Full second-order cold-fluid theory of the diocotron and magnetron resonances

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The full second-order, cold-fluid theory of the diocotron and magnetron resonances for nonrelativistic, non-neutral electron flow in planar, crossed-field geometry is investigated using the method of multiple-scales perturbation theory. We find that, for both resonances, the zeroth-order density evolves in the slow time and spatial plane according to a diffusion equation. The quasilinear treatment of the diocotron instability [R. C. Davidson, Phys. Fluids **28**, 1937 (1985)] is recovered from the full treatment near the wave-particle, or diocotron, resonance. We also conjecture that, under constant-voltage conditions, this diffusion will cause a small dc current to be set up from the anode to the cathode and that this second-order diffusion equation could possess equilibrium solutions as well. Such solutions would have negative density gradients between all unstable resonances (magnetron as well as diocotron), with plateau formation occurring at the spatial position of all these resonances. Although we cannot show stabilization, we can argue that the introduction of negative density gradients, which would occur as the density profile seeks to re-form into an equilibrium profile, should tend to reduce the linear-instability growth rates, thereby causing the system to tend toward stabilization.

I. INTRODUCTION

There have recently been several theoretical studies[1-7] of the equilibrium and stability properties of sheared, non-neutral electron flow in various planar and cylindrical high-voltage diode and crossed-field amplifier configurations of interest in microwave and inertial-confinement-fusion applications. Such configurations are characterized by crossed fields and high densities. These studies have included relativistic, kinetic, electromagnetic, and cylindrical-geometry effects on stability properties at moderately high density. In particular, Davidson and Tsang[1,2] have studied the stability of nonrelativistic cylindrical magnetic diodes, and the influence of density profile shape on stability in planar diodes. Also, Davidson, Tsang, and Swegle[3] have studied the stability properties of relativistic planar diodes; Swegle[4] has studied the stability of a general relativistic laminar flow of electrons in crossed fields, and Chernin and Lau[5] have studied the stability of cylindrical laminar layers of electrons in the presence of crossed fields. Davidson has developed a kinetic stability theorem for the relativistic planar diode,[6] a quasilinear theory of the diocotron instability (a long-wavelength instability which occurs when the phase velocity of the wave matches the velocity of the electrons in the presence of a positive density gradient) in planar geometry,[7] and has also used a macroscopic guiding-center theory to study the stability of the cylindrical magnetic diode.[8] Prasad, Morales, and Fried[9] have studied the modification of electron gyrofrequencies inside a non-neutral plasma. There have also been several other kinetic studies of non-neutral plasmas,[10-16] mainly in homogeneous, low-density regimes. These have recently been extended into both the inhomogeneous and high-density regimes by Kaup, Hansen, and Thomas[17] using a singular perturbation expansion for the equilibrium orbits.

Our interest in such studies is to better understand the operation of such devices as the magnetron, crossed-field amplifiers (CFA's), and other related crossed-field devices. In particular, we wish to study the nonlinear regime in such devices so as to place the current nonlinear theory[18] on a firm mathematical basis by the use of multiple-scale expansions. Of course, before this can be done, it is necessary to have a well-developed linear theory to expand about. The above studies have provided such a linear theory for the cold-fluid equations.

A study based on the Fokker-Planck equation was done in 1966 by Monthaan and Sussind[18] which predicted a dc current in the smooth-bore magnetron. They obtained a diffusion equation describing vertical particle transport in the magnetron, and their results were shown to give good qualitative agreement with experiments.[19] In the first nonlinear multiple-scale treatment of this problem in 1981, Thomas[18] obtained rf-field-dependent coefficients for a nonlinear Schrödinger equation. In 1982, Swegle and Ott[20] obtained the Kortweg–de Vries (KdV) equation for long-wavelength perturbations on a cold-fluid Brillouin flow. Most recently, Davidson's quasilinear theory[7] gave a detailed quasilinear description of the classical diocotron instability, treating the electrons as a massless ($m=0$) guiding-center fluid. The quasilinear diffusion of the time-averaged (or dc) plasma density leads to a smoothing out of the density profile (the so-called spatial "plateau formation") and the stabilization of the instability. Davidson[7] considers general features of the stabilization process in detail.

We are interested in the nonlinear evolution of the zeroth-order or dc parameters of the crossed-field device, such as the electric field, and the electron flow density. Thus we must consider the magnetron instability[3,4,20,21] which occurs for sheared non-neutral flow at higher densities where space-charge effects are stronger; and we cannot treat electrons as Davidson's massless guiding-center fluid.

The quasilinear theory of Davidson is equivalent to the second order of the "method of multiple scales,"[22] a systematic technique for solving nonlinear ordinary differential equations (ODE's) or partial differential equations (PDE's) by a singular perturbation expansion. We use the method of multiple scales in this paper to study fully the second order of the nonlinear regime of crossed-field device operation using the full cold-fluid equations. We assume that the ratio of the plasma frequency to the cyclotron frequency is sufficiently less than unity so that the linear instability is weak; and that the device configuration is planar. We retain the full effect of electron inertia, and consider in detail the effect of diocotron and magnetron resonances in driving the nonlinear evolution. For now, we do not consider electromagnetic, relativistic, or cylindrical-geometry effects. Our equations generalize Davidson's treatment by including the effects of high electron density and electron inertia.

