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Quantum-field superpositions via self-phase modulation of coherent wave packets

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With the use of a quantum theory of optical propagation, a set of nonlinear stochastic partial differential equations may be derived which describes the quantum-statistical properties of traveling waves due to self-phase modulation in a nonlinear medium. We calculate exact moments for the field, which exhibit classical self-phase modulation in the short-interaction limit and periodic quantum evolution through field-superposition states in the long-interaction limit. Reversible and irreversible behaviors of the stochastic description are reviewed. The relation of the present work to the corresponding single-mode nonlinear oscillator is discussed.

I. INTRODUCTION

The quantum-statistical properties of traveling waves have received relatively little attention in the optics literature.[1] However, recently, two quantum theories of optical propagation have been discussed.[2,3] The motivation behind these is the need to describe quantum fluctuations in an optical fiber, and thus give a realistic description of the generation and detection of squeezed states in such a medium. Classically, wave propagation in a fiber is described by the nonlinear Schrödinger equation, and it is well known that the balance between group-velocity dispersion and nonlinearity (four-wave mixing), allows the propagation of solitary wave pulses. The use of pulses for the generation of squeezed states of light in fibers has potential advantages over the continuous-wave case, as suppression of troublesome stimulated Brillouin scattering losses may be possible.[4] Using perturbation theory on a c-number stochastic nonlinear Schrödinger equation, derived by generalizing the phase-space methods of Drummond and Gardiner,[5] Carter *et al.*[2] have shown that considerable broadband squeezing is possible in the anomalous dispersion regime. Similar results have been obtained by Potasek and Yurke[3] using perturbation theory on an operator nonlinear Schrödinger equation. We note that traveling-wave parametric conversion has recently been used to generate short pulses of squeezed light.[6]

Here we describe the application of stochastic methods[2] to the optical generation of quantum-superposition states. The observation of macroscopic-superposition states remains an outstanding problem in fundamental physics. A suggestion that such a state may be generated by unitary evolution of a single-mode nonlinear oscillator from an initial coherent state has been recently made.[7] The superposition state gives a characteristic interference signal in homodyne detection, but is washed out rapidly if losses are present. The interference may persist for long times, however, if the loss modes are squeezed.[8] To implement such a scheme experimentally and avoid medium losses one envisages the propagation of a coherent pulse through a nonlinear medium of certain length. However, the combination of small nonlinearity and medium losses makes it unrealistic at present to generate macroscopic superpositions in, for example, an optical fiber. Nevertheless, a model exhibiting macroscopic quantum effects in an extended system is interesting in its own right. This can be viewed as a fully nonlinear quantum theory of self-phase modulation, as the usual classical self-phase modulation is recovered in the short-interaction limit. In the long-interaction limit periodic evolution through quantum superpositions takes over.

Since a single-mode theory cannot describe propagation effects, a multimode model is necessary. The theory of Drummond and Carter[2] is adequate for our purposes since it includes the four-wave-mixing (Kerr-type) nonlinearity responsible for the self-phase modulation of optical pulses. Group-velocity dispersion is, however, ignored in our idealized model. This is a reasonable first approximation for an optical fiber operating at the zero first-order dispersion wavelength, although strictly higher-order perturbations should be considered. In the present analysis a major theoretical problem is that to adequately describe the generation of a superposition state, the fully nonlinear nature of the quantum fluctuations must be included. A perturbation procedure about some classical solution will not predict the characteristic quantum-interference signal of a superposition state. It is for this reason that we find it necessary to drop group-velocity dispersion and choose a relatively simple model, which, however, exhibits very interesting features. Without the smoothing effect of the group-velocity dispersion the model is somewhat singular, and we find it necessary to explicitly include the finite bandwidth of interacting modes in order to regularize the results. The role of dispersive phase shifts remains open.

Application of stochastic methods leads to nonlinear stochastic partial differential equations (SPDE's) with multiplicative noise. Such equations have received relatively little attention in the physics literature.[9] An interesting feature of the stochastic equations obtained here is their application to a time-reversible process implicit in the treatment of unitary evolution. Often stochastic equations in physics arise through coupling an open system to its reservoirs, and signal the loss of information caused by neglecting reservoir-system correlations. In the present problem stochastic equations are obtained by doubling the dimension of the classical phase space. This allows the stochastic description to apply to intrinsically time-reversible behavior. We find that all time-irreversible behavior is localized to those moments of the stochastic process having no corresponding physical observable. By comparison, the observable moments have reversible behavior. The advantage of this description is that the addition of irreversible losses and sources of classical noise which may be present is relatively straightforward.

The remainder of the paper is organized as follows. In Sec. II we write down the SPDE which governs the quantum-statistical properties of traveling waves in a parametric converter. We apply the method of characteristics to reduce the problem to ordinary stochastic differential equations (SDE's), and deduce that these may be interpreted in the Ito sense, so that standard techniques may be applied.[10] To elucidate we review the single-mode model[7] in Sec. III, and rederive some results using Ito's formula.[10] From the point of view of the single-mode problem this is unnecessary, as Schrödinger's equation may be solved exactly. However, the same method turns out to be applicable to the propagation model also. With these insights we obtain solutions for a hierarchy of normally ordered moments for arbitrary input states in Sec. IV, and discuss the particular case of coherent wave packets in Sec. V. In Sec. VI we indicate how dissipation affects these results.

