Computationally induced "irregularity" in the spectra of integrable quantum systems

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The nearest-neighbor spacing distributions $P(S)$ for the variationally determined eigenvalues of two separable model Hamiltonians (two-dimensional quartic oscillators) are studied as functions of the dimension n of the variational basis. For large basis sizes the spacing histograms for a given energy range can be well fitted by a Poisson distribution, reflecting the regularity of the spectrum of a completely separable Hamiltonian. The decrease of n leads to histograms of qualitatively different spacing (for the same energy range as before), which can be well fitted by Brody distributions $P_q(S)$. The Brody parameter q controls the transition between a Poisson ($q=0$) and a Wigner distribution ($q=1$). For large n, $q=0$ holds. When n falls below a critical value, $q=1$ results, i.e., $P_q(S)$ turns over into a Wigner distribution. The latter is assumed to characterize a completely irregular spectrum in the classically chaotic energy range of a nonseparable Hamiltonian. The transition between a regular and an irregular spectrum is shown to be induced only by a change of numerical accuracy. We conjecture that similar behavior should also be observed in nonseparable systems.

I. INTRODUCTION

Since the elegant studies of Kolmogorov, Arnold, and Moser[1] (KAM), it is well known that the classical dynamics of even small systems can display two types of motion, regular or irregular (often termed chaotic), within the same range of the total energy of the system. KAM showed that a collection of nonlinearly coupled oscillators can display regular behavior (similar to that of separable systems) as long as the value of a control parameter λ does not exceed a critical value λ_c. In simple examples such as the Hénon-Heiles[2] or the Barbanis system,[3] the control parameter is simply a linear factor in front of the nonlinear coupling term in the potential-energy function of the system. As predicted by KAM the threshold for regularity can be observed in the model systems either with a given total energy and a critical λ value or with fixed λ at a critical energy E_c.

Several attempts have been made to carry over the well-understood concepts of regular and chaotic dynamic behavior of classical systems to quantum mechanics in order to establish quantum chaos,[4-41] but there is still no consensus on its definition or nature, nor even on its existence.

Following the arguments of Percival,[4] a regular state can be labeled by a vector quantum number $\underline{n}=(n_1,\ldots,n_f)$ if the system has f degrees of freedom. A state with quantum number $\underline{n}$ corresponds to those phase-space trajectories of the corresponding classical system that lie on an f-dimensional invariant toroid with action constants I_k. These are given by the semiclassical quantum condition $I_n=\hbar(\underline{n}+\alpha/4)$, where the components α_i of $\alpha=(\alpha_1,\ldots,\alpha_f)$ are integers equal to the number of "turning points" of the projection of the trajectory onto the position space of the ith irreducible circuit Γ_i of the torus, i.e., the number of places on Γ_i where the torus is "normal" to the position space.[5] The regular states are very much like the quantum states of a completely separable system. Percival predicted[4] that the energies of an irregular spectrum should be more sensitive to a slowly changing perturbation than are those of a regular spectrum. Indeed, Pomphrey,[6] in numerical studies of the quantum Hénon-Heiles system, found a drastic change of the eigenvalue sensitivity to slight changes in the perturbation (measured by recording the second differences Δ with respect to the nonlinear coupling parameter) with increasing energy. Eigenvalues of the model Hamiltonian with energies less than a critical value E_c were found to be insensitive, while above E_c eigenvalues were very sensitive to small changes in the perturbation. The latter were assigned to belong to the irregular spectrum, while the former were termed regular. The critical energy for this transition was found to be roughly equal to the energy for the onset of chaotic motion in the corresponding classical system.

Pomphrey's sensitivity analysis was applied by several authors[7-11] in numerical treatments of nonlinearly coupled quantum oscillators. In most of these studies a more or less monotonic increase of the number of eigenvalues with high Δ_i value with increasing energy was reported, but it should also be noted here that Weissman and Jortner[12] did not find any irregular behavior of the eigenstates of the Hénon-Heiles system. They conjectured that Pomphrey's findings may be related to improper computation or to a lack of a suitable reference to the possible values of Δ_i. In many papers[13-18] the irregularity of a quantum state was also related to the occurrence of avoided crossings in the energy spectrum as a function of the coupling parameter. Noid and co-workers[8,18] explicitly related avoided crossings or crossings of eigenvalues in plots of $E_i(\lambda)$ versus λ to extremely large second-difference results, and they conjectured that avoided