One important difference between Davidson's treatment and ours is that we consider an operating configuration with constant anode voltage rather than the constant-charge operating configuration of Davidson. Because of this, in second order we can obtain a nonzero flux of particles from cathode to anode: a small dc current can exist in such a device. In second order, we obtain an equation for the density profile which is a diffusion equation (24) describing how electrons diffuse from the high-density region near the cathode to the low-density region near the anode, producing the dc current. This equation is a direct generalization of Davidson's quasilinear equation, except that the constant-voltage conditions require an additional term, which generates the dc current. Furthermore, under the constant-voltage condition, we conjecture that static equilibrium profiles may exist. By equating the particle flux by diffusion to the net current, one can determine an equilibrium profile if it exists. (The equations defining an equilibrium profile are integro-differential equations and we have no proof of the existence of solutions.) This equilibrium profile (if it exists) has a negative definite density gradient inversely proportional to the diffusion coefficient. Thus, where the diffusion coefficient is large, the profile flattens (the plateau formation of Davidson), and where it is small, the profile steepens.

This picture of the density profile differs considerably from the conventional Brillouin flow, wherein the density is constant up to an edge at which it drops precipitously to zero. Diffusion makes profiles with sharp edges nonstatic and density gradients minimal only at the position of a resonance, with steeping between the resonances.

We define the diocotron resonance as the resonance where the phase velocity of the wave matches the local electron velocity ($\omega = kv_0$), and the magnetron resonance as the resonance where the local frequency of the wave, as seen by the moving electrons, is equal to the electron cyclotron frequency Ω. Thus $(\omega - kv_0)^2 = \Omega^2$ at the magnetron resonance. The magnetron instability apparently occurs when both the diocotron resonance and the magnetron resonance occur inside the electron sheath, with the diocotron resonance positioned near the edge.[1,2,4]

The diffusion coefficient, determined by the eigenmodal structure of the linear problem, is actually nonlinear, being linear in the magnitude squared of the vertical velocities of each unstable linear mode. It peaks strongly at all resonances of any unstable mode. At any diocotron resonance, its structure is exactly as detailed by Davidson[7] with a peak value inversely proportional to the growth rate and a width directly proportional to the growth rate. We find the diffusion coefficient at the magnetron resonance to be exactly the same form as at the diocotron resonance: strongly peaked with amplitude inversely and width directly proportional to the growth rate. Consequently, plateau formation[7] occurs at the magnetron, as well as at the dicotron, resonance.

But will stabilization occur? In other words, what happens to the growth rates as the initial density profile relaxes toward an equilibrium profile? We do not know, and the existence of the above-mentioned equilibrium profiles hinges on the answer to this critical question. Davidson[7] was able to show that, for the diocotron instability when the density profile has positive gradients, stabilization would occur in the low-density, massless, guiding-center limit with growth rates vanishing to zero as the positive density gradients diffuse away. Although we do not know what happens in the general case, numerical results indicate that if the profile drops sufficiently, growth rates do vanish.[2] This is encouraging, because we would expect equilibrium profiles to have negative density gradients away from any resonances. More work must be done here, particularly on the evaluation of growth rates of any linear unstable modes for any equilibrium profiles.

Our primary concern is not with the question of the existence of equilibrium density profiles, but rather with potential nonlinear instabilities such as the modulational instability, which is a third-order phenomenon. In preparation for studying these, we have considered this second-order problem, and in the process have both found new questions to be answered and developed a basic understanding of what does happen at this order. We find that unstable linear modes initially drive a second-order diffusion process. We conjecture then that the initial density profile will relax toward some possible equilibrium profile which has negative density gradients between resonances and plateaus at resonances. As this process occurs, the linear eigenmodal structure shifts because the density profile is changing. We do not know what will happen to the growth rates then. However, density gradient steeping does reduce the growth rates,[2] which leads toward stabilization.

In Sec. II, the zeroth-order equilibrium configuration which we consider for the crossed-field device is described, and the relevant equations describing the linear theory (first-order perturbations) are given. The equations describing the nonlinear (second-order) evolution of the zeroth-order electron flow velocity and electric field are derived in Sec. III. In Sec. IV we discuss these equations and present the results. We discuss the effects of the diocotron and magnetron resonances in driving the nonlinear evolution of the zeroth-order quantities.

II. THEORETICAL MODEL—EQUILIBRIUM AND FIRST-ORDER EQUATIONS

The cold-fluid macroscopic equations (with pressure 0) describing the nonrelativistic flow of a non-neutral pure electron plasma are

$$\partial_t n + \nabla\cdot(n\mathbf{v}) = 0 \ , \tag{1a}$$

$$(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{v} + \mathcal{E} + \mathbf{v}\times\Omega = 0 \ , \tag{1b}$$

$$\mathcal{E} = -\nabla\phi \ , \tag{1c}$$

$$\nabla^2\phi = \omega_p^2 \ . \tag{1d}$$

Here, n, m, and v denote the electron number density, the electron mass, and the velocity, respectively. The plasma frequency is $\omega_p^2 = 4\pi n e^2/m$. The equilibrium we consider corresponds to the planar configuration of Fig. 1 with crossed electric and magnetic fields. The cathode is at $y=0$, and the anode at $y=l$. The stability of nonrelativistic and relativistic non-neutral electron flows in planar and cylindrical diodes has recently been considered by several authors.[1–4] We define the normalized electric field $\mathcal{E}_0 = e\mathbf{E}_0/m = -eE_0\hat{\mathbf{y}}/m$ and the gyrofrequency $\mathbf{\Omega} = \mathbf{B}_0 e/mc = -e(B_0/mc)\hat{\mathbf{z}}$.