We consider direct solutions for the stochastic equations in Sec. VII. The resulting evolution in phase space is discussed in terms of its reversible and irreversible components. The results are compared to the moment equations. It is shown that reversible time evolution can be correctly described with stochastic methods. Typically quantum behavior is exhibited by trajectories exploring dimensions of the phase space outside of the classical phase space. This is expected, in view of the nonclassical nature of quantum superpositions. Our results and conclusions are discussed in Sec. VIII.

II. TRAVELING-WAVE PARAMETRIC CONVERSION

The aim of this section is to show how standard stochastic methods may be applied to discuss the quantum fluctuations of traveling waves using the example of parametric frequency conversion. This is a particularly helpful example, as the associated SPDE may be solved by the method of characteristics, and also exact solutions for the local-field operators may be found. Comparison of the two solutions clarifies interpretation of the stochastic characteristic equations. The material presented here is not directly related to the generation of superposition states, but is included for the purpose of illustrating methods which will be used later in that context.

A quantum theory may be formulated by starting with the multimode Hamiltonian

$$H=\sum_k \hbar\omega_k a_k^\dagger a_k+\frac{\chi_p}{\epsilon^2}\int d\mathbf{x}\,{:}D^2(\mathbf{x}){:}\,E_p(\mathbf{x},t)\ , \qquad (1)$$

where a_k and $a_k^\dagger$ are the usual boson operators for longitudinal mode k, with frequency ω_k, χ_p is proportional to the parametric nonlinearity, and the notation :: denotes normal ordering with the electric displacement field operator given by

$$D(x)=i\sum_k\left[\frac{\epsilon\hbar\omega_k}{2V}\right]^{1/2}(a_k e^{ikx}-\text{H.c.})\ , \qquad (2)$$

where ϵ is the permittivity, $V=AL$ is the quantization volume, with A the effective mode cross section and L the length of the medium. Transverse guiding modes are assumed spatially uniform for simplicity (but not necessity), the field is assumed to be linearly polarized, and the coordinate x is measured along the length L of the medium. The pump field $E_p(x,t)$ is treated classically

$$E_p(x,t)=Fe^{i2(k_0x-\omega_0t)}+\text{c.c.}\ , \qquad (3)$$

and for simplicity F is chosen to be real.

The electric field operator in equation (3) may be expressed in terms of field operators $\Psi(x)$ and its Hermitian conjugate $\Psi^\dagger(x)$, where

$$\Psi(x)=\frac{1}{\sqrt{L}}\sum_k a_{k_0+k}e^{ikx} \qquad (4)$$

and the sum over modes k is taken relative to some reference wave vector chosen here to be k_0. By considering a finite number of modes, discretizing spatially, and using phase-space methods,[5] c-number stochastic differential equations of Ito type may be derived for a set of complex variables $\psi_l(t)$ and $\psi_l^\dagger(t)$ on the lattice sites l.[2] By taking the continuum limit one obtains the SPDE

$$\left[\frac{\partial}{\partial t}+\omega'\frac{\partial}{\partial x}\right]\psi(x,t)=is\psi^\dagger(x,t)+\sqrt{is}\,\eta(x,t)\ , \qquad (5)$$

where ω' is the phase velocity, $s=\omega_0\chi_pF/\epsilon$, and ψ and $\psi^\dagger$ are complex random fields (associated with the local-field operators Ψ and $\Psi^\dagger$, respectively) with dimensions of the inverse square root of length, and which are complex conjugate in the mean.[5] A similar equation for $\psi^\dagger$ is found by replacing i by $-i$, ψ by $\psi^\dagger$ (and vice versa), and η by $\eta^\dagger$. The zero-mean-noise fields $\eta(x,t)$ and $\eta^\dagger(x,t)$ are δ correlated in time and space, i.e.,

$$\langle\eta(x,t)\rangle=\langle\eta^\dagger(x,t)\rangle=\langle\eta(x,t)\eta^\dagger(x',t')\rangle$$
$$=0\ , \qquad (6)$$
$$\langle\eta(x,t)\eta(x',t')\rangle=\langle\eta^\dagger(x,t)\eta^\dagger(x',t')\rangle$$
$$=\delta(x-x')\delta(t-t')\ . \qquad (7)$$

Note that the derivation places certain restrictions on the bandwidth of the field for the derivation of Eq. (5) to be valid [see Eq. (37)]. However, in practice only states of the field in a finite bandwidth $(k_0-K/2,k_0+K/2)$ which we take to be symmetrical around k_0, will participate in the interaction. The bandwidth will be much less than an optical wave vector, i.e., $K \ll k_0$. This finite bandwidth must be kept in mind when one evaluates any field correlations.