crossings provided a mechanism for the origin of such large second differences. Isolated avoided crossings offer a way of substantial mixing of the nodal pattern of two wave functions and may also be responsible for the irregular pattern found by Stratt *et al.*[13] from numerical studies of some nonlinearly coupled oscillators. However, isolated avoided crossings do not lead to an irregularity in the level spectrum. Only if a particular state participates simultaneously in many—overlapping—avoided crossings, would irregularity be generated. This was recently demonstrated by Noid and co-workers and other authors.[8,14–16,18] Brickmann and Levine[17] conjectured that if irregularity is identified with the occurrence of "overlapping avoided crossings," one should expect a decrease rather than an increase of the second derivatives of the eigenvalues (or second difference) in going from the low-energy to the high-energy range after passing a domain where isolated avoided crossings (with extremely high Δ_i values) dominate. This conjecture was numerically tested by Özkan *et al.*[19] who studied the Hénon-Heiles system and this three-dimensional analog. The arguments given above may be summarized as follows.

(i) A regular eigenstate of a nonintegrable Hamiltonian may always be correlated to one eigenstate of an integrable Hamiltonian which, however, is generically not identical for all the regular eigenstates of the system, as has been demonstrated by Hose and Taylor.[20,21] Each regular state is related to its own effective separable Hamiltonian. Hose and Taylor developed a theoretical concept which is the quantum analog to that of KAM in classical physics.

(ii) On the other hand, an irregular state always needs contributions with comparable magnitudes from several eigenstates of some integrable Hamiltonian in order to be adequately described.

This characterization is closely related to a criterion of Rice[7] and Nordholm and Rice.[22–24] These authors have assigned regularity to those states which remain "isolated" in action space upon addition of a nonlinear mode coupling term to *one* particular separable zero-order Hamiltonian, and irregularity to a delocalization in this space. The Nordholm and Rice criterion has to be modified because of the arbitrariness of the separable reference Hamiltonian.

All the concepts described above may well be suited to identify regularity for individual states from analytical or numerical considerations—even when there is no straightforward procedure available to find a separable Hamiltonian for this state—but they generate only very weak criteria for identifying individual irregular states. There are many open questions. How many zeroth-order states—they may be eigenstates of different zeroth-order Hamiltonians—are necessary to compose an irregular state? Is there generically a smooth transition between regularity and irregularity? Is it possible, in principle, to identify *individual* irregular states or can irregularity be assinged only to a certain energy range, wherein all states show qualitatively similar behavior? The questions may be at least partly answered from the quantum phase-space treatments of Heller and co-workers[25–30] or from similar criteria of Kay,[31,32] from entropy-type measurements of Brickmann and co-workers for the population of eigenstates representing a wave packet,[33,42–45] or from statistical analysis of the energy eigenvalue spectrum[46–58] (see Ref. 46 for a recent review), which is mostly based on methods used for the study of random matrix ensembles.[47]

All these criteria, however, are not rigorously proved for "real" quantum systems, i.e., for systems which are *not* close to the classical limit ($\hbar \to 0$). Nevertheless, they are frequently applied in numerical studies of model systems.[12,16–19,25–33,42–45,48–58] In these studies the eigenstates of a Hamiltonian containing nonlinear coupling terms between different degrees of freedom are approximately determined by applying standard methods like the linear variation scheme. The observed qualitative behavior (for example, that of the spectrum) is consequently always related to approximate quantities, and it is by no means clear in all these cases whether an identified irregularity it a generic property of the system or just the result of the method of calculation. It is therefore not surprising that different authors sometimes find completely different qualitative results for identical systems.

It is the aim of this work to demonstrate, with a few numerical examples, that the approximately determined eigenstates of a separable Hamiltonian may show regular as well as irregular behavior and that a transition between both types (as identified with methods mentioned above) can be generated in a certain energy range just by a change of the numerical accuracy of the calculation. We focus our intention on the spectral properties of the system. In Sec. II some basic results necessary for the characterization of spectral fluctuations are reviewed, while Sec. III deals with the description of the model systems and of the numerical procedure. In Sec. IV the numerical results are presented and discussed in detail, while the final section, Sec. V, gives some conclusions.

II. STATISTICAL ANALYSIS OF THE EIGENVALUE SPECTRUM

Berry and Tabor[39] have postulated that the distribution of eigenvalues for regular states should substantially differ from that of irregular ones. In the regular regime a Poisson distribution,[39]

$$P(S)=\exp(-S) , \tag{1}$$

for the nearest-neighbor distance S of the deconvoluted spectrum is expected. The latter is obtained from the original spectrum by deconvolution, i.e., by scaling with the aid of a smooth function such that the average density becomes equal to unity.

