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Covariant phase-space representation for harmonic oscillators

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It is shown that, in the phase-space representation of quantum mechanics, the uncertainty relation can be stated in terms of the integral invariant of Poincaré. The uncertainty relation for spreading free wave packets is discussed as an illustrative example. This phase-space approach can be extended to the relativistic regime. It is shown that Lorentz boosts are area-preserving canonical transformations in the phase space of the light-cone variables. The harmonic oscillator is discussed in detail as an illustrative example for the covariant realization of the uncertainty principle.

I. INTRODUCTION

The phase-space representation of quantum mechanics[1] is of current interest. We have shown in our previous paper[2] that the light-cone coordinate system is the natural language for the Lorentz-covariant phase-space representation of quantum mechanics. The localized light wave was discussed as an illustrative example. However, the light wave depends on only one of the two light-cone variables.

The purpose of this paper is to discuss a physical example depending on both of the light-cone variables. The covariant harmonic oscillator will serve this purpose. The harmonic oscillator occupies a very prominent place in the physics of phase space. Unless we know how to deal with the covariance of the harmonic oscillator, we are not likely to understand the covariance of phase space.

In the phase-space representation, the uncertainty relation is stated in terms of the integral invariant of Poincaré,[3] which is called the "error box" in the current literature.[4] The area of the error box cannot be smaller than Planck's constant. In order to illustrate the advantage of using the phase-space representation, we start with the problem of wave-packet spreads. In the Schrödinger picture, the uncertainty product increases as time progresses or regresses. On the other hand, the volume of the error box remains constant for the spreading wave packet, even though its shape is deformed.[5] This means that the wave-packet spread is a canonical transformation in phase space.

The phase-space representation of nonrelativistic quantum mechanics has a built-in symmetry which is mathematically equivalent to that of the $(2+1)$-dimensional Lorentz group. Since the position and momentum variables are c numbers in the phase-space representation, it is possible to formulate canonical transformations in a manner identical to the case in classical mechanics. Indeed, the group of linear canonical transformations is Sp(2) which is isomorphic to the $(2+1)$-dimensional Lorentz group.[3,6]

In this paper, we shall show that Lorentz boosts are canonical transformations in the phase space consisting of the light-cone variables. Thus, from the mathematical point of view, the Lorentz covariance does not add new complications in phase space. However, from the physical point of view, we are dealing with a realization of the uncertainty principle which is different from that in the Schrödinger picture of quantum mechanics. The phase-space representation enables us to state the uncertainty relation in a Lorentz-covariant manner.

In Sec. II, we study the phase-space representation for wave-packet spreads and compare it with the case of harmonic oscillators. It is pointed out that the uncertainty relation can be stated in terms of the area of the error box in phase space. The area of the error box remains unchanged as time progresses or regresses. It is shown that the error boxes for the spreading wave packet and the harmonic oscillator coincide with each other in the large-time and weak-spring-constant limits, respectively. In Sec. III, the wave-packet spread is formulated in terms of homogeneous linear canonical transformations in phase space.

Sections IV and V consist mostly of reviews which are needed for the covariant formulation given in Sec. VI. Section IV deals with the phase-space representation for nonrelativistic harmonic oscillators. Section V is based on the covariant harmonic-oscillator formalism which constitutes a representation of the Poincaré group, and which effectively describes the basic phenomena of relativistic hadrons in the quark model.[7]

In Sec. VI, we combine Secs. IV and V to formulate the covariant phase-space representation of harmonic oscillators. The conclusion of the present paper is that the concept of the covariant error box in phase space gives the physical basis for the phenomenology based on the mathematics of the covariant harmonic-oscillator formalism.

II. WAVE-PACKET SPREADS

Let us start with a one-dimensional harmonic oscillator in its ground state. The uncertainty product remains invariant when the spring constant is gradually reduced. On the other hand, when the oscillator force is suddenly removed, the uncertainty product becomes dependent on time. Although these two different cases are two different manifestations of one physical principle, the wave-packet spread has been one of the agonizing features of the present form of nonrelativistic quantum mechanics.

In the phase-space representation of quantum mechanics, the uncertainty relation can be stated in a time-independent manner for both the harmonic oscillator and the spreading wave packet. If $\psi(x,t)$ is the physical solution of the Schrödinger equation, the phase-space distribution function is defined as[1]

$$P(x,p,t)=\frac{1}{\pi}\int \psi^*(x+y,t)\psi(x-y,t)e^{2ipy}dx \ , \tag{2.1}$$

which we shall hereafter call the PSD function. This is a function of t, x, and p, which are c numbers. This function is real but is not necessarily positive everywhere in the phase space of x and p. We can, however, recover the positive distribution functions in the position and momentum coordinates

$$\begin{aligned}\rho(x,t)&=|\psi(x,t)|^2=\int P(x,p,t)dp \ ,\\ \sigma(p,t)&=|\phi(p,t)|^2=\int P(x,p,t)dx \ ,\end{aligned} \tag{2.2}$$

where $\phi(p)$ is the momentum wave function.

