e^2}{dx^2}+\frac{\gamma}{x}\frac{dn_e^2}{dx}+2n_e^2=0 \quad (\gamma=0,1,2) , \tag{33}$$

which is *linear* in n_e^2.

We obtain the following expressions for the equilibria ($|x|<|x_0|$, where x_0 is the first zero from the center), namely,

$$n_e=n_{e0}[\cos(\sqrt{2}x)]^{1/2} \quad (|x|\le 1.11) \tag{34}$$

for 1D,

$$n_e=n_{e0}[J_0(\sqrt{2}x)]^{1/2} \quad (|x|\le 1.70) \tag{35}$$

for 2D,

$$n_e=n_{e0}[j_0(\sqrt{2}x)]^{1/2} =n_{e0}\frac{(2)^{1/4}}{2}\frac{\sqrt{\pi}}{\sqrt{x}}[J_{1/2}(\sqrt{2}x)]^{1/2} \quad (|x|\le 2.22) \tag{36}$$

for 3D, or if the real space coordinates are reintroduced

$$n_e=n_{e0}\left\{\cos\left[\left[\frac{2c}{a}\right]^{1/2}x\right]\right\}^{1/2} \quad (|x|\le 1.1\sqrt{a/c}) \tag{37}$$

for 1D,

$$n_e=n_{e0}\left\{J_0\left[\left[\frac{2c}{a}\right]^{1/2}r\right]\right\}^{1/2} \quad (|r|\le 1.70\sqrt{a/c}) \tag{38}$$

for 2D,

$$n_e=n_{e0}\sqrt{\pi}/2\left[\frac{2a}{c}\right]^{1/4}\frac{1}{\sqrt{r}}\left\{J_{1/2}\left[\left[\frac{2c}{a}\right]^{1/2}r\right]\right\}^{1/2} \quad (|r|)\le 2.22\sqrt{a/c} \tag{39}$$

for 3D. We notice from the exact solutions, (37)–(39), that the equilibria become narrower as the coefficient c, representing the nonlinearity becomes larger, or when the diffusion, represented by a becomes smaller.

For explosive-type solutions we take

$$n(x,t)=(t_0-t)^{-1}\phi(x) . \tag{40}$$

We then obtain the following ordinary differential equation for $\phi(x)$:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\gamma}{x}\frac{\partial}{\partial x}\right]\phi^2=2(\phi-\phi^2) , \tag{41}$$

which for $\gamma=0$, i.e., for the 1D case, has a particular solution

$$\phi=\tfrac{2}{3}[1+\cos(x/\sqrt{2})] , \tag{42}$$

or

$$\phi=\tfrac{4}{3}\cos^2[x/(2\sqrt{2})] , \tag{43}$$

corresponding to

$$n=\frac{n_0}{1-\frac{3}{4}cn_0t}\cos^2\left[x\left[\frac{c}{8a}\right]^{1/2}\right] . \tag{44}$$

We notice the interesting fact that this simple exact solution describes an explosive growth, which develops in time with a preservation in shape of a certain spatial distribution.

For the corresponding 2D and 3D situations, the following solutions can be constructed, namely:

$$\phi=\frac{4+\gamma}{3}\cos^2(x/L),\quad L=[2(4+\gamma)]^{1/2}\,, \tag{45}$$

which for $\gamma=0$ is identical to the exact solution (43), but for $\gamma=1,2$ are approximative such that $\partial\phi/\partial x=0$ for $x=0$ and $x=(\pi/2)L$ satisfying to lowest order Eq. (41) in the neighborhood of these points and bridging the intermediate domain in an approximate way.

It is possible to improve the solutions in the 2D and 3D cases. For our forthcoming purpose of studying the dynamics of the central region it suffices to use the expression

$$\phi=\frac{4+\gamma}{3}\cos^2\left[x/L-\frac{1}{12}\frac{\gamma}{3+\gamma}(x/L)^3\right],$$

$$L=[2(4+\gamma)]^{1/2} \tag{46}$$

which satisfies Eq. (41) to order x^2 in the central region, $x\approx 0$, for $\gamma=1,2$ and which coincides with the exact solution for $\gamma=0$.