In this paper we consider electrostatic modes where the magnetic field remains equal to the equilibrium or zeroth-order value at all orders. Hence we will omit the subscript 0 on the Ω even though this is a zeroth-order quantity. We shall also do the same for the plasma frequency shortly. We consider a two-dimensional diode structure, with translational invariance in the z direction both in the equilibrium and for the perturbation quantities. In addition, we assume translational invariance in the x direction for the equilibrium quantities. From Poisson's equation, we have

$$\partial_y \mathcal{E}_0 = \omega_p^2 \ , \tag{2a}$$

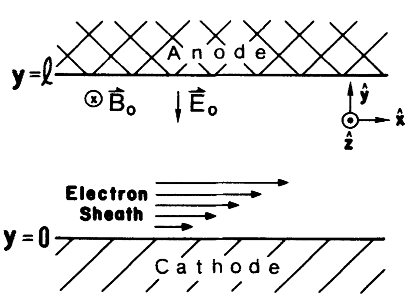

FIG. 1. Geometry and the shear flow in the planar magnetron.

where now $\omega_p^2 = 4\pi n_0 e^2/m$, and n_0 is the equilibrium electron density. The electron density n_0 may be specified to be of any form and, hence, is arbitrary. For n_0 or ω_p^2 a monotonic decreasing function of y, the linear diocotron mode is stable for a cold massless low-density electron plasma.[8] This criterion has recently been generalized to warm plasmas without assuming the guiding-center approximation.[23] The equilibrium electron flow velocity is

$$\mathbf{v}_0 = v_0\hat{\mathbf{x}} = \left[\frac{\mathcal{E}_0}{\Omega}\right]\hat{\mathbf{x}} \ . \tag{2b}$$

Equations (2a) and (2b) imply that v_0 increases monotonically with y, and thus we have a nonzero velocity shear in the equilibrium.

We next consider first-order perturbations of Eqs. (1). In general, all physical quantities X may be written as

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots \ , \tag{3a}$$

where ϵ is a measure of the (small) deviation from the equilibrium value X_0. Here, X may denote the density, electric field, or velocity. In first order, all quantities are expanded as

$$X = X_0 + \epsilon X_1 = X_0 + \epsilon(\tilde{X}_1 e^{i(kx-\omega t)} + \text{c.c.}) \ , \tag{3b}$$

with $\text{Im}\omega > 0$ for instability (in the initial-value sense). The perturbations X_1 are Fourier decomposed in the x coordinate, which is of infinite extent, with $\tilde{X}_1$ being the Fourier amplitude. The linearized equations for the Fourier amplitudes of the first-order perturbation quantities $\tilde{X}_1$ will be combined into a composite equation for a single first-order physical variable (the perturbed electrostatic potential $\tilde{\phi}_1$ or the perturbed y velocity $\tilde{v}_{1y}$). This equation, with appropriate boundary conditions at the electrodes, is then to be solved for the eigenfunctions $\tilde{\phi}_1$ (or $\tilde{v}_{1y}$) and the eigenvalues ω for various perturbation wave numbers k. The stability of the modes is then determined from the sign of $\text{Im}\omega$. The stability properties of electrostatic- and extraordinary-mode perturbations, as well as the structure of the eigenfunctions, have recently been investigated in considerable detail.[1–7] We will not consider this here. However, we give below the equations satisfied by linear electrostatic modes for nonrelativistic non-neutral electron flow in the planar device configuration. These equations will be necessary in deriving the second-order theory of Sec. III.

For electrostatic perturbations of the form given in Eq. (3), the set of Eqs. (1) may be linearized. Translational invariance in the z direction implies that $\partial_z = 0$. In the nonrelativistic limit, the z component of the perturbed electric field ($\mathcal{E}_{1z}$) satisfies the free-space equation $(\partial_y^2 - k^2)\mathcal{E}_{1z} = 0$. Hence $\mathcal{E}_{1z}$ decouples from the electrostatic wave. So we shall take $\mathcal{E}_{1z} = 0$, which implies $\tilde{v}_{1z} = 0$. Introducing the first-order perturbed potential $\phi_1 = \tilde{\phi}_1 e^{i(kx-\omega t)} + \text{c.c.}$, such that

$$\tilde{\mathcal{E}}_{1x} = -ik\tilde{\phi}_1, \quad \tilde{\mathcal{E}}_{1y} = -\partial_y\tilde{\phi}_1 \ . \tag{4}$$

Then for the electrostatic modes, the linearized perturbation equations obtained from Eq. (1) for nonrelativistic, non-neutral electron flow are

$$\tilde{v}_{1x}=\frac{1}{R}\left[k\omega_e+\frac{\Delta^2}{\Omega}\partial_y\right]\tilde{\phi}_1 , \tag{5a}$$

$$\tilde{v}_{1y}=-\frac{ik}{R}\left[\Omega+\frac{\omega_e}{k}\partial_y\right]\tilde{\phi}_1 , \tag{5b}$$

$$\tilde{n}_1=-\frac{in_0\Gamma}{\omega_p^2\omega_e}\tilde{v}_{1y} , \tag{5c}$$

and

$$(\partial_y^2-k^2)\tilde{\phi}_1-\omega_p^2\left[\frac{\tilde{n}_1}{n_0}\right]=0 . \tag{5d}$$