The time-dependent Schrödinger equation leads to the differential equation[1]

$$\frac{\partial}{\partial t}P(x,p,t)=-\left[\frac{p}{m}\right]\frac{\partial}{\partial x}P(x,p,t)+\sum_{n=0}^{\infty}\left[\frac{1}{2}\right]^{2n}\frac{1}{(2n+1)!}\left[\left[\frac{\partial}{\partial x}\right]^{2n+1}V(x)\right]\left[\frac{\partial}{\partial x}\right]^{2n+1}P(x,p,t) \ , \tag{2.3}$$

where m is the mass of the particle, and $V(x)$ is the potential. In the case of the harmonic oscillator with $V(x)=Kx^2/2$, the above differential equation becomes

$$\frac{\partial}{\partial t}P(x,p,t)=-\left[\frac{p}{m}\right]\frac{\partial}{\partial x}P(x,p,t)+Kx\frac{\partial}{\partial p}P(x,p,t) \ . \tag{2.4}$$

If the particle is free, this differential equation becomes

$$\frac{\partial}{\partial t}P(x,p,t)=-\left[\frac{p}{m}\right]\frac{\partial}{\partial x}P(x,p,t) \ . \tag{2.5}$$

The solution of the above differential equation is[5]

$$P(x,p,t)=P(x-pt/m,p,0) \ . \tag{2.6}$$

The time evolution of this solution is illustrated in Fig. 1. Indeed, the error box undergoes a shear. The volume of the error box is invariant under time evolution. This is a more precise statement of the uncertainty relation than is given in the Schrödinger picture.

If we start with a free-particle wave packet with a Gaussian momentum wave function

$$g(k)=\left[\frac{b}{\pi}\right]^{1/4}e^{-bk^2/2} \tag{2.7}$$

at $t=0$, the time-dependent Schrödinger wave function becomes

$$\psi(x)=\left[\frac{b}{\pi}\right]^{1/4}\left[\frac{1}{b+it/m}\right]^{1/2}e^{-x^2/2(b+it/m)} \ . \tag{2.8}$$

If we construct the PSD function for the above wave function, its form is

$$P(x,p,t)=\frac{1}{\pi}\exp\{-[(x-pt/m)^2/b+bp^2]\} \ . \tag{2.9}$$

This distribution is concentrated within the region where the exponent is less than 1 in magnitude. This region is described by the tilted ellipse described in Fig. 2. This is the error box for the spreading wave packet.

Since x and p are c numbers in the phase-space representation, the PSD function $P(x,p,t)$ can be canonically transformed in phase space, as is done in classical mechanics. The concept of the error box is already in the Poisson-bracket formalism of classical mechanics. Its volume is invariant under canonical transformations. This is called the integral invariant of Poincaré. However, classical mechanics does not give the lower limit on the size of the error box.

Let us go back to the wave-packet spread. How is the spreading Gaussian wave packet different from the ground-state harmonic oscillator whose mass and spring constant become adiabatically weak? This is a situation

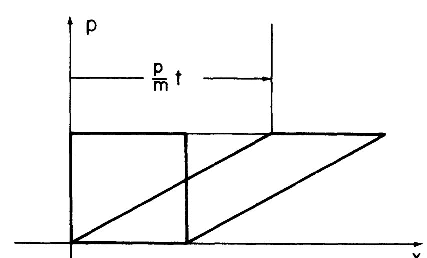

FIG. 1. Shear in phase space. Every point in phase space moves horizontally in the x direction with velocity proportional to p. This is an area-preserving transformation.

very familiar to us, and is described in Fig. 2, in which the x axis expands and the p axis contracts. This deformation is also an area-preserving canonical transformation, and is commonly called "squeeze" in the literature.[8]

The spread of the Gaussian wave packet is also illustrated in Fig. 2. This is consistent with the shear effect given in Fig. 1. It is possible to take the "projection" of this elliptic distribution to the x and p spaces using the formulas for $\rho(x)$ and $\sigma(p)$ given in Eq. (2.2). These probability densities lead to the uncertainty product in the Schrödinger picture, which expands as time progresses or regresses.