C. Evolution of narrow deformations of the initial shape of ELS

Consider a perturbation of the following initial form:

$$\Delta n(x,0)=\varepsilon n_0\cos[(x-x_1)/L_1]\,, \tag{47}$$

where $\varepsilon\ll 1$, $L_1\ll L=[2(4+\gamma)]^{1/2}$, and $|x-x_1|/L\ll\pi/2$. Assume that the perturbed state can be described by

$$n_p(x,t)=n(x,t)+\Delta n(x,t)\,, \tag{48}$$

where $n(x,t)$ denotes the unperturbed state. Introducing the expression (48) into Eq. (32), neglecting the $(\Delta n)^2$ terms, and considering L_1 as a constant, one obtains for $x\approx x_1$,

$$\frac{\Delta n(x_1,t)}{\Delta n(x_1,0)}=\left[1-\frac{3}{4+\gamma}n_0t\right]^q\,, \tag{49}$$

where

$$q=\left[L_1^{-2}-2-\frac{d^2\phi}{dx^2}\Big/\phi-\frac{\gamma}{x}\frac{d\phi}{dx}\Big/\phi+\frac{\gamma}{L_1}\frac{\sin[(x-x_1)/L_1]}{x_1}\right]_{x=x_1}\phi(x_1) \tag{50}$$

and where the last term in the large parentheses of (50) is zero if $x_1\neq 0$.

Taking for ϕ in expression (50) the form (43) we obtain

$$q=\left[L_1^{-2}(1+\gamma)-2+\frac{1+\gamma}{4+\gamma}\right]\frac{4+\gamma}{3}\,, \tag{51}$$

whereas if we consider the region of maximum gradient of ϕ, where $x_1=(\pi/4)[2(4+\gamma)]^{1/2}$, we have

$$q=\left[L_1^{-2}-2+\frac{4}{\pi}\frac{\gamma}{4+\gamma}\right]\frac{4+\gamma}{6}\,. \tag{52}$$

We notice that the sign of q determines whether or not the perturbation will grow in time. For $q>0$ the perturbation will vanish as $t\to[(4+\gamma)/3]n_0$, whereas for $q<0$ the tendency is that the perturbation will grow on a time scale that characterizes the explosive growth of the main profile.

It follows that a narrow perturbation will develop according to the size of its width L_1. If L_1 is smaller than a critical value L_{1c} the perturbation will vanish in the course of time, whereas if $L_1>L_{1c}$ the perturbation will grow.

In the center of the main profile the critical limit of L_1 is

$$L_{ic}=[2(1+\gamma)^{-1}-(4+\gamma)^{-1}]^{-1/2}\quad(x_1=0)\,, \tag{53}$$

which should be compared to the width L of the main profile $L=[2(4+\gamma)]^{1/2}$ for different dimensions $(\gamma=0,1,2)$. We then obtain $(x_1=0)$, $L_{1c}/L=0.27$ $(\gamma=0)$, 0.35 $(\gamma=1)$, 0.41 $(\gamma=2)$, in the 1D, 2D, and 3D cases, respectively, which means that the profile is "stable" for narrow perturbations in the center if L_1/L is less than the values given for L_{1c}/L.

In the region of maximum gradient $(x_1=\frac{1}{4}\pi L)$ of the main profile the corresponding critical limits of L_1 become

$$L_{1c}=\left[2-\frac{4}{\pi}\frac{\gamma}{4+\gamma}\right]^{-1/2}\,. \tag{54}$$

The critical widths of the narrow perturbations as related to the width of the main profile are the following: $L_{1c}/L=0.25$ $(\gamma=0)$, 0.24 $(\gamma=1)$, 0.23 $(\gamma=2)$, which means that the profile is stable for narrow perturbations in the region of steep gradient of the main profile if $L_1/L\lesssim\frac{1}{4}$, and that the limit is about the same for all three dimensions.

D. Dynamic confluence with ELS for profiles of various initial widths and amplitudes; the ELS as a natural nonlinear entity

When an initial profile has a width that does not coincide with an ELS it is an interesting problem to consider whether or not the profile will have a tendency to approach that of an ELS. Let us try to describe the evolution in time of a profile, which has an initial shape

$$F(x/L_0)=\cos^2\left[x/L_0-\frac{1}{12}\frac{\gamma}{3+\gamma}(x/L_0)^3\right]\,, \tag{55}$$

where L_0 differs from that of an ELS, i.e., $L_0^2 \neq 2(4+\gamma)$, and to consider the change in time of $L = L(t)$ as well as of an amplitude $A = A(t)$, such that

$$n = AF(x/L) . \tag{56}$$

Introducing the expression (56) into the original equation (32), we obtain

$$\frac{\partial A}{\partial t}F - \frac{x}{L^2}AF'\frac{\partial L}{\partial t} = A^2\left[\frac{1}{L^2}\left[(F')^2 + FF'' + \frac{\gamma}{x/L}FF'\right] + F^2\right] , \tag{57}$$

where the prime indicates derivation with regard to x/L.