The characteristics of (5) are defined by the straight lines $x=x_0+\omega' t$, where x_0 is a constant defining a characteristic. The characteristic equation corresponding to (5) is formally

$$\omega' \frac{d\psi(x,\xi)}{dx} = is\psi^\dagger(x,\xi)+\sqrt{is}\,\eta(x,\xi) , \tag{8}$$

where $(x,\xi)\equiv(x,(x-x_0)/\omega')$. Taking the mean of equation (8), and eliminating $\psi^\dagger$ by using the conjugate equation, one finds

$$\frac{d^2\langle\psi(x,\xi)\rangle}{dx^2} = \frac{s^2}{\omega'^2}\langle\psi(x,\xi)\rangle , \tag{9}$$

with solution

$$\langle\psi(x,t)\rangle = \langle\psi(0,t-x/\omega')\rangle\cosh\left[\frac{sx}{\omega'}\right] + i\langle\psi^\dagger(0,t-x/\omega')\rangle\sinh\left[\frac{sx}{\omega'}\right] . \tag{10}$$

Using linearity the operator equation corresponding to (8) may be written down immediately as

$$\omega' \frac{d\Psi(x,\xi)}{dx} = is\Psi^\dagger(x,\xi) , \tag{11}$$

and can be solved with its Hermitian conjugate to yield

$$\Psi(x,t) = \Psi(0,t-x/\omega')\cosh\left[\frac{sx}{\omega'}\right] + i\Psi^\dagger(0,t-x/\omega')\sinh\left[\frac{sx}{\omega'}\right] . \tag{12}$$

Taking the expectation value of (12) gives the result (10) as it should. This also agrees with the results obtained in Ref. 11 in the limit that the pump amplitude is constant, as we have assumed here.

To understand the stochastic noise source in (5) it is very instructive to compare the operator and stochastic equations for $\langle\Psi^2(x,\xi)\rangle$ and $\langle\psi^2(x,\xi)\rangle$. These quantities should be identical. We write Eq. (8) in differential form

$$d\psi(x,\xi) = \frac{is}{\omega'}\psi^\dagger(x,\xi)dx + \frac{\sqrt{is}}{\omega'}dY(x,\xi) , \tag{13}$$

and assuming that this is to be interpreted in the Ito sense[9] $dY(x,\xi)$ is an infinitesimal of order $\sqrt{dx}$. For this reason Ito equations do not obey the usual rules of calculus, so we require the Ito formula to show

$$d\psi^2 = 2\psi\, d\psi + (d\psi)^2 \tag{14}$$

and

$$d\langle\psi^2\rangle = \frac{is}{\omega'}[2\langle\psi^\dagger\psi\rangle dx + \frac{1}{\omega'}\langle(dY)^2\rangle] , \tag{15}$$

respectively.

Equation (11) leads to the corresponding result, i.e.,

$$\frac{d\langle\Psi^2\rangle}{dx} = \frac{is}{\omega'}(\langle\Psi\Psi^\dagger\rangle + \langle\Psi^\dagger\Psi\rangle) . \tag{16}$$

Noting that the stochastic equation can give only normally ordered averages of quantum operators, we find by equating (15) and (16) that

$$\langle[dY(x,\xi)]^2\rangle = \omega' dx\langle[\Psi(x,\xi),\Psi^\dagger(x,\xi)]\rangle , \tag{17}$$

consistent with the assumption that dY is an infinitesimal of order $\sqrt{dx}$. This is a generalization of the usual one-dimensional Ito rule for the Weiner process, where $\langle[dW(t)]^2\rangle = dt$. The phase velocity factor ω' comes from the characteristic transformation and just gives dY the correct units. The presence of the commutator in (17) is not surprising, as generally quantum-noise sources arise from the need to preserve commutation relations. Most importantly it is evaluated at a single space-time point.

The operator $\Psi(x)$ defined in Eq. (4) contains a summation over k. As we have remarked, only states of the field within a finite range $(k_0-K/2,k_0+K/2)$ participate in the interaction. Thus when one evaluates the expectation value of the commutator in (17), the sums are effectively restricted to this range. Using Eq. (4) we then find

$$\langle[\Psi(x),\Psi^\dagger(x)]\rangle = \frac{K}{2\pi} , \tag{18}$$

and hence (17) may be written

$$\langle[dY(x,\xi)]^2\rangle = \frac{\omega' K}{2\pi}dx . \tag{19}$$

The simplicity of this result indicates that Eq. (8) may be consistently interpreted as an Ito-type SDE. This interpretation may be verified by comparing the equations of motion for other moments calculated via operator and stochastic methods. Note that instead of writing Eq. (8) we could choose to parametrize the characteristic by t instead of x, in which case one finds an Ito SDE, with independent variable t and the corresponding noise correlation

$$\langle[dY(x_0+\omega' t,t)]^2\rangle = K dt/2\pi .$$

We are free to choose either x or t as independent variable of the characteristic equations. Having achieved our goal in this section we do not further discuss the parametric converter here. A discussion of the generation and detection of short pulses of squeezed light by traveling-wave parametric conversion is given in Refs. 6 and 11.

III. GENERATION OF QUANTUM SUPERPOSITIONS BY UNITARY EVOLUTION OF A NONLINEAR OSCILLATOR

Before turning our attention to the main problem of interest, namely, the generation of quantum-field-superposition states in an extended nonlinear medium, we review the seminal work of Yurke and Stoler[7] on the nonlinear single-mode oscillator. We also introduce the method based on Ito's formula[10] which is used in Sec. IV to calculate a hierarchy of moments for the traveling-wave problem.