The error box for the spreading wave packet in the infinite-time limit coincides with the error box in the limit of zero spring constant. In the phase-space representation, the magnitude of uncertainty is the same for both cases. In the Schrödinger representation, one is finite while the other is infinite.

III. WAVE-PACKET SPREADS IN TERMS OF CANONICAL TRANSFORMATIONS

In order to study canonical transformation properties of the shear described in Fig. 1, let us write the solution of Eq. (2.5) given in Eq. (2.6) as

$$\begin{bmatrix} x' \\ p' \end{bmatrix} = \begin{bmatrix} 1 & t/m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} . \tag{3.1}$$

On the other hand, the group of homogeneous linear canonical transformations is Sp(2),[3,6] which is locally isomorphic to the $(2+1)$-dimensional Lorentz group.[9] Indeed, the group of homogeneous linear canonical transformations consists of rotations around the origin generated by

$$L = \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix} , \tag{3.2}$$

and squeezes[8] along the x axis and along the $x=p$ line generated by

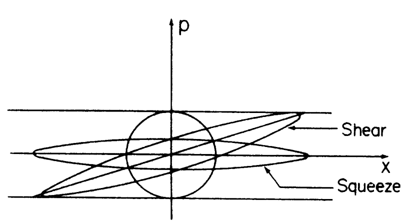

FIG. 2. Ground-state harmonic oscillator and the spread of the Gaussian wave packet in phase space. In terms of the integral invariant of Poincaré, the uncertainty relation can be stated in the same manner for both cases. Their projections to the x and p axes are different. In the oscillator case, the p distribution contracts while the x distribution expands. On the other hand, in the case of spreading wave packets, the p distribution does not change. This is why the uncertainty product for spreading wave packets increases as time progresses or regresses.

$$B_1 = \begin{bmatrix} i/2 & 0 \\ 0 & -i/2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & i/2 \\ i/2 & 0 \end{bmatrix} , \tag{3.3}$$

respectively. These generators satisfy the commutation relations

$$[B_1,B_2] = -iL, \quad [B_1,L] = -iB_2, \quad [B_2,L] = iB_1 . \tag{3.4}$$

This set of commutation relations is identical to the Lie algebra for the $(2+1)$-dimensional Lorentz group.[9] On the other hand, the shear transformation of Eq. (2.9) is generated by the matrix

$$N = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} , \tag{3.5}$$

with

$$e^{-i(t/m)N} = \begin{bmatrix} 1 & t/m \\ 0 & 1 \end{bmatrix} .$$

Then, where does the matrix N stand among the generators of canonical transformations given in Eqs. (3.2) and (3.3)?

In order to answer this question, let us construct first the squeeze operator

$$S(\eta) = e^{-i\eta B_1} = \begin{bmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{bmatrix} . \tag{3.6}$$

This operator expands the x axis while contracting p. This is of course a canonical transformation. Thus the following operators also generate homogeneous linear canonical transformations:

$$B_1' = B_1 = S(\eta)B_1S(-\eta), \quad B_2'(\eta) = e^{-\eta}S(\eta)B_2S(-\eta) , \tag{3.7}$$

$$L'(\eta) = e^{-\eta}S(\eta)LS(-\eta) . \tag{3.8}$$

In the limit of large η, the above operators become B_1, N, and N, respectively. These operators do not form a closed Lie algebra of any group, but give a partial view of a more complete group-theoretical picture that both the Lorentz-boosted O(3) and O(2,1) become a group locally isomorphic to the two-dimensional Euclidean group in the large-η limit.[7,10]

In the present case, the O(3)-like group is generated by L, L_1, and L_2, where

$$L_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} . \tag{3.9}$$

The O(2,1)-like group is generated by L_1, B_2, and B_3, where

$$B_3 = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} . \tag{3.10}$$

The mathematics of this process is called the contraction of O(3) and O(2,1) to E(2).[7,10] The physical content of this process is that a unified description can be given for the O(3)-like, E(2)-like, and O(2,1)-like internal space-time symmetries of massive, massless, and imaginary-mass particles, respectively, as Einstein's $E=(m^2+p^2)^{1/2}$ unifies the energy-momentum relation for all relativistic particles. This point has been discussed in the literature,[10,11] and is illustrated in the first and second rows of Fig. 3.