Theoretical arguments[38] suggest that the statistics of irregular states (at least in the semiclassical limit) are the same as the ones of ensembles of real symmetric matrices with elements distributed according in a Gaussian way.[59] For such systems $P(S)$ is closely approximated by the Wigner distribution,[46,60–62]

$$P(S)=\frac{\pi}{2}S\exp(-\tfrac{1}{4}\pi S^2) . \tag{2}$$

Generic systems do not conform to these special cases,[60] and one should expect a superposition of (1) and (2) in spectra where both regular and irregular terms occur in

the same energy range. Such intermediate cases can be described by the Berry-Robnik distributions,[46,55,60]

$$P(q;S)=\exp\left[-(1-q)S-\frac{\pi}{4}q^2S^2\right]\times\left[1-q^2+\frac{\pi}{2}q^3S-(1-q)^2R(qS)\right] \tag{3}$$

with

$$R(z)=1-\exp(\pi z^2/4)\operatorname{erfc}(\pi^{1/2}z/2)\ , \tag{4}$$

which is obtained from the semiclassical limit for systems whose classical energy surface is divided into a number of separate regions in which the motion is regular or chaotic. Another approach—the statistical analysis of nearest-neighbor spacings in nuclear spectra—leads to the Brody distribution,[63,47]

$$\begin{aligned}&P_q(S)=\alpha S^q\exp(-\beta S^{1+q})\ ,\\&\alpha=(1+q)\beta\ ,\\&\beta=\Gamma^{1+q}[(2+q)/(1+q)]\ ,\end{aligned} \tag{5}$$

which has also been successfully applied numerical treatments of nonlinearly coupled quantum systems.[50] For both intermediate distributions the parameter q specifies a continuous change between the Poisson distribution ($q=0$) and the Wigner distribution ($q=1$), i.e., q roughly measures the amount of irregularity in a certain energy range.

In this work we applied the Brody distribution for the numerical analysis for two reasons. Firstly, this distribution generates analytical expressions within the integral fit of histogram distributions (see Sec. IV) and, secondly, the systems studied here do not have classical counterparts (see Sec. III), so that the semiclassical Berry-Robnik distribution has no preference.

III. GENERATION OF A MODEL HAMILTONIAN FROM A SEPARABLE SYSTEM

In most of the discussions on regularity and irregularity in nonlinear quantum systems, the total Hamiltonian is treated as a sum of two parts,

$$H=H_0+W_\lambda \tag{6}$$

where H_0 is an integrable Hamiltonian (with purely regular spectrum) and W_λ is a nonlinear operator (dependent on a control parameter λ) which is responsible for the occurrence of chaotic trajectories in the corresponding classical case. As was mentioned above, the special properties of H can only be analyzed on the basis of approximately determined eigenfunctions of this operator and, consequently, it seems to be questionable whether the identified regularity (or irregularity) is a property of H or whether it is simply related to the quality of the approximation.

We wish to demonstrate that inappropriate choices of the numerical procedure can generate that type of behavior which is commonly termed "irregularity." For this reason we start from a completely separable Hamiltonian H_0 with discrete spectrum $\sigma(H)$, and we assume that there is no symmetry in the system [in order to avoid nongeneric structures in $\sigma(H_0)$]. Let us consider a second separable Hamiltonian H_1 with $[H_0,H_1]\neq 0$ and a complete set of eigenstates $\{\,|\,i\,\rangle\}$, i.e.,

$$H_1\,|\,i\,\rangle=E_i\,|\,i\,\rangle\ , \tag{7}$$

such that H_1 also generates only discrete eigenvalues E_i. Then the projection operator

$$P_n=\sum_{i=1}^{n}|\,i\,\rangle\langle\,i\,| \tag{8}$$

is used to partition the Hamiltonian H_0 according to

$$\begin{aligned}H_0&=P_nH_0P_n+Q_nH_0Q_n+P_nH_0Q_n+Q_nH_0P_n\\&=H_n-W_n\ ,\end{aligned} \tag{9}$$