Yurke and Stoler[7] considered a nonlinear oscillator governed by a Hamiltonian, which may be written in the normally ordered form

$$H = \hbar\omega a^\dagger a - \hbar\chi a^{\dagger 2} a^2 . \tag{20}$$

This Hamiltonian has been previously used by Drummond and Walls[12] as the basis of a quantum theory of dispersive optical bistability, and by Milburn and Holmes[13] in studies of the correspondence between classical and quantum dynamics. The latter works also indicate a general relationship between the destruction of macroscopic quantum coherence and classical dynamical behavior. In particular, since $a^\dagger a$ is conserved, and if the initial state of the oscillator is a coherent state $|\alpha\rangle$, the state vector evolves periodically in time with period $2\pi/\chi$ through superpositions of coherent states. A most interesting state is reached after a time $\pi/2\chi$ (one quarter period) and periodically thereafter, when the oscillator is in a superposition state with possibly "macroscopic" amplitude α, given by

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(e^{-i\pi/4}|\alpha\rangle + e^{i\pi/4}|-\alpha\rangle) . \tag{21}$$

Generally the oscillator evolves through a series of quantum-superposition states which are "macroscopically" distinguishable if α is large enough. These results were obtained by a simple analysis of the Schrödinger equation. Since the state vector is known, any observable may be calculated straightforwardly. We now show how a hierarchy of normally ordered moments may be calculated exactly using the Ito SDE's derived in Ref. 12. This is a purely academic exercise in the present context, as the Schrödinger equation is easily solved for this example. However, there is a close relationship between the present model and self-phase modulation of traveling-wave packets, our main interest here. Identical methods turn out to be fruitful there also, and agree with operator methods.[14]

From Ref. 12 the Ito equations for the independent variables α and $\alpha^\dagger$ associated with the operators a and $a^\dagger$, respectively, are given in a frame rotating at frequency ω_L by

$$\begin{aligned} d\alpha(t) = \{ & [-i(\omega-\omega_L) + i2\chi\alpha^\dagger(t)\alpha(t)]dt \\ & + \sqrt{i2\chi}\, dW(t)\}\alpha(t) \end{aligned} \tag{22}$$

and

$$\begin{aligned} d\alpha^\dagger(t) = \{ & [i(\omega-\omega_L) - i2\chi\alpha^\dagger(t)\alpha(t)]dt \\ & + \sqrt{-i2\chi}\, dW^\dagger(t)\}\alpha^\dagger(t) , \end{aligned} \tag{23}$$

where the linear drift term involving the arbitrary frequency ω_L arises here, as the Hamiltonian in Eq. (20) is normally ordered. The independent Weiner increments $dW(t)$ and $dW^\dagger(t)$ have zero mean and satisfy[10]

$$\langle [dW(t)]^2 \rangle = \langle [dW^\dagger(t)]^2 \rangle = dt . \tag{24}$$

We ignore any losses or thermal fluctuations in this section, but we will indicate how losses affect our results in Sec. VI.

To obtain the moments of interest we consider the time dependence of the function $f_{q,p}(\alpha,\alpha^\dagger) = \langle (\alpha^\dagger\alpha)^q \alpha^p \rangle$ with q and p arbitrary integers, by using Eqs. (22) and (23) and the multivariate Ito formula.[10] We find $f_{q,p}$ obeys the hierarchy

$$\frac{d}{dt} f_{q,p} = i\Omega_{q,p} f_{q,p} + i2\chi p f_{q+1,p} \tag{25}$$

with

$$\Omega_{q,p} = \chi p (2q+p) . \tag{26}$$

The definition (26) assumes that the choice $\omega_L = \omega + \chi$ has been made. The origin of this choice comes from reordering the Hamiltonian (20) so that its interaction part is given by $\hbar\omega(a^\dagger a)^2$. Frequency ω_L then corresponds to the oscillator frequency in Ref. 7. Equation (25) may be solved exactly to give

$$f_{q,p}(t) = e^{i\Omega_{q,p}t} \sum_{n=0}^{\infty} f_{q+n,p}(0) \frac{1}{n!}(e^{i2\chi p t} - 1)^n . \tag{27}$$

For an initial coherent state $|\beta\rangle$, $f_{q+n,p} = |\beta|^{2(n+q)}\beta^p$ and Eq. (27) gives

$$f_{q,p}(t) = e^{i\Omega_{q,p}t} \beta^p |\beta|^{2q} \exp[-|\beta|^2(1 - e^{i2\chi p t})] \tag{28}$$

with, for example,

$$\langle \alpha^p \rangle = \langle a^p \rangle = \beta^p \exp[i\chi p^2 t - |\beta|^2(1 - e^{i2\chi p t})] \tag{29}$$

and the complex conjugate of (29) for $\langle \alpha^{\dagger p} \rangle = \langle a^{\dagger p} \rangle$. These results agree with the Schrödinger analysis of Yurke and Stoler.[7] In particular, at time $t = \pi/2\chi$ (and periodically thereafter) the system is in the superposition state (21), and the form of the moments at this time are a signature of the superposition state.