IV. HARMONIC OSCILLATORS

The Hamiltonian for the one-dimensional nonrelativistic harmonic oscillator with unit frequency can be written as

$$H=\tfrac{1}{2}(P^2+x^2) . \tag{4.1}$$

The normalized solutions of the Schrödinger equation are

$$\psi_n(x)=[1/(\sqrt{\pi}2^n n!)]^{1/2}H_n(x)\exp(-x^2/2) , \tag{4.2}$$

where $H_n(x)$ is the Hermite polynomial of nth order. These wave functions are in the energy eigenstates. It is possible to evaluate the quantum PSD function

$$P_n(x,p)=\frac{1}{\pi}\int \psi_n^*(x+y)\psi_n(x-y)e^{2ipy}dy . \tag{4.3}$$

The result of the calculation is[12]

$$P_n(x,p)=\left(\frac{n!}{\pi}\right)[\exp(-r^2)]\sum_{k=0}^{\infty}(-1)^k 2^{n-k}r^{2(n-k)}/\{[(n-k)!]^2 k!\} , \tag{4.4}$$

where $r^2=(x^2+p^2)$.

The above form is defined in the two-dimensional phase space spanned by x and p axes. Since it depends on x and p only through the variable r, the function is invariant under rotations around the origin. We can thus write $P_n(x,p)$ as

$$P_n(x,p)=P_n(r) . \tag{4.5}$$

As is expected, this function is positive in some regions and is negative in other regions in phase space. It vanishes on the circles on which the polynomial contained in $P_n(r)$ of Eq. (4.5) is zero.

In order to study this more systematically for the harmonic oscillator, let us see whether $P_n(x,p)$ of Eq. (3.5) can be interpreted in terms of another equation. The PSD function given in Eq. (3.5) indeed satisfies the differential equation[13]

$$-\frac{1}{2\rho}\left[\frac{d}{d\rho}\rho\left[\frac{d}{d\rho}P_n(r)\right]\right]+\tfrac{1}{2}\rho^2P_n(r)=(2n+1)P_n(r) , \tag{4.6}$$

where $\rho=\sqrt{2}r$. This is the radial part of the rotation-invariant Schrödinger equation for the harmonic oscillator in two-dimensional space spanned by the variables $\sqrt{2}x$ and $\sqrt{2}p$. If we use $R_k(\rho)$ for the normalized radial equation for the kth excited state with the eigenvalue $(k+1)$ with the orthogonality relation

$$\int_0^{\infty}\rho R_n(\rho)R_m(\rho)d\rho=\delta_{nm} , \tag{4.7}$$

then the PSD function is

$$P_n(r)=(1/\sqrt{4\pi})R_{2n}(\rho) . \tag{4.8}$$

Therefore P_n satisfies the orthogonality relation

$$\int P_m^*(x,p)P_n(x,p)dx\,dp=2\pi\int P_n(r)P_m(r)r\,dr =\delta_{nm} . \tag{4.9}$$

For the ground state, the PSD function is

$$P_0(x,p)=\frac{1}{\pi}\exp[-(x^2+p^2)] . \tag{4.10}$$

If $n=1$,

$$P_1(x,p)=\frac{1}{\pi}(x^2+p^2-\tfrac{1}{2})\exp[-(x^2+p^2)] . \tag{4.11}$$

We can then use the Schmidt orthogonalization procedure to construct the PSD function for higher values of n.

In this paper, we are interested in homogeneous linear canonical transformations in phase space consisting of rotations and squeezes along a given direction. The above

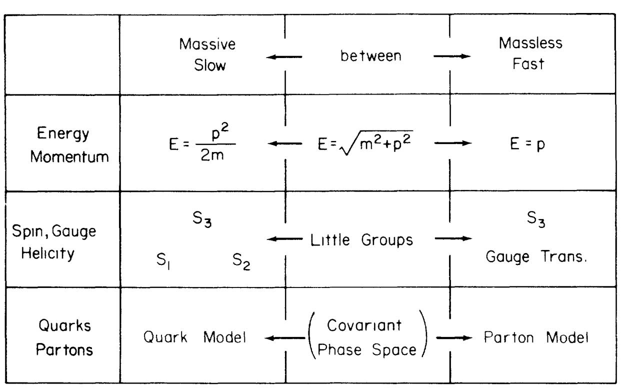

FIG. 3. Slow and fast particles. Einstein's $E=(P^2+m^2)^{1/2}$ unifies the energy-momentum relations for massive (nonrelativistic) particles and for massless particles. The second row indicates that the little group of the Poincaré group unifies the internal space-time symmetries of massive and massless particles, as is discussed in Ref. 11. The third row states that the covariant phase-space representation forms the physical basis for the covariant harmonic-oscillator formalism which has been shown to give a unified picture of quark model and the parton picture at the phenomenological level. This point is discussed in Sec. VI of the present paper.