Limiting our study to the central region and retaining terms up to second order in x/L the following equations result.

(i) From the constant terms,

$$\frac{\partial A}{\partial t} = A^2\left[1 - \frac{1(1+\gamma)}{L^2}\right] . \tag{58}$$

(ii) From the x^2 terms,

$$\frac{2}{L}\frac{\partial L}{\partial t} = \frac{1}{A}\frac{\partial A}{\partial t} + 2A\left[\frac{1}{L^2}(5+2\gamma) - 1\right] . \tag{59}$$

By combining Eqs. (58) and (59) we can conveniently write the coupled equations in the following forms:

$$\frac{1}{A}\frac{\partial A}{\partial t} = -A[2(1+\gamma)/L^2 - 1] \tag{60}$$

and

$$\frac{1}{L^2}\frac{\partial L^2}{\partial t} = A[2(4+\gamma)/L^2 - 1] . \tag{61}$$

Introducing the amplitude A from Eq. (61) into the right-hand side (rhs) of Eq. (6) and integrating we obtain the following relation between A and L^2, namely,

$$\frac{A}{A_0} = \left[\frac{L_0^2}{L^2}\right]^{(1+\gamma)/(4+\gamma)} \left[\frac{2(4+\gamma) - L_0^2}{2(4+\gamma) - L^2}\right]^{3/(4+\gamma)} , \quad [L^2 \leq 2(4+\gamma)] \tag{62}$$

where A_0 is the initial amplitude, and L_0^2 corresponds to the initial width. From Eq. (61) it furthermore follows with the aid of Eq. (62) that

$$\frac{\partial L^2}{\partial t} = A_0\left[\frac{L_0^2}{L^2}\right]^{(1+\gamma)/(4+\gamma)} [2(4+\gamma) - L_0^2]^{3/(4+\gamma)} \times [2(4+\gamma) - L^2]^{(1+\gamma)/(4+\gamma)} . \tag{63}$$

Therefore $\partial L^2/\partial t \to 0$ when $L^2 \to 2(4+\gamma)$, i.e., when $A \to \infty$.

From Eq. (60) we notice that $\partial A/\partial t \to 0$ when $L^2 \to 2(1+\gamma)$, which corresponds to a *minimum* in A, since from relation (60) we have

$$(\partial^2 A/\partial t^2)_{L^2=2(1+\gamma)} = 2A^2(1+\gamma)(L^{-4}\partial L^2/\partial t)_{L^2=2(1+\gamma)} > 0 .$$

The minimum value of A can be found from expression (62) with $L^2 = 2(1+\gamma)$,

$$\frac{A_{\min}}{A_0} = \left[\frac{L_0^2}{2(1+\gamma)}\right]^{(1+\gamma)/(4+\gamma)} \{(\tfrac{1}{6})[2(4+\gamma) - L_0^2]\}^{3/(4+\gamma)} . \tag{64}$$

From relation (64) we notice that for small L_0^2, i.e., very narrow initial profiles, the ratio $A_{\min}/A_0$ depends on L_0 as $A_{\min}/A_0 = 1.04L_0^{1/2}$; $0.78L_0^{4/5}$; and $0.58L_0$ for 1D, 2D, and 3D ($\gamma = 0,1,2$), situations, respectively.