IV. QUANTUM FLUCTUATIONS OF TRAVELING WAVES IN A KERR MEDIUM

We now focus our attention on the quantum-statistical properties of traveling waves in a Kerr medium. A quantum theory may be formulated by starting with the Hamiltonian[2]

$$H = \sum_k \hbar\omega_k a_k^\dagger a_k - \frac{\epsilon_0 \chi^{(3)}}{4\epsilon^4} \int d\mathbf{x} : D^4(\mathbf{x}) , \tag{30}$$

where $\chi^{(3)}$ is the third-order susceptibility, and the displacement field operator is as defined in Eq. (2). In common with Sec. II we consider propagation along a medium of length L and uniform mode cross section A. The SPDE's for this model have been given in Ref. 2, where a stochastic nonlinear Schrödinger equation was derived as the continuum limit of a set of ordinary Ito SDE's. Here we do not wish to consider the effects of group-velocity dispersion, and so we drop the second-order derivative in the spatial coordinate x, as well as any higher-order derivatives. The resultant SPDE may be written in a frame rotating at frequency $\omega_0 + \Delta\omega$ in the form

$$\left[\frac{\partial}{\partial t}+\omega'\frac{\partial}{\partial x}\right]\psi(x,t)=[\,i\Delta\omega+i\sigma\psi^{\dagger}(x,t)\psi(x,t)+\sqrt{i\sigma}\,\eta(x,t)]\psi(x,t)\ , \tag{31}$$

where $\sigma\equiv 3\epsilon_0\chi^{(3)}\hbar\omega_0^2/4A\epsilon^2$, $\Delta\omega$ is for the moment arbitrary, and other symbols may be interpreted in the same sense as in Sec. II, with $\omega_0=\omega' k_0$. Note that Eq. (31) is considerably more complicated than Eq. (5), as it has both a nonlinear drift (self-phase modulation) and multiplicative noise.

Assuming the method of characteristics may be applied, we proceed in the same manner as in Sec. II and find an equation formally similar to Eqs. (13) and (22), i.e.,

$$d\psi(\mu,t)=\{\,[i\Delta\omega+i\sigma\psi^{\dagger}(\mu,t)\psi(\mu,t)]dt+\sqrt{i\sigma}\,dY(\mu,t)\}\psi(\mu,t)\ , \tag{32}$$

where $(\mu,t)\equiv(x_0+\omega' t,t)$. A corresponding equation exists for $\psi^{\dagger}(\mu,t)$. The equation of motion for the field operator $\Psi(\mu,t)$ is found to be given by (32) without the noise term, and the substitution of the operators Ψ and $\Psi^{\dagger}$ for ψ and $\psi^{\dagger}$, respectively. By applying the same comparisons as in Sec. II, with restriction of the field states to the finite range $(k_0-K/2,k_0+K/2)$, one may again show that

$$\langle [dY(\mu,t)]^2\rangle=\langle [\Psi(\mu,t),\Psi^{\dagger}(\mu,t)]\rangle dt =Kdt/2\pi$$

(technically we also need to use the nonanticipating property of the Ito calculus[10] to prove this relationship since the noise in (32) is multiplicative). This is essentially the regularization procedure mentioned in the Introduction. Hence the characteristic equation (32) may be consistently interpreted as an Ito SDE, and therefore standard stochastic techniques are applicable. In particular, following the method of Sec. III we derive an equation for

$$F_{q,p}(\psi^{\dagger},\psi)=\langle [\psi^{\dagger}(\mu,t)\psi(\mu,t)]^q[\psi(\mu,t)]^p\rangle\ ,$$

using the Ito formula.[10] We find

$$\frac{d}{dt}F_{q,p}=i\sigma pF_{q+1,p}+i\Lambda_{q,p}F_{q,p}\ , \tag{33}$$

where

$$\Lambda_{q,p}=p\left[\Delta\omega+\frac{\sigma K}{4\pi}(2q+p-1)\right]. \tag{34}$$

Equation (33) has the solution

$$F_{q,p}(x,t)=e^{i\Lambda_{q,p}t}\sum_{n=0}^{\infty}F_{q+n,p}(x-\omega' t,0)\frac{1}{n!}\left[\left[\frac{2\pi}{K}\right](e^{ip\sigma Kt/2\pi}-1)\right]^n . \tag{35}$$

This is an exact solution for a certain hierarchy of normally ordered moments. Note the formal similarity between Eqs. (35) and (27) for the nonlinear oscillator. With an initial coherent state the oscillator evolves periodically to a superposition state whose normally ordered moments may be found from (27). In Sec. V we pursue an analogous situation for the traveling-wave problem in which a localized coherent pulse is propagated in the Kerr medium. We find that the coherent pulse evolves into a field of coherent superpositions periodically as it propagates into the medium.

V. COHERENT-WAVE-PACKET INPUT

We are now in the position to discuss the effects of self-phase modulation on the quantum fluctuations of coherent pulses as they propagate in a lossless Kerr medium. First we define the coherent input state and then use it to evaluate initial conditions for Eq. (35), so that normally ordered moments for the localized pulse may be calculated.