PSD functions are rotationally invariant. The squeeze along the x direction means that the coordinate variable x is multiplied by a real positive number b, while the p variable is divided by b. The integral measure $(dx\,dp)$ remains invariant during this process.

In this case, the group of homogeneous linear canonical transformations is generated by

$$L=-\frac{i}{2}\left[x\frac{\partial}{\partial p}-p\frac{\partial}{\partial x}\right],$$

$$B_1=\frac{i}{2}\left[x\frac{\partial}{\partial x}-p\frac{\partial}{\partial p}\right],\quad B_2=\frac{i}{2}\left[x\frac{\partial}{\partial p}+p\frac{\partial}{\partial x}\right], \tag{4.12}$$

which satisfy the commutation relations for the generators of the $(2+1)$-dimensional Lorentz group given in Eq. (3.4). While this group is one of the fundamental symmetry groups in the physics of phase space,[6] we shall see in Sec. VI that the Lorentz boost is a squeeze in the phase space of the light-cone coordinate position and momentum variables.

V. COVARIANT HARMONIC OSCILLATORS

The covariant harmonic oscillator has a long history. It was Dirac who suggested in 1945 the use of harmonic oscillators to construct representations of the Lorentz group.[14] Yukawa in 1953 studied the possibility of using the oscillator for studying relativistic composite particles.[15] However, its physical relevance was not revealed until the successful calculation of the proton form factor by Fujimura, Kobayashi, and Namiki in 1970.[16] In the 1971 paper of Feynman, Kislinger, and Ravndal, the authors point out the need for relativistic bound-state models, such as the harmonic oscillator model, in order to supplement the traditional Feynman-diagram approach which is not always effective in dealing with covariant bound-state problems.[17]

The covariant oscillator formalism has been extensively discussed in the literature.[7] It serves as one of the physical representations of the Poincaré group. At the same time, the formalism allows us to explain the peculiarities of Feynman's parton picture in terms of the bound-state quark model.[7,18]

Let us start with the differential equation of Feynman *et al.*[17] for a hadron of two quarks bound together by a harmonic-oscillator potential of unit strength

$$\left\{-2\left[\left[\frac{\partial}{\partial x_a^{\mu}}\right]^2+\left[\frac{\partial}{\partial x_b^{\mu}}\right]^2\right]+\left(\tfrac{1}{16}\right)(x_a^{\mu}-x_b^{\mu})^2+m_0^2\right\}\varphi(x_a,x_b)=0, \tag{5.1}$$

where x_a and x_b are space-time coordinates for the first and second quarks, respectively. This partial differential equation has many different solutions depending on the choice of variables and boundary conditions.

In order to simplify the above differential equation, let us introduce new coordinate variables[17]

$$X=(x_a+x_b)/2,\quad x=(x_a-x_b)/2\sqrt{2}. \tag{5.2}$$

The four-vector X specifies where the hadron is located in space-time, while the variable x measures the space-time separation between the quarks. In terms of these variables, Eq. (5.1) can be written as

$$\left[\frac{\partial^2}{\partial X_\mu^2}-m_0^2+\frac{1}{2}\left[\frac{\partial^2}{\partial x_\mu^2}-x_\mu^2\right]\right]\varphi(X,x)=0. \tag{5.3}$$

This equation is separable in the X and x variables. Thus

$$\varphi(X,x)=f(X)\psi(x), \tag{5.4}$$

and $f(X)$ and $\psi(x)$ satisfy the following differential equations, respectively:

$$\left[\frac{\partial^2}{\partial X_\mu^2}-m_0^2-(\lambda+1)\right]f(X)=0, \tag{5.5}$$

$$\frac{1}{2}\left[\frac{\partial^2}{\partial x_\mu^2}-x_\mu^2+(\lambda+1)\right]\psi(x)=0. \tag{5.6}$$

Equation (5.5) is a Klein-Gordon equation, and its solution takes the form

$$f(X)=\exp(\pm iP_\mu X^\mu), \tag{5.7}$$

with

$$-P^2=-P_\mu P^\mu=M^2=m_0^2+(\lambda+1),$$

where M and P are the mass and four-momentum of the hadron, respectively. The eigenvalue λ is determined from the solution of Eq. (5.6). We are using the same notation for the operator and the eigenvalue for the hadronic four-momentum. This should not cause any confusion since we are dealing only with free hadronic states with a definite four-momentum.