with the block-diagonal Hamiltonian H_n,

$$H_n=P_nH_0P_n+Q_nH_0Q_n\ , \tag{10}$$

and the perturbation

$$W_n=-P_nH_0Q_n-Q_nH_0P_n\ . \tag{11}$$

Rewriting (9) so that it is similar to (6),

$$H_n=H_0+W_n\ , \tag{12}$$

we are formally in the same situation as in the case of the coupling between a regular system H_0 and a nonlinear perturbation. The control parameter is now the number of eigenfunctions n used for the construction of P_n [Eq. (8)], but we are in a much better position than in most nonlinear Hamiltonians; the lowest n eigenstates of H_n can be exactly determined by diagonalizing $P_nH_0P_n$, i.e., the accuracy in the determination of the eigenvalue spectrum is only limited by the numerical errors of the diagonalization procedure. This fact enables us to follow the spectral properties of H_n in the low-energy regime as a function of the control parameter $\lambda=1/n$. For $\lambda=0$ the spectrum is exactly that of the separable Hamiltonian H_0, i.e., we expect a completely regular sequence. With increasing λ (or decreased size of the secular matrix) a change of the qualitative behavior may take place, reflecting an improper choice of the secular basis. In Sec. IV the results of numerical treatments of two model systems of noninteracting quartic oscillators are presented in order to demonstrate these spectral changes with explicit examples.

IV. RESULTS AND DISCUSSION

In order to ensure that the spectrum of H_0 is purely regular, we select two model potentials which are expressed as simple sums of quartic oscillators in x and y directions,

$$\begin{aligned}&H_0=-\tfrac{1}{2}(d^2/dx^2+d^2/dy^2)+V(x,y)\ ,\\&V(x,y)=P_4(x)+Q_4(y)\ .\end{aligned} \tag{13}$$

Here and in the following, we use generalized units. The

energy is measured in multiplies of $\hbar\omega$, while distances occur as multiples of $(m\omega/\hbar)^{1/2}$, where m is the mass of the particle and ω an arbitrary frequency.

The parameters for the one-dimensional potentials $P_4(x)$ and $Q_4(x)$ are given as follows. For model 1,

$$P_4(x)=0.50x^2-0.0080x^3+0.000\,018x^4\,,$$
$$Q_4(y)=0.72y^2-0.0096y^3+0.000\,018y^4\,. \tag{14}$$

For model 2,

$$P_4(x)=0.5000x^2-\tfrac{2}{90}x^3+\tfrac{1}{3600}x^4\,,$$
$$Q_4(y)=0.5408y^2-0.024\,996\,978y^3+0.000\,324\,961y^4\,. \tag{15}$$

Both polynomials of model 1 have double minima, but for the discussion presented in this work, we focus on the motion in the region of the local minimum at $x=y=0$ (see Fig. 1). This minimum is not an absolute minimum of the potential. We, nevertheless, only consider one-well states around the origin of the coordinate system, because of the enormously high barriers of the potentials separating this well from the others (approximately $N_L=10^5$ eigenvalues are expected within this well). Any reasonable variational calculation for a limited number of eigenvalues $N \ll N_L$ can be easily restricted to the well region.

In model 2 the potential has an absolute minimum at $x=y=0$ and inflection points in x and y directions. The parameters for $P_4(x)$ and $Q_4(y)$ are chosen such that both inflection points have identical energy values. The Figs. 2(a) and 2(b) depict the two-dimensional potential surfaces.

For separable problems such as those of models 1 and 2, the eigenvalues are given as $E_{n,m}=E_n^{(1)}+E_m^{(2)}$, where $E_n^{(1)}$ and $E_m^{(2)}$ are the eigenvalues of the one-dimensional Hamiltonians for the motion along the x and y directions, respectively. The one-dimensional eigenvalue equations can be approximately solved with arbitrary accuracy by expanding the trial wave function in terms of the eigenfunctions of the harmonic oscillator. Since these functions form a complete set, all the eigenvalues uniformly converge to the exact results upon increasing the number n_B of basis functions. We analyzed the eigenvalues of the one-dimensional systems as functions of n_B. Only those that did not change within an absolute error of 10^{-6} GEU (generalized energy units) are considered for our further analysis. Note that the energy difference of two adjacent states in the one-dimensional models is in the order of 1 GEU. Once all possible combinations from these two sets are formed, a maximum energy $E_{\rm lim}$, the "exact" eigenvalue in the two-dimensional problems, is determined, which guarantees eigenvalues $E_{n,m}<E_{\rm lim}$ with at least six-digit accuracy. This upper limit is defined as

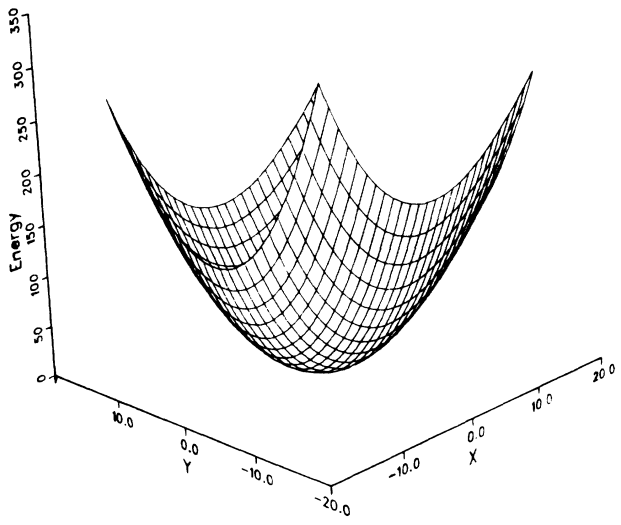

FIG. 1. Potential $V(x,y)$ of model 1 in the region of the local minimum at $x=y=0$.