Here, we define

$$\omega_e\equiv(\omega-kv_0) , \tag{6a}$$

$$R\equiv\Omega^2-\omega_p^2-\omega_e^2 , \tag{6b}$$

$$\Delta^2\equiv\Omega^2-\omega_p^2 , \tag{6c}$$

$$\Gamma\equiv\partial_y\omega_p^2-\frac{2k\omega_e\omega_p^4}{\Omega(\Omega^2-\omega_e^2)} . \tag{6d}$$

Equations (5a) and (5b) invert to

$$-ik\tilde{\phi}_1=i\omega_e\tilde{v}_{1x}+\frac{\Delta^2}{\Omega}\tilde{v}_{1y} \tag{7a}$$

and

$$\partial_y\tilde{\phi}_1=\Omega\tilde{v}_{1x}-i\omega_e\tilde{v}_{1y} . \tag{7b}$$

Using Eqs. (5a),(5b) and (7a),(7b), we have

$$\partial_y\tilde{v}_{1x}=\left[ik-\frac{i\Gamma'}{\omega_e\Omega}\right]\tilde{v}_{1y} \tag{7c}$$

and

$$\partial_y\tilde{v}_{1y}=-ik\tilde{v}_{1x}-\frac{2k\omega_e\omega_p^2}{\Omega(\Omega^2-\omega_e^2)}\tilde{v}_{1y} , \tag{7d}$$

where

$$\Gamma'\equiv\partial_y\omega_p^2+\frac{2k\omega_e\Delta^2\omega_p^2}{\Omega(\Omega^2-\omega_e^2)} . \tag{8}$$

The first-order perturbation equations (5) for nonrelativistic flow may be combined into the following equation for the Fourier amplitude of the first-order perturbation potential $\tilde{\phi}_1$:

$$\partial_y\left[\left[1+\frac{\omega_p^2}{R}\right]\partial_y\tilde{\phi}_1\right]-k^2\left[1+\frac{\omega_p^2}{R}\right]\tilde{\phi}_1+\frac{k\Omega}{\omega_e}\tilde{\phi}_1\partial_y\left[\frac{\omega_p^2}{R}\right]=0 , \tag{9a}$$

where the boundary conditions are $\tilde{\phi}_1(y=0)=0=\tilde{\phi}_1(y=l)$. This equation may be written in the equivalent useful form

$$\left[\partial_y^2-k^2+\frac{\Gamma}{R}\left[\partial_y+\frac{k\Omega}{\omega_e}\right]\right]\tilde{\phi}_1=0 . \tag{9b}$$

Equations (9a) and (9b) have been considered previously by Davidson and Tsang,[1] and in generalized form by Davidson, Tsang, and Swegle.[3] In particular, Davidson and Tsang have derived a sufficient condition for the stability of low-frequency flute perturbations in a moderate aspect-ratio cylindrical diode configuration. They have also considered the detailed structure of weak and strong instabilities driven by velocity shear for electrostatic modes in a cylindrical diode configuration with various density profiles $n(y)$, and various values of the ratio ω_p^2/Ω^2. Analogous results have been obtained for extraordinary-mode stability of relativistic non-neutral electron flow in a planar diode by Davidson, Tsang, and Swegle.[3]

An important aspect of these equations is the fact that $R=0$ is not a real singularity but is only an apparent one.[24] This may be made obvious by considering the composite equation for the Fourier amplitude of the perturbed y velocity $\tilde{v}_{1y}$. From Eqs. (4),

$$\partial_y\left[\partial_y\tilde{v}_{1y}+2k\left[\frac{\omega_e\omega_p^2}{\Omega(\Omega^2-\omega_e^2)}\right]\tilde{v}_{1y}\right]=k^2\left[1-\frac{2(\omega_p^2\Omega^2-\omega_p^4)}{\Omega^2(\Omega^2-\omega_e^2)}\right]\tilde{v}_{1y}-\frac{k}{\Omega}\frac{(\partial_y\omega_p^2)}{\omega_e}\tilde{v}_{1y} , \tag{10}$$

which has no singularity at $R=0$. Equation (10) generalizes the equation considered earlier by Buneman, Levy, and Linson[23] to the case with $\partial_y\omega_p^2\neq0$. It does not appear to have been derived earlier in the literature. Also, as noted by Buneman *et al.*, the singularity at $R=0$ in Eqs. (9a) and (9b) is only "apparent" and does not cause a blowup of the physical quantities. However, the diocotron resonance $\omega_e=0$ present in Eqs. (9a), (9b), and (10), and the magnetron resonance $\omega_e^2-\Omega^2=0$ which occurs in Eqs. (9b) and (10) are "real" singularities. Hence Eq. (10) for $\tilde{v}_{1y}$ contains only genuine or real singularities, in contrast to Eqs. (9a) and (9b), which contain the apparent singularities at $R=0$.