As for the four-momenta of the quarks p_a and p_b, we can combine them into the total four-momentum and momentum-energy separation between the quarks[17]

$$P=p_a+p_b,\quad q=\sqrt{2}(p_a-p_b), \tag{5.8}$$

where P is the hadronic four-momentum conjugate to X. The internal momentum-energy separation q is conjugate to x provided that there exist wave functions which can be Fourier transformed. If the momentum-energy wave functions can be obtained from the Fourier transformation of the space-time wave function, the differential equation in the q space is the same as the harmonic oscillator equation for the x space given in Eq. (5.6).

Since the three-dimensional harmonic oscillator is quite familiar to us, we are naturally led to consider the separation of the space and time variables in Eq. (5.6). However, the $(\mathbf{x}\, t)$ system is not the only coordinate system in which the differential equation is separable. If the hadron moves along the z direction with velocity parameter β, the hadronic rest frame is important. In this frame, the coordinate variables are

$$\begin{aligned} &x'=x,\quad y'=y\ ,\\ &z'=(z-\beta t)/(1-\beta^2)^{1/2}\ ,\\ &t'=(t-\beta z)/(1-\beta^2)^{1/2}\ . \end{aligned} \tag{5.9}$$

The differential equation of Eq. (5.6) is separable also in these variables:

$$\frac{1}{2}\left[-\nabla'^2+\frac{\partial^2}{\partial t'^2}-[(\mathbf{x}')^2-t'^2]\right]\psi(x)=(\lambda+1)\psi(x)\ . \tag{5.10}$$

The solution of this equation consists of a product of four one-dimensional oscillator wave functions. The x' and y' components are not affected by the boost along the z direction. Thus we can drop them from our consideration. As for the t' component, the excitation contributes a negative number to λ. However, this excitation can be suppressed on the grounds that the time and energy variables are c numbers. Indeed, the time-energy uncertainty relation is a c-number relation.[19] This suppression of timelike excitations can be achieved by the subsidiary condition

$$P_\mu\left[x^\mu-\frac{\partial}{\partial x_\mu}\right]\psi_\beta(x)=0\ . \tag{5.11}$$

Then the wave function takes the form

$$\psi_\beta^n(z,t)=[1/(\pi 2^n n!)]^{1/2}H_n(z')\exp[-\tfrac{1}{2}(z'^2+t'^2)]\ . \tag{5.12}$$

This normalizable wave function describes the internal space-time structure of the hadron moving along the z direction. If $\beta=0$, then the wave function becomes

$$\psi_0^n(z,t)=[1/(\pi 2^n n!)]^{1/2}H_n(z)\exp[-\tfrac{1}{2}(z^2+t^2)]\ . \tag{5.13}$$

Thus

$$\psi_\beta^n(z,t)=\psi_0^n(z',t')\ . \tag{5.14}$$

VI. COVARIANT PHASE-SPACE REPRESENTATION FOR HARMONIC OSCILLATORS

One of the most outstanding problems in modern physics is how to formulate covariantly interactions between two elementary particles. For instance, we still do not know exactly what force is responsible for keeping the quarks inside a hadron. It may be possible to make the interaction invariant by postulating that it depends on the distance in the coordinate system at rest with the temporary center of mass of the particles, or that it depends on the two positions at the time when their relativistic distance is zero—when one is on the light cone of the other.[20]

In this paper, we take the light-cone approach. While it is not possible to solve all the problems at this time, we can discuss the uncertainty principle applicable to the space-time separation between the quarks in a harmonic system, using the light-cone variables. The covariant harmonic oscillator discussed in Sec. V serves as a theoretical tool for this purpose. The harmonic-oscillator wave function consists of a Gaussian factor and Hermite polynomials. Since the Gaussian factor determines the localization property of the wave function, let us study first the ground-state wave function, whose form is

$$\psi_0^0(z,t)=\left[\frac{1}{\pi}\right]^{1/2}\exp[-(z^2+t^2)/2]\ . \tag{6.1}$$

We have dropped the x and y variables which are not affected by the Lorentz boost along the z direction.

This wave function can be written in the light-cone coordinate system.[21] If the hadron moves along the z direction, the light-cone variables are defined to be

$$u=(t+z)/\sqrt{2},\quad v=(t-z)/\sqrt{2}\ . \tag{6.2}$$

Their Fourier conjugate variables are[18]

$$q_u=(q_z-q_0)/\sqrt{2},\quad q_v=(q_z+q_0)/\sqrt{2}\ . \tag{6.3}$$

The major advantage of using these variables is that the Lorentz boost of Eq. (5.9) takes a very simple form:[18]

$$\begin{aligned} &u'=\left[\frac{1-\beta}{1+\beta}\right]^{1/2}u,\quad v'=\left[\frac{1+\beta}{1-\beta}\right]^{1/2}v\ ,\\ &q_u'=\left[\frac{1+\beta}{1-\beta}\right]^{1/2}q_u,\quad q_v'=\left[\frac{1-\beta}{1+\beta}\right]^{1/2}q_v\ . \end{aligned} \tag{6.4}$$

Under this transformation, the products uq_u and uq_v remain invariant.