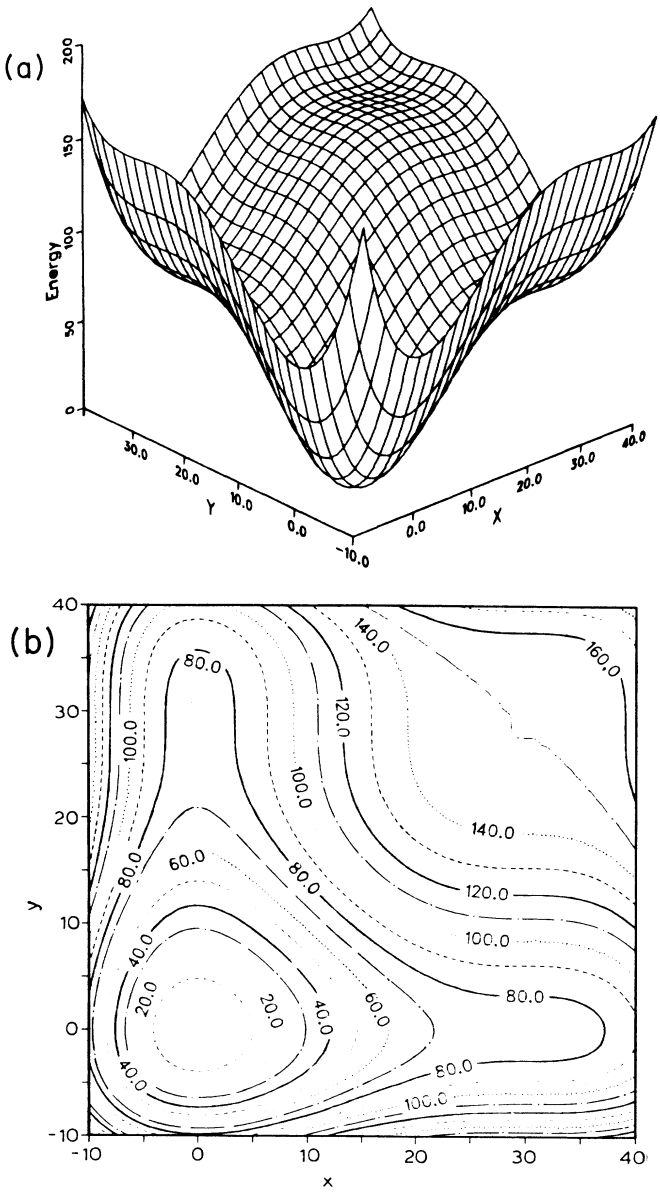

FIG. 2. Potential $V(x,y)$ of model 2. (a) Three-dimensional view. (b) Equipotential lines.

$$E_{\lim}=\min(E^{(1)},E^{(2)}) ,$$
$$E^{(1)}=E_0(x)+E_{\max}(y), \quad E^{(2)}=E_{\max}(x)+E_0(y) , \tag{16}$$

where $E_0(x)$ is the lowest one-dimensional eigenvalue in the x direction and $E_{\max}(y)$ is the highest accurate eigenvalue in the y direction, and vice versa. Even though it is possible to obtain accurate eigenvalues above this limit, the statistical analysis cannot be carried out, as there would be contaminations into the spectrum from inaccurate eigenvalues and the spacings would not actually correspond to the one of the exact solution. In our calculations 2200 eigenvalues below $E=72.0$ GEU and 1100 eigenvalues below $E=45.0$ GEU are used to the first and second potentials, respectively.

In principle, the eigenvalues of nonseparable Hamiltonians can be approximately determined as in the one-dimensional cases with the aid of the linear variational scheme. Now the many-dimensional trial function is approximated as a linear combination in the product space of basis functions in each dimension. If no symmetry conditions are imposed, then a simple direct-product set is built and the resulting matrix equations are solved to the desired accuracy. However, this procedure can be very slowly convergent, depending on the nonseparable part of the potential and the amount of anharmonicity in the Hamiltonian. In analogy to the