A coherent wave packet may be defined as the tensor product of single-mode coherent states

$$|\{\beta\}\rangle=\prod_k|\beta_k\rangle\ , \tag{36}$$

where $a_k|\{\beta\}\rangle=\beta_k|\{\beta\}\rangle$, for each k. The modes are assumed to be symmetrically distributed in the range $(k_0-K/2,k_0+K/2)$ around wave vector k_0, with $K\ll k_0$. As already indicated in Eq. (4), the local-field operator Ψ is closely related to the positive-frequency component of the displacement field in the medium; explicitly using the approximation

$$\left[\frac{\epsilon\hbar\omega_k}{2V}\right]^{1/2}\cong\left[\frac{\epsilon\hbar\omega_{k_0}}{2V}\right]^{1/2}, \tag{37}$$

we find

$$D^{+}(x,0)\cong i\left[\frac{\epsilon\hbar\omega_{k_0}}{2A}\right]^{1/2}\Psi(x)e^{ik_0x}\ , \tag{38}$$

where we have used the definitions (2) and (4). The initial spatial envelope propagated into the medium is then given by

$$\langle\{\beta\}|\Psi(x)|\{\beta\}\rangle=\frac{1}{\sqrt{L}}\sum_k\beta_{k_0+k}e^{ikx}\ . \tag{39}$$

Writing the sum over k as an integral

$$\sum_k\beta_{k_0+k}e^{ikx}=\frac{L}{2\pi}\int_{-K/2}^{K/2}dk\,B(k)e^{ikx}\ , \tag{40}$$

we may perform the integral for various functions $B(k)$. Here we choose the Lorentzian[15]

$$B(k)=\frac{2\kappa\beta_0}{L}\frac{1}{\kappa^2+k^2}\ , \tag{41}$$

with $\kappa \ll K \ll k_0$, so that the integration limits in (40) may be extended to $\pm\infty$ without error to give

$$\langle \{\beta\} \mid \psi(x) \mid \{\beta\} \rangle = \frac{1}{\sqrt{L}}\beta_0 e^{-\kappa|x|} \equiv \beta(x) \ . \tag{42}$$

Using these methods it is straightforward to calculate the initial condition for our moment hierarchy which corresponds to the coherent wave packet. In terms of the last result this may be written

$$F_{n,p}(x,0) = |\beta(x)|^{2n}\beta^p(x) \ . \tag{43}$$

Substituting this into Eq. (35) and fixing $\Delta\omega = K\sigma/4\pi$ (compare with the discussion in Sec. III) we find

$$F_{q,p}(x,t) = e^{ip\Delta\omega(2q+p)t}|\beta(x-\omega' t)|^{2q}\beta^p(x-\omega' t)\exp\left[-\frac{2\pi}{K}|\beta(x-\omega' t)|^2(1-e^{2ip\Delta\omega t})\right] , \tag{44}$$

where, for example,

$$\langle \psi^p(x,t)\rangle = e^{ip^2\Delta\omega t}\beta^p(x-\omega' t)\exp\left[-\frac{2\pi}{K}|\beta(x-\omega' t)|^2(1-e^{i2\Delta\omega p t})\right] . \tag{45}$$

Comparison of Eqs. (44) and (45) with the corresponding single-mode results, (28) and (29), indicates that superposition states analogous to Eq. (21) are produced at positions x centrally localized around the space points $x_r = \omega' t_r$ defined by

$$x_r = \omega'\frac{\pi}{2\Delta\omega}(1+4r) \quad (r \text{ integer}) \ . \tag{46}$$

Generally the state of the wave packet will evolve through a series of quantum-field superpositions periodically as it propagates, in analogy with the single-mode problem. The principal difference is that in the present case the coherent amplitude which characterizes the state of the wave packet is a function of a continuous variable x. This parametric dependence leads to a field of superposition states which periodically recur. In the single-mode problem the coherent-state amplitude is related to a single superposition state.

It is interesting to note that for $\Delta\omega t \ll 1$, the short-interaction limit, the moments reduce to the classical self-phase modulation solutions.

VI. EFFECT OF DISSIPATION

Here we address the problem of how dissipation affects the quantum coherences responsible for the periodic evolution. The role of dissipation on the nonlinear oscillator has already been considered in detail by Milburn and Holmes,[13] who discussed how loss destroys quantum coherences and leads to dynamical behavior similar to the classical Liouvillian evolution. Inclusion of linear losses with decay rate γ modifies our Eqs. (22) and (23) by the substitutions $\omega-\omega_L \to \omega-\omega_L - i\gamma/2$, and complex conjugate, respectively.[12] The analysis then proceeds as before with the replacement $\Omega_{q,p} \to \Omega'_{q,p} \equiv \Omega_{q,p} + i\gamma(2q+p)/2$. We then find the loss-modified solution of Eq. (25) to be

$$f_{q,p}(t) = e^{i\Omega'_{q,p}t}\sum_{n=0}^{\infty} f_{q+n,p}(0)\frac{1}{n!}\left[\frac{(e^{i2\chi p t(1+i\gamma/2\chi p)}-1)}{1+i\gamma/2\chi p}\right]^n , \tag{47}$$

which for the oscillator initially in a coherent state $|\beta\rangle$ yields

$$f_{q,p}(t) = e^{i\Omega'_{q,p}t}|\beta|^{2q}\beta^p\exp\left[-\frac{|\beta|^2}{1+i\gamma/2\chi p}(1-e^{i2\chi p t(1+i\gamma/2\chi p)})\right] . \tag{48}$$

With $q=0, p=1$ this agrees with Eq. (15) of Ref. 13. Following the discussion of Milburn and Holmes,[13] the deleterious effects of loss are minimized when the nonlinearity and coherent-state amplitude satisfy $\gamma/2\chi \ll 1$ and $\gamma t|\beta|^2 \ll 1$. For large-amplitude coherent states dissipation destroys quantum coherence fastest, and thus places an upper bound on the macroscopic amplitude for which any periodic evolution is observable. It is expected, however, that this can be enhanced by squeezing the loss modes.[8] Note also that pth-order moments decay exponentially with a rate $p\gamma$.