In terms of the light-cone variables, the wave function of Eq. (6.1) can be written as

$$\begin{aligned} \psi_0^0(z,t)&=\psi_0^0(u,v)\\ &=\left[\frac{1}{\pi}\right]^{1/2}\exp[-(u^2+v^2)/2]\ . \end{aligned} \tag{6.5}$$

If the system is boosted, the wave function becomes

$$\begin{aligned} \psi_\beta^0(z,t)&=\left[\frac{1}{\pi}\right]^{1/2}\exp[-(u'^2+v'^2)/2]\\ &=\left[\frac{1}{\pi}\right]^{1/2}\exp\left[-\left[\frac{1}{2}\right]\left[\frac{1-\beta}{1+\beta}u^2+\frac{1+\beta}{1-\beta}v^2\right]\right]\ . \end{aligned} \tag{6.6}$$

This wave function undergoes a Lorentz deformation as β increases.[18] The momentum-energy wave function is

$$\varphi_\beta^0(q_u,q_v)=\left[\frac{1}{2\pi}\right]\int \psi_\beta^0(x,t)e^{-i(q_z z-q_0 t)}dz\,dt\ . \tag{6.7}$$

The evaluation of this integral leads to

$$\varphi_\beta^0(q_u,q_v)=\left[\frac{1}{\pi}\right]^{1/2}\exp\left[-\left[\frac{1}{2}\right]\left[\frac{1+\beta}{1-\beta}q_u^2+\frac{1-\beta}{1+\beta}q_v^2\right]\right]\ . \tag{6.8}$$

For the ground state, the PSD function can now be defined as

$$P_\beta^0(u,q_u;v,q_v)=\left[\frac{1}{\pi}\right]^2\int[\psi_\beta^0(u+x,v+y)]^*\psi_\beta^0(u-x,v-y)\exp[2i(q_u x+q_v y)]dx\,dy\ . \tag{6.9}$$

After the evaluation of this integral, the PSD function becomes

$$P_\beta^0(u,q_u;v,q_v)=\left[\frac{1}{\pi}\right]^2\exp\left[-\left[\frac{1}{2}\right]\left[\frac{1-\beta}{1+\beta}u^2+\frac{1+\beta}{1-\beta}q_u^2\right]\right]\exp\left[-\left[\frac{1}{2}\right]\left[\frac{1+\beta}{1-\beta}v^2+\frac{1-\beta}{1+\beta}q_v^2\right]\right]\ . \tag{6.10}$$

The above PSD function is defined in two independent phase spaces consisting of (u,q_u) and (v,q_v), respectively. When the hadron is at rest with $\beta=0$, the above PSD function is localized in the regions

$$(u^2+q_u^2)<1,\quad (v^2+q_v^2)<1\ . \tag{6.11}$$

These localization regions are described in Fig. 4. When the hadron moves, these regions undergo elliptic deformations.

This PSD function reproduces the distributions $|\psi_\beta^0(u,v)|^2$ and $|\varphi_\beta^0(q_u,q_v)|^2$ after the appropriate integrals:

$$\begin{aligned}|\psi_\beta^0(u,v)|^2&=\int P_\beta^0(u,v;q_u,q_v)dq_u\,dq_v\ ,\\ |\varphi_\beta^0(q_u,q_v)|^2&=\int P_\beta^0(u,v;q_u,q_v)du\,dv\ .\end{aligned} \tag{6.12}$$

As for the excited states, there are no timelike oscillations in the hadronic rest frame, and the oscillations in the transverse direction are not affected. Therefore, the only factor we have to consider is the Hermite polynomial $H_n(z')$ to be multiplied to the ground-state wave function. The nth excited-state wave function is given in Eq. (5.13). In terms of the u' and v' variables, $H_n(z')$ can be written as[22]

$$\begin{aligned}H_n(z')&=H_n((u'+v')/\sqrt{2})\\ &=\left[\frac{1}{2}\right]^{n/2}\sum_{m=0}^{n}\begin{bmatrix}n\\ m\end{bmatrix}H_{n-m}(u')H_m(v')\ .\end{aligned} \tag{6.13}$$