III. SECOND-ORDER EQUATIONS

In this section we will derive the equations satisfied by the second-order perturbation quantities. These quantities contain terms independent of t, as well as second-harmonic $e^{2i(kx-\omega t)}$ terms as per the usual multiple-scale analysis.[22] We are interested in the slow-time variation of the zeroth-order physical quantities characterizing the cross-field device, e.g., the average density. These evolutions occur on the slow-time and spatial scales $t_2=\epsilon^2 t$, and $x_2=\epsilon^2 x$ of multiple-scale analysis, with $\partial_t=\partial_t+\epsilon^2\partial_t+\cdots$, and $\partial_x=\partial_{x_0}+\epsilon^2\partial_x+\cdots$. We will therefore concentrate on parts of the second-order perturbation equations which are independent of the fast-time scale and obtain evolution equations for the variations of the zero-order velocity v_0, normalized electric field $\mathcal{E}_0$, and plasma frequency ω_p^2 on the slow-time and spatial scales t_2 and x_2. (We do not consider first-order time or spatial scales t_1 and x_1, in the zeroth-order quantities, because if they were present, they would drive the first-order equations as inhomogeneous terms. Rather, we are interested in how the first-order fields could drive the zeroth-order quantities away from the equilibrium values.) We also assume any linear instabilities to be small and on the order of the time scale $t_2=\epsilon^2 t$.[25]

Writing the second-order physical quantities χ_2 (fields, velocities, density, etc.) as the sum of zeroth-harmonic and second-harmonic terms,[22] we have [cf. Eq. (3a)]

$$\chi_2=\tilde{\chi}_2^{(0)}+(\tilde{\chi}_2 e^{2i(kx-\omega t)}+\text{c.c.}) \ . \tag{11}$$

Here, the subscript 2 indicates a second-order perturbation quantity. The superscript 0 on the $\tilde{\chi}_2^{(0)}$ indicates that $\tilde{\chi}_2^{(0)}$ is the zeroth-harmonic or the fast-time–independent part of the second-order perturbation quantity χ_2. As with $\tilde{\chi}_1$ in first order, $\tilde{\chi}_2$ denotes the Fourier amplitude of the second-harmonic part of χ_2.

Perturbing Eqs. (1) and neglecting terms of order ϵ^3 or higher, we obtain the equations satisfied by the second-order perturbed quantities χ_2. Using Eq. (11), the zeroth-harmonic component of the second-order perturbation equations, describing nonrelativistic electron flow, may be derived.

The second-order perturbations to the Maxwell equations (1c)–(1f) yield

$$n_2^{(0)}=-\frac{n_0}{\omega_p^2}\partial_y \tilde{\mathcal{E}}_{2y}^{(0)} \ , \tag{12a}$$

$$0=-\partial_{x_2}\mathcal{E}_0-\partial_y \tilde{\mathcal{E}}_{2x}^{(0)} \ , \tag{12b}$$

as well as $\Omega_2^{(0)}$ and $\tilde{\mathcal{E}}_{2z}^{(0)}$ being constants, which we take to be zero. We have also taken Ω to be independent of the slow time.

Similarly, the zeroth-harmonic part of the second-order perturbations to the continuity equation (1a) gives

$$\partial_y(n_0\tilde{v}_{2y}^{(0)})=-\partial_{t_2}n_0-\partial_{x_2}(n_0 v_0)-\partial_y\langle n_1 v_{1y}\rangle \ . \tag{13}$$

Equation (13), together with approximated versions of Eqs. (7a) and (7b), have been used by Davidson[7] to develop his quasilinear theory for the slow-time evolution of the zeroth-order quantities. His approximations will be considered in Sec. IV, where we compare his results to our exact theory. It is shown there that Davidson's quasilinear theory is exactly valid near the diocotron resonance. For the case where one is away from the diocotron resonance, or slow spatial variations are present, a generalization of his theory results.

We also have, from the zeroth-harmonic part of the second-order momentum equations (1b), that

$$\tilde{v}_{2x}^{(0)}=-\frac{1}{\Omega}(\tilde{\mathcal{E}}_{2y}^{(0)}+\langle \mathbf{v}_1\cdot\nabla_0 v_{1y}\rangle) \ , \tag{14a}$$

$$\tilde{v}_{2y}^{(0)}=\frac{\Omega}{\Delta^2}[(\partial_{t_2}+v_0\partial_{x_2})v_0+\tilde{\mathcal{E}}_{2x}^{(0)}+\langle \mathbf{v}_1\cdot\nabla_0 v_{1x}\rangle] \ , \tag{14b}$$

$$\tilde{v}_{2z}^{(0)}=0 \ . \tag{14c}$$

In Eqs. (12)–(14) quantities in angular brackets denote the fast-time averaged part.

Equation (13) may be integrated over y to give

$$\omega_p^2\tilde{v}_{2y}^{(0)}+\partial_{t_2}\mathcal{E}_0+\tfrac{1}{2}\partial_{x_2}\left[\frac{\mathcal{E}_0^2}{\Omega}\right]+\langle n_1 v_{1y}\rangle\frac{\omega_p^2}{n_0}=c(t_2,x_2)\Omega^2 \ , \tag{15}$$

where $c(t_2,x_2)$ is a constant of integration, and Eq. (2a) has been used. Equations (12), (14), and (15) are the set of equations we will use.

The case of zero net flux studied by Davidson[7] may be treated easily only in the limit $m=0$, corresponding to $c(t_2,x_2)=0$. In the general case, the electron flow is not an exact $\mathbf{E}\times\mathbf{B}$ drift and we cannot then argue that $\partial_y\mathcal{E}_0=0$ at both cathode and anode. Thus we see no advantage in treating the zero flux here and instead will concentrate on the operating configuration where the voltage at the anode is to be kept constant. Then $c(t_2,x_2)$ is determined by the anode voltage (see Sec. IV).

It appears that $\tilde{\mathcal{E}}_{2y}^{(0)}$ or $\tilde{n}_2^{(0)}$ is arbitrary. The arbitrariness of $\tilde{n}_2^{(0)}$ is simply due to the zeroth-order density n_0 being arbitrary. Without loss of generality, we may take $\tilde{n}_2^{(0)}=0$.