A similar analysis may be carried out for a propagating coherent wave packet with distributed losses in the medium. There are several different situations which can be envisaged, for example, inhomogeneous losses or losses which occur over certain frequency ranges which may be large or small compared with the pulse bandwidth. For simplicity here we restrict our analysis to a homogeneous loss γ, which is uniform over the pulse bandwidth. The loss modifies Eq. (32) by the replacement $\Delta\omega \to \Delta\omega + i\gamma/2$. This can be justified by master equation methods. The analysis then proceeds as before, with the substitution $\Lambda_{q,p} \to \Lambda'_{q,p} \equiv \Lambda_{q,p} + i\gamma(2q+p)/2$. We then find that for the coherent wave packet, losses may be incorporated in the moment equation (44) by the replacements $\Lambda_{q,p} \to \Lambda'_{q,p}$ and $K/2\pi \to K/2\pi + i\gamma/p\sigma$ (note that $\Delta\omega$ depends on K).

To estimate the conditions necessary to generate such a superposition state we note that the interaction length x_0 [Eq. (46)] is independent of the pump amplitude. This contrasts with, for example, the generation of optical squeezed states where the effective interaction lengths (in cavity or traveling-wave devices) may be reduced by increasing the pump power, provided induced losses are not too great. Of course the destruction of the quantum superposition by loss does depend on the pump intensity. Squeezed states are generated in regimes where quantum fluctuations act as small perturbations on the classical evolution, in contrast to the fully nonlinear quantal analysis here for superposition states.

We require the inequalities $\gamma \ll \Delta\omega$, $x_0 \ll \omega'\gamma$, and $x_0 \ll \omega'/\gamma\,|\beta_0|^2$ to be satisfied so that loss does not destroy the superposition generated after a propagation length x_0 [Eq. (46)]. For a silica glass fiber[16] [$\chi^{(3)} \cong 2.4\times 10^{-23}$ mks; $\gamma \cong 10^6$ s^{-1}, $A=\pi(2\mu m)^2$] $\sigma \cong 10^{-4}$ ms^{-1}. This gives $\Delta\omega \cong K$ (in m^{-1})$\times 10^{-5}$ s^{-1}. Since $K \ll$ optical wave vector 10^7 m^{-1}, $\Delta\omega \ll 100$ s^{-1}; the first of the inequalities is therefore not satisfied. As $\omega'/\gamma \cong 300$ m and $x_0 = \omega'\pi/2\Delta\omega > 10^6$ m $\gg 300$ m, the second inequality is not satisfied either. In principle, if σ could be increased by several orders of magnitude, by operating at higher frequency ω_0 ($\sigma \propto \omega_0^2$) in a fiber with much larger $\chi^{(3)}$, and smaller mode cross section A, then the generation of such superpositions is more feasible. This assumes that losses are not increased. Moreover, since we have entirely neglected group-velocity dispersion, ω_0 should correspond to a zero first-order dispersion wavelength, and this is fixed by the medium concerned. Over such large distances neglect of higher-order dispersive phase shifts is somewhat dubious, though our estimates do indicate the experimental difficulties involved in the optical regime. Resonant enhancement of material nonlinearities is usually accompanied by increased losses through spontaneous emission so that the situation is difficult to improve (in two-photon resonant excitation of atomic levels in the Λ configuration, however, this is not necessarily the case[17]).

VII. PHASE SPACE AND REVERSIBILITY

We next wish to discuss the problem of time reversibility in our stochastic equations. The existence of time-reversed solutions to the unitary time evolution of a closed quantum system apparently distinguishes the normal quantum theory of time evolution from our stochastic theory. It is certainly true that our transformed equations are intrinsically non-time-reversible. This occurs because the distributions satisfy a Fokker-Planck equation with semi-positive-definite diffusion. At a trivial level, one can just reverse the sign of time in this Fokker-Planck equation to obtain time-reversed results. However, this does not take into account the fact that the diffusion array is always transformed into an equivalent positive definite form prior to solving the equations of motion. The corresponding stochastic equations are apparently not time reversible.

In order to clarify this situation, we reconsider the nonlinear oscillator. In this case we have from Eq. (22)

$$\frac{d\alpha}{dt} = [-i(\omega-\omega_L) + 2i\chi\alpha^\dagger\alpha + \sqrt{2i\chi}\,\xi_I(t)]\alpha \;, \tag{49}$$

where we have written the Ito equation as a differential equation in $\alpha(t)$, with $\xi_I(t)$ the Ito white noise. next, as we wish to make a change of variable, it is simplest to use the Stratonovich stochastic calculus. Transforming Eq. (49) to the equivalent Stratonovich form, we find

$$\frac{d\alpha}{dt} = [-i(\omega+\chi-\omega_L) + 2i\chi\alpha^\dagger\alpha + \sqrt{2i\chi}\,\xi_S(t)]\alpha \;, \tag{50}$$

where $\xi_S(t)$ is the Stratonovich white noise. In the Stratonovich calculus, variable changes follow the standard rules of ordinary calculus. We make a change to polar coordinates, on defining $\alpha \equiv e^z$,

$$\frac{dz}{dt} = -i\Delta + 2i\chi\exp(z+z^\dagger) + \sqrt{2i\chi}\,\xi_S(t) \;, \tag{51}$$

where $z \equiv \ln\alpha$, $z^\dagger \equiv \ln\alpha^\dagger$, and $\Delta \equiv \omega - \omega_L + \chi$. Note that in the case where losses are included we must replace Δ as in Sec. VI, by $\Delta - i\gamma/2$.