Thus the explicit form of the physical wave function becomes

$$\psi_\beta^n(u,v)=\left[\frac{1}{2}\right]^{n/2}\left[\frac{1}{\pi n!}\right]^{1/2}\left[\sum_{m=0}^{n}\begin{bmatrix}n\\ m\end{bmatrix}H_{n-m}(u')H_m(v')\right]\exp[-(u'^2+v'^2)]\ . \tag{6.14}$$

This means that we need off-diagonal PSD functions for the one-dimensional harmonic oscillator, such as

$$P_{nm}(x,p)=\frac{1}{\pi}\int\psi_n^*(x+y)\psi_m(x-y)e^{2ipy}dy\ , \tag{6.15}$$

to evaluate the PSD function for covariant harmonic oscillator. It is possible to evaluate this integral using the generating function of Hermite polynomials. The result is[23]

$$P_{nm}(x,p)=\frac{(n!m!)^{1/2}}{\pi}\left[\sum_{k=0}^{s}\frac{(-1)^k[\sqrt{2}(x+ip)]^{n-k}[\sqrt{2}(x-ip)]^{m-k}}{k!(n-k)!(m-k)!}\right]\exp[-(x^2+p^2)]\ , \tag{6.16}$$

where s is n or m, whichever is smaller. We can then go back to Eqs. (6.9) and (6.14) to complete the evaluation of the PSD function. The localization and deformation properties of the PSD function for excited states are essentially the same as those of the ground-state oscillator.

Let us go back to the localization problem. Unlike the case of light waves,[2] we have to deal with two phase spaces. If the hadron is at rest with $\beta=0$, the localization region can be specified by a circle in both the phase spaces of (u,q_u) and of (v,q_v). If the hadron moves, the u and q_v distributions expand, while those of v and q_u be-

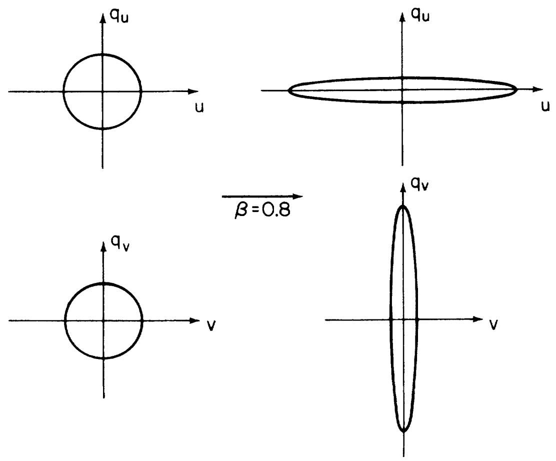

FIG. 4. Lorentz deformations in the light-cone phase space consisting of two pairs of conjugate variables. The major (minor) axis in the uv coordinate system is conjugate to the minor (major) axis in the $q_u q_v$ coordinate system. The Lorentz boost is an area-preserving canonical transformation in both phase spaces. For the case of localized light waves, which was discussed in Ref. 2, there is only one phase space. The covariant phase space given in Ref. 2 is the lower half of this figure consisting of v and q_v.

come contracted.

These deformations are canonical transformations, and are illustrated in Fig. 4. If the hadron's speed becomes close to the speed of light with $\beta \rightarrow 1$,

$$\begin{aligned} &t = z, \quad u = \sqrt{2} z \ , \\ &q_t = q_z, \quad q_v = \sqrt{2} q_z \ , \end{aligned} \tag{6.17}$$

according to Eqs. (6.2)–(6.4). The simultaneous expansions of z and q_z are observed universally in high-energy laboratories. This is called Feynman's parton picture,[24] and the calculation based on the ground-state oscillator gives a good agreement with the observed proton structure function.[25] Indeed, the parton phenomenon is a manifestation of the Lorentz covariance of the uncertainty relation which can best be stated in terms of the phase-space representation.

VII. CONCLUDING REMARKS

As we pointed out in Ref. 2, the phase-space representation of quantum mechanics serves useful purposes in many branches of modern physics. In this paper, we emphasized the fact that it can give a more precise interpretation of the uncertainty principle, as is manifested in the case of wave-packet spread. The phase-space representation allows us to formulate the uncertainty relation in a covariant manner. For this purpose, we have discussed in this paper the covariant phase-space representation for harmonic oscillators.

The major advantage of using the covariant oscillator formalism is that there is an experimental observation of the effect of covariance, as is explained in Sec. VI. Indeed, the covariant phase space is the physical basis for the covariant harmonic oscillator. This is illustrated in the third row of Fig. 3.

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