In order to relate the dynamics of the system to time explicitly, we use Eq. (63) and choose to express time as a function of L^2, accordingly,

$$t = \Lambda(L_0^2)I(L^2) , \tag{65}$$

where

$$I(L^2) = \int_{L_0^2}^{L^2}\left[\frac{L^2}{2(4+\gamma) - L^2}\right]^{(1+\gamma)/(4+\gamma)} dL^2 \tag{66}$$

and

$$\Lambda(L_0^2) = A_0^{-1}(L_0^2)^{-(1+\gamma)/(4+\gamma)} \times [2(4+\gamma) - L_0^2]^{-3/(4+\gamma)} . \tag{67}$$

Introducing $y = L^2$, $y_0 = L_0^2$, $y_\infty = 2(4+\gamma)$, and $\kappa = (1+\gamma)/(4+\gamma)$, the integral I can be expressed as

$$I = \int_{y_0}^{y}\left[\frac{y'}{y_\infty - y'}\right]^{\kappa} dy' , \tag{68}$$

where $\kappa = \frac{1}{4}$, $\frac{2}{5}$, or $\frac{1}{2}$, for 1D, 2D, or 3D ($\gamma = 0,1,2$), respectively, and $y_\infty = 8$, 10, or 12, correspondingly. By means of suitable transformations, e.g., $[y/(y_\infty - y)]^\kappa = s$, it is possible to carry out the integration of (68) for the various values of κ.

For $\gamma = 0$, $\kappa = \frac{1}{4}$, the 1D case, we obtain

$$I=8\left[\frac{s_0}{1+s_0^4}-\frac{s}{1+s^4}\right]+\frac{\sqrt{2}}{8}\left\{\ln\left|\frac{s^2+s\sqrt{2}+1}{s^2-s\sqrt{2}+1}\right|-\ln\left|\frac{s_0^2+s_0\sqrt{2}+1}{s_0^2-s_0\sqrt{2}+1}\right|+2\left[\arctan\left[\frac{s\sqrt{2}}{1-s^2}\right]-\arctan\left[\frac{s_0\sqrt{2}}{1-s_0^2}\right]+\varphi\right]\right\}, \quad (69)$$

where $\varphi=\pi$ if $s_0<1<s^2$, but $\varphi=0$ otherwise.

For $\gamma=1$, $\kappa=\frac{2}{5}$, the 2D case, it is convenient to introduce the transformation $[y(y_\infty-y)]^{1/5}=u$. In this case we then obtain

$$I=y_\infty\left[\frac{u_0^2}{1+u_0^5}-\frac{u^2}{1+u^5}+2\int_{u_0}^{u}du'\frac{u'}{1+(u')^5}\right]. \quad (70)$$

We notice that for small u_0 and $y=y_\infty$, i.e., $u\to\infty$, we have

$$\int_0^\infty du'\frac{u'}{1+(u')^5}=\frac{\pi}{5\sin(2\pi/5)}=0.65 ,$$

and that for $u<1$ we can expand the integrand in the integral of (70) to determine $I_{y=y_{A_{\min}}}=2.18$, whereas $I_{y=y_\infty}=13.0$.

For $\gamma=2$, $\kappa=\frac{1}{2}$, the 3D case, we find the exact expression

$$I=[y_0(12-y_0)]^{1/2}-[y(12-y)]^{1/2}+12\left\{\arctan\left[\left[\frac{y}{12-y}\right]^{1/2}\right]-\arctan\left[\left[\frac{y_0}{12-y_0}\right]^{1/2}\right]\right\}. \quad (71)$$

From the above formulas, (65)–(71), the times corresponding to $A_{\min}$ and A_∞ can be calculated in each separate case and for any dimension ($\gamma=0,1,2$).

If we consider again small values of L_0^2, i.e., very narrow initial profiles, we find for the ratio of the time $t_{A_{\min}}$ for reaching $A_{\min}$, and the time t_∞ for reaching infinite values of A the following figures:

$$\frac{t_{A_{\min}}}{t_\infty}=0.13 \quad (\gamma=0)$$
$$=0.17 \quad (\gamma=1)$$
$$=0.18 \quad (\gamma=2) \quad (72)$$

for the 1D, 2D, and 3D cases, respectively, in full agreement with direct numerical solution of the coupled equations (60) and (61) by computer.

The main features of the evolution of a certain profile, that is, for example, more narrow than an ELS in the initial state, can thus be understood from purely analytic considerations. Assume that we start with a narrow profile, i.e., $L^2\ll 2(1+\gamma)$ of high amplitude, i.e., $A_0\gg A_{\min}$. According to relations (60) and (61) A will start to decrease with time until it reaches a minimum $A=A_{\min}$ for $L^2=2(1+\gamma)$, then progress and increase to infinity for $L^2=2(4+\gamma)$, whereas in the same period of time L^2 will increase smoothly and reach a maximum where $L^2=2(4+\gamma)$. It follows from relation (60) with $L^2=2(4+\gamma)$ that $\partial A/\partial t=[3/(4+\gamma)]A^2$, i.e., the profile, the dynamics of which we consider, will approach the growth rate as well as the width of the ELS and become asymptotically confluent with it at a finite time.