It is convenient to define new variables Z and Φ by

$$Z \equiv \ln(\alpha^\dagger\alpha) = z + z^\dagger \;, \tag{52}$$

$$\Phi \equiv \frac{1}{2i}\ln\left[\frac{\alpha^\dagger}{\alpha}\right] = \frac{1}{2i}(z^\dagger - z) \;. \tag{53}$$

These satisfy the following differential equations:

$$\frac{dZ}{dt} = -\gamma + \sqrt{2i\chi}[\xi_S(t) + i\xi_S^\dagger(t)] \;, \tag{54}$$

$$\frac{d\Phi}{dt} = \Delta - 2\chi e^Z + \sqrt{i\chi/2}[\xi_S^\dagger(t) + i\xi_S(t)] \;. \tag{55}$$

Since the equation for Z does not involve Φ, it can be treated directly. The exact formal solutions are

$$Z(t) = Z_0 - \gamma t + \sqrt{2i\chi}\int_0^t [\xi_S(t') + i\xi_S^\dagger(t')]dt' \;, \tag{56}$$

$$\Phi(t) = \Phi_0 + \int_0^t \{\Delta - 2\chi\exp[Z(t')] + \sqrt{i\chi/2}[\xi_S^\dagger(t') + i\xi_S(t')]\}dt' \;. \tag{57}$$

We note that moments may be written in the form

$$\langle \alpha^n \alpha^{\dagger n'} \rangle = \langle \exp[(n'+n)Z/2 + i\Phi(n'-n)] \rangle \;. \tag{58}$$

There are only analytic moments in the physical observables; any moments of the form $(\alpha)^*$ or $(\alpha^\dagger)^*$ do not correspond with the physical observables. The simplest example is the photon number $\langle n \rangle = \langle a^\dagger a \rangle = \langle \alpha^\dagger \alpha \rangle$. In the case where $\gamma = 0$, this reduces to

$$\langle n \rangle = \langle e^Z \rangle \;. \tag{59}$$

Since $Z(t)$ is a Gaussian process with mean Z_0, the result reduces to

$$\langle n \rangle = \exp[Z_0 + \tfrac{1}{2}\langle (Z-Z_0)^2 \rangle] \;. \tag{60}$$

It is significant that the fluctuation term $\langle (Z-Z_0)^2 \rangle$ is written in terms of analytic functions of Z. These average to zero, since

$$\langle (Z-Z_0)^2\rangle=\left\langle 2i\chi\int_0^t\int_0^t[\delta(t'-t'')-\delta(t'-t'')]dt'dt''\right\rangle=0\ . \tag{61}$$

Hence $\langle n\rangle=\langle e^{Z_0}\rangle$ is a constant of motion for $\gamma=0$, and so is reversible; the time-reversed stochastic equations have an identical behavior.

When losses are included, the solution for $\langle n\rangle$ is just $\exp(Z_0-\gamma t)$. In this case the solutions are not time reversible, for obvious reasons. Now the system in question includes an irreversible loss to a reservoir. Tracing out the dynamical correlations between system and reservoir degrees of freedom leads to time-irreversible evolution for the system.

In summary, the stochastic moments of physical significance have a time-reversible behavior in the absence of loss. However, there also exist nonphysical moments of the stochastic equations which can exhibit irreversible behavior. These do not interfere with the physically observable time-reversible behavior, hence allowing stochastic methods to be utilized without violating unitarity.

VIII. DISCUSSION

We have investigated the quantum fluctuations of traveling waves using the method of characteristics applied to a pair of stochastic equations. The characteristics in the examples presented here are straight lines, and we have shown that the equations may be interpreted as Ito-type SDE's. Furthermore, we have observed that the noise is proportional to the square root of the volume of momentum space K, which the interacting field states occupy. This arises from the connection between the noise source and the expectation value of the quantum-mechanical commutator expressed in Eq. (17).

Our major result was to show that self-phase modulation can in principle produce a field of quantum-superposition states when a coherent pulse is propagated in a lossless Kerr medium. The pulse localization in configuration space x is mapped parametrically to a phase-space interference pattern by which a quantum-superposition state is manifest; for each position x, the superposition is related to the retarded coherent amplitude $\beta(x-\omega' t)$ in a way reminiscent of the single-mode nonlinear oscillator.[7,13] Recent discussions of macroscopic-superposition states[18] have indicated their sensitivity to fluctuations and dissipation, which tend to cause rapid decay of quantum coherence. This extreme sensitivity may be reduced somewhat if the (radiative) loss modes are squeezed.[8] In principle, macroscopic-superposition states may be observed by characteristic interference signals in homodyne detection.[7] Due to the smallness of optical nonlinearities, however, candidate media such as optical fibers cannot realistically present a long enough interaction length to generate quantum superpositions with present technology. Resonant enhancement of nonlinearities is a partial remedy, though this introduces excess quantum fluctuations and concomitant increase in losses,[19] which are not treated in the present idealized model.

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