Introducing the electrostatic potential ϕ_0, where

$$\phi_0=\int_0^y \mathcal{E}_0(y)dy \tag{16}$$

and the potential is chosen to be zero at the cathode, we may solve Eq. (12b) for $\tilde{\mathcal{E}}_{2x}^{(0)}$. Then our second-order relations can be reduced to Eq. (14b) and

$$\tilde{\mathcal{E}}_{2x}^{(0)}=-\partial_{x_2}\phi_0 \ , \tag{17a}$$

$$\omega_p^2\tilde{v}_{2y}^{(0)}=-(\partial_{t_2}+v_0\partial_{x_2})\mathcal{E}_0-\omega_p^2\left\langle\frac{n_1}{n_0}v_{1y}\right\rangle+c(t_2,x_2)\Omega^2 \ , \tag{17b}$$

$$\partial_y\mathcal{E}_{2y}^{(0)}=0 \ , \tag{17c}$$

$$\Omega\tilde{v}_{2x}^{(0)}=-\tilde{\mathcal{E}}_{2y}^{(0)}-\langle \mathbf{v}_1\cdot\nabla_0 v_{1y}\rangle \ . \tag{17d}$$

Using Eqs. (2a), (14b), and (17b), we then obtain the equations for the evolution of the zero-order velocity v_0, and the normalized zero-order electric field $\mathcal{E}_0$ on the slow temporal and spatial scales t_2 and x_2. The equation for v_0 results by using Eq. (2a) to eliminate $\mathcal{E}_0$ from Eq. (17b) in favor of v_0. On the other hand, eliminating v_0 in favor of $\mathcal{E}_0$ in Eq. (17b) by using Eq. (2c) results in the equations for $\mathcal{E}_0$. These are

$$(\partial_{t_2}+v_0\partial_{x_2})v_0=\frac{\omega_p^2}{\Omega^2}\partial_{x_2}\phi_0-\frac{S}{\Omega}+\frac{\Delta^2}{\Omega}c(t_2,x_2) \tag{18a}$$

and

$$(\partial_{t_2}+v_0\partial_{x_2})\mathcal{E}_0=\frac{\omega_p^2}{\Omega}\partial_{x_2}\phi_0-S+\Delta^2 c(t_2,x_2) \ . \tag{18b}$$

Here, the nonlinear source term in Eqs. (18) is

$$S\equiv\frac{\omega_p^2}{\Omega}\langle(\mathbf{v}_1\cdot\nabla_0)v_{1x}\rangle+\frac{\Delta^2}{\Omega^2}\frac{\omega_p^2}{n_0}\langle n_1 v_{1y}\rangle \ . \tag{19}$$

Equation (18b) describes how $\mathcal{E}_0$ will evolve on the slow-time scale, while (2a) defines how it must vary vertically. The integrability condition for these two equations to have the same common solution is

$$(\partial_{t_2}+v_0\partial_{x_2})\omega_p^2=\left[\frac{\partial_y\omega_p^2}{\Omega}\right]\partial_{x_2}\phi_0-c(t_2,x_2)\partial_y\omega_p^2-\partial_y S+\frac{\omega_p^2}{\Omega}\partial_{x_2}\mathcal{E}_0 \ . \tag{20}$$

Lastly, we shall work out the components of the nonlinear source term S in Eq. (19). We have

$$\langle(\mathbf{v}_1\cdot\nabla_0)v_{1x}\rangle=\sum_k\tilde{v}\,^*_{1y}(\partial_y\tilde{v}_{1x})+\text{c.c.}=-\sum_k(\tilde{v}\,^*_{1y}\tilde{v}_{1y})\left[\frac{i\Gamma'}{\omega_e\Omega}+\text{c.c.}\right]e^{2\omega_i t} \ , \tag{21a}$$

$$\frac{\omega_p^2}{n_0}\langle n_1 v_{1y}\rangle=-\sum_k(\tilde{v}\,^*_{1y}\tilde{v}_{1y})\left[\frac{i\Gamma}{\omega_e}+\text{c.c.}\right]e^{2\omega_i t} \ , \tag{21b}$$

where the sum over k indicates a sum over all linear eigenmodes of the system. Hence, using Eqs. (8), the source term reduces to simply

$$S=\sum_k\left[-i\,|\,\tilde{v}_{1y}\,|^2\frac{\partial_y\omega_p^2}{\omega_e}+\text{c.c.}\right]e^{2\omega_i t} \ , \tag{22}$$

where ω_i is the imaginary part of the eigenfrequency ω. This is the nonlinear term driving the evolution of the zero-order velocity v_0 and normalized electric field $\mathcal{E}_0$, on the slow scales t_2 and x_2, via Eqs. (18). We will consider this term in detail in Sec. IV.

IV. RESULTS AND CONCLUSIONS

In this section, we discuss the evolution of the zero-order density ω_p^2, normalized electric field $\mathcal{E}_0$, and velocity v_0 on the slow-time and spatial scales t_2 and x_2. Our treatment up to this point has been exact, and Eqs. (18), (20), and (22) describe the general nonlinear evolution of the zero-order quantities, which we now consider.

Equation (20) may be written in the equivalent form

$$[\partial_{t_2}+c(t_2,x_2)\partial_y+v_0\partial_{x_2}]\omega_p^2=\partial_y\left[\frac{\omega_p^2}{\Omega}\partial_{x_2}\phi_0-S\right] \ . \tag{23}$$

The changes on the slow spatial scale x_2 in Eq. (23) in the $\mathbf{E}\times\mathbf{B}$ drift direction correspond to possible long-wavelength fluctuations in the density. These changes would be important for a pulse which had a finite spatial extent.