It is important to emphasize, however, that a minimum reached for the amplitude in the central region does not correspond to a situation where local amplitude minima occur simultaneously for all points in space. The equilibria described by expressions (34)–(39) will thus *not* be reached in the dynamic process we are here considering. It therefore seems that the ELS, here introduced, plays a more fundamental role than the time-independent equilibria. ELS may, in fact, be considered, at least as regards small-scale perturbations or dynamic deviations, as dynamic equilibria with pronounced properties of stability.

So far we have emphasized the behavior of narrow deviations from ELS or the evolution of narrow initial profiles. It is easy to argue in an analogous way about the dynamic evolution of a profile which is initially broader than ELS, i.e., for which $L_0^2>2(4+\gamma)$. For such a profile the width will shrink in the process of evolution, in accordance with relation (61) until it reaches the "dynamic equilibrium" where $L^2=2(4+\gamma)$.

It thus seems that the dynamic evolution of a profile which differs from ELS initially, and which is governed in its evolution by Eq. (31) will approach ELS asymptotically. Computer experiments support this view and indicate that in the course of dynamic evolution the profile may develop from an initial form and finally adjust to that of ELS. It therefore seems justified to regard ELS as a natural entity of a nonlinear dynamic system governed by the reaction-diffusion rate equation (31).

IV. CONCLUSIONS

It is demonstrated analytically that the plasma dynamic evolution of density profiles of various initial widths and amplitudes will approach asymptotically to characteristic profiles of specific shapes, widths, and amplitudes, as determined by particular solutions of reaction-diffusion equations in one, two, and three dimensions. The dynamics is assumed to be governed by simultaneous processes of diffusion and reactions, represented in the models by quadratic nonlinearities. The profiles may represent distributions in configuration space of particles, e.g., electrons and ions, or plasmons as well as species of other populations. Particular annihilation types as well as creation types of reactions, which may lead to certain explosive localized structures, are considered. The results, which have been confirmed by computer simulations,[9] are of physical significance for practical applications concerning, e.g., hot tritium-deuterium burning fusion plasmas or free-energy plasma systems of, e.g., beam-plasma interaction type. Other applications regarding the case of creative nonlinearity are found, e.g., in studies on ionized media, where ionization of ions by hot electrons produces new electrons.

The basic equations (1) and (29) do not have the Painlevé property[4] (absence of movable singularities other than simple poles) for the corresponding reduced ordinary differential equations. It would, accordingly, seem as if the possibilities of finding explicit solutions for these equations were limited. The inverse scattering transform technique (IST), extensively used for solitons would not be applicable either.[4] Nonetheless, exact particular solutions are found for the reaction-diffusion equation, representing states which are of physical significance.[3,4] The particular solution discovered in this investigation for the reaction-diffusion equation with creative reaction term, for one, two, and three spatial dimensions, represents an explosive localized state, ELS, the amplitude of which grows explosively in time while the shape of the ELS is being preserved in the process of evolution. The ELS, (43)–(46), may accordingly be extended to represent periodic structures, *polytons,* in one, two, and three dimensions, by repetition in space of a single localized ELS, a *singleton.*

The ELS has been discovered to be of fundamental significance in that other states "nearby" are attracted to it. The ELS has, furthermore, been found to be stable against "narrow" perturbations (the width of the perturbation $\lesssim \frac{1}{4}$ the width of the main profile) in one, two, and three dimensions. The analysis of the reaction-diffusion equation, presented here, and the role of the ELS, discovered here, are important steps forward for understanding the dynamic properties of media described by this equation.

New possibilities are foreseen for further investigations of several new issues such as studies of the interaction between two or several ELS's, influence of initial and boundary conditions, generalized forms, and corresponding generalized solutions of the original equation, including more general diffusion and reaction terms, and extensions to differential-integral reaction-diffusion equations.[10] These questions are being addressed by the author and are due to appear in forthcoming publications. The results of the present work, furthermore, give indications and check points for new computer experiments of principal interest and practical significance for laboratory experiments.

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