From this point on, we shall consider devices and operating conditions where the x_2-spatial structure is unimportant. Thus, all quantities become independent of x_2. Equation (23) then reduces to the form

$$[\partial_{t_2}+c(t_2)\partial_y]\omega_p^2=\partial_y(D\partial_y\omega_p^2) \ , \tag{24}$$

where we have set $S=-D\partial_y\omega_p^2$. The quantity D in (24) is a diffusion coefficient and is given by

$$D=\sum_k D_k \ , \tag{25a}$$

$$D_k=i\frac{|\,\tilde{v}_{1k}\,|^2}{\omega_e}e^{2\omega_i t}+\text{c.c.} \ , \tag{25b}$$

where D_k is the contribution to D from the kth mode. Obviously, in general, ω_i is a function of k also.

In order to interpret (24), let us consider the two separate parts individually. First, let D be zero. Then the solution would be $\omega_p^2=f(\chi)$, where for c constant, $\chi=y-ct_2$. If c is positive, the time evolution of ω_p^2 is described as a uniform lateral shift of the plasma sheath upwards towards the anode. Now consider (24) when $c=0$. It then becomes the standard diffusion equation, but with a y-dependent coefficient. This part will seek to smooth out sharp density gradients. If we put two parts together, then we have (for $c>0$) a general upward motion of the plasma sheath accompanied simultaneously by a smoothing of any density gradients.

Due to (18b), the value of c is actually determined by the diffusion coefficient and the density profile. Requiring ϕ_0 in (16) to be fixed at the anode, integrating (18b) from $y=0$ (cathode) to $y=l$ (anode) gives

$$c(t_2)=\int_0^l(-D\partial_y\omega_p^2)dy\Big/\int_0^l\Delta^2 dl \ . \tag{26}$$

In (26), the denominator is positive definite. Since we only consider here profiles with negative gradients, the sign of c is the same as the sign of D. If $D>0$ (normal diffusion), then $c>0$ and, except for modifications due to the diffusion, the general sheath motion is upwards toward the anode. If $D<0$ (reverse diffusion), then $c<0$ and the sheath would tend to contract. If $D(y)$ is of mixed signs, the sign of c will depend on the exact distribution of $D(y)$.

Directly from (24), we see the possibility for the existence of an equilibrium distribution static to second order in time. (Recall that to zeroth order and first order, the density profile was arbitrary.) Setting $\partial_{t_2}\omega_p^2=0$ in (24) gives

$$c\omega_p^2-D\partial_y\omega_p^2=\Gamma(t_2) \ , \tag{27}$$

where Γ is the constant of integration. From (18b), one may evaluate Γ, obtaining

$$\Gamma=c\Omega^2 \ . \tag{28}$$

One may combine (27) and (28) to obtain

$$D\partial_y\omega_p^2=-c(\Omega^2-\omega_p^2) \ , \tag{29}$$

which shows that under the conditions of normal diffusion, the equilibrium density gradient will be negative definite and inversely proportional to D. Thus, where D is large, $\partial_y\omega_p^2$ is small and vice versa. This is equivalent to the plateau formation observed by Davidson.[7]

Note that (29) predicts that any equilibrium density distribution must fill the entire interelectrode space, under the conditions of normal diffusion. Since c and D are nonzero, both the density and the density gradient cannot simultaneously vanish. Thus ω_p^2 may only vanish at the anode, with the equilibrium dc current being proportional to $c\Omega^2$.

The equation for c, (26), and the equilibrium density profile (29) require values for $D(y)$. In general, these are obtainable only in an iterative process where by one starts with a density profile, obtains the linear eigenmodal structure, and then constructs D as in (25). One could take the profile obtained from (29) (which now will differ from the original), calculate its linear eigenmodel structure, and reconstruct the D for it. Although one suspects this procedure will converge, its convergence is not known. We also do not know if there is only one equilibrium profile, since several may satisfy (29). One way to determine what does occur is to integrate (24) forward in t_2, simultaneously calculating $D(y,t_2)$ at each t_2 from the linear eigenmodal structure.

Let us turn our attention to the structure of D in (25). The results of Davidson's quasilinear theory are contained in this, as is seen by applying the conditions

$$\omega_e^2 \ll \Omega^2 \tag{30a}$$

and

$$|\omega_e k_y| \ll k\Omega \tag{30b}$$

to (5b) and (25b). In (30b), k_y is a typical value for a wavevector in the y direction near the diocotron resonance, $\omega_e=0$. One obtains for a cold, massless, electron fluid that

$$D_k \cong \frac{2\omega_i}{\Omega^2}\frac{k^2|\tilde{\phi}_1|^2 e^{2\omega_i t}}{(\omega_r-kv_0)^2+\omega_i^2} , \tag{31}$$

where ω_r (ω_i) is the real (imaginary) part of the eigenfrequency. [The D in (31) is exactly twice that of Davidson's Eq. (36) due to different conventions for real parts.] Note that for instability where $\omega_i>0$, we have $D_k>0$. If $\omega_i=0$, then $D_k=0$ and if $\omega_i<0$, then $D_k<0$ (reverse diffusion). However, the latter case is exponentially damped in time so that after appearing as an initial transient, it vanishes from the system.

The structure of D in (25) is very suggestive. If we define a "vertical displacement," ζ, via

$$(\partial_t+\mathbf{v}\cdot\nabla)\zeta=v_y , \tag{32}$$

then in the first order we have

$$\tilde{v}_{1y}=-i\omega_e\tilde{\zeta}_1 , \tag{33}$$

which gives

$$D_k=(\partial_t+\mathbf{v}_0\cdot\nabla)\langle\zeta_1^2\rangle . \tag{34}$$

This shows that this diffusion is a random-walk process because the diffusion coefficient is equal to the time rate of change of the average square of the vertical displacement. This provides an alternate mechanism for calculating D_k when one performs particle simulations. One simply follows one fluid particle and averages over several periods, the time derivative of which will give the total diffusion coefficient D.

Let us now consider the shape of D_k at the two resonances. If we evaluate D_k exactly from (25b), we have

$$D_k=\frac{2\omega_i|\tilde{v}_{1y}|^2 e^{2\omega_i t}}{(\omega_r-kv_0)^2+\omega_i^2} . \tag{35}$$

At the diocotron resonance where $\omega_r=kv_0$, from (10), one may show that $\tilde{v}_{1y}$ is regular.

Thus, around the diocotron resonance, we may treat $\tilde{v}_{1y}$ as a constant. Then D_k is Lorentzian shaped with a peak amplitude inversely proportional to ω_i and a spatial width of $\omega_i\Omega/(k\omega_p^2)$.

At the magnetron resonance, Eq. (10) shows that $\tilde{v}_{1y}$ is singular and is of the form

$$\tilde{v}_{1y}\to\frac{2\alpha\Omega^2}{\Omega^2-\omega_e^2}+\cdots , \tag{36}$$

where α is some constant. Now, in the region around the magnetron resonance, which is where $(\omega_r-kv_0)=\pm\Omega$, (35) becomes

$$D_k\simeq\frac{2\omega_i|\alpha|^2 e^{2\omega_i t}}{(\omega_r-kv_0\mp\Omega)^2+\omega_i^2}+\cdots . \tag{37}$$

Again, we have a Lorentzian shape with an amplitude inversely proportional to ω_i and a spatial width of $\omega_i\Omega/(k\omega_p^2)$. As in the diocotron resonance, this profile of D will seek to drive the density gradient toward zero at the magnetron resonance, simply because of the nature of diffusion. And if an equilibrium is ever approached, (29) assures that plateau formation will also occur at the magnetron resonance.

Will an equilibrium ever be reached? Davidson's[7] cold, massless, guiding-center fluid at low densities and not-too-high wavelengths did give stabilization and a final equilibrium by diffusing away any positive density gradients. As the positive density gradients melted away, the unstable growth rates decreased, eventually reaching zero and giving stabilization. While it is possible to extend Davidson's result into the high-density general case for k not too large, the magnetron instability makes this of little practical value. For the diocotron resonance located at the edge of the sheath, as k increases, the magnetron resonance eventually moves above the cathode and into the plasma sheath. When this occurs, all modes with a value of k above this value seem to be unstable and to have $\omega_i>0$.[4,23] If any unstable modes exist, then D becomes time dependent. However, usually one unstable mode dominates. Then D becomes proportional to $\exp(2\omega_i t)$, where now ω_i is the imaginary part of the most unstable mode. And by (26), c becomes proportional to *exactly the same factor,* which when factored out of (29), leaves (29) as time independent. Thus, although D may have an exponential growth, one could still have a static background equilibrium.

With few exceptions,[1,2] calculations of growth rates of the magnetron instability have been based on flat density profiles with no density gradients except at the very edge of the sheath. This box-shaped profile is contrary to the second-order diffusion equation (24). Because of diffusion, no such profile could be static. The sharp edges would immediately diffuse away, and the remaining smoothed profile would, more slowly, relax to some equilibrium solution of (29).

What will happen to the growth rates as the profile relaxes and negative density gradients appear between the resonances? We cannot yet answer this question but observe that, considering (29), if a resonance does not occur exactly at the cathode, then D will be small there, forcing a negative gradient into the density profile at the cathode. This will drive ω_p^2 away from its maximum possible value of Ω^2. Once ω_p^2 is less than about $\frac{1}{2}\Omega^2$, at least for flat profiles, growth rates do become very, very small.[4,23]

In conclusion, our viewpoint of the evolution of the sheath is that linear instabilities grow and initiate the second-order diffusion instability described by (29). A small dc current is the consequence of the diffusion of electrons from high density near the cathode to low density near the anode. The time scale for the diffusion to be initiated is $t_2(=\epsilon^2 t)$ which, for very practical reasons, we want to be faster than the linear instability growth time $1/\omega_i$). Thus our analysis fails if wide, flat density profiles with $\Omega^2 > \omega_p^2 > \frac{1}{2}\Omega^2$ occur because of the large linear growth rates.[1,2,4,23] On the other hand, if our average density has $\langle \omega_p^2 \rangle < \frac{1}{2}\Omega^2$, then we can expect very small linear growth rates and the second-order diffusion process could then dominate. And if diffusion sufficiently smooths the profile, the linear growth rates may decrease and lead to stabilization.

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[25] Since growth rates ω_i are frequently very small compared to the real part of frequencies ω_r in such situations as ours (see Ref. 24), we take $\omega_i t$ to be on the order of the slow time t_2, or even slower. Of course, if $\omega_i t$ is much faster than this, then our second-order analysis would not be required due to the linear term being so unstable as to dominate all second-order (and even higher-order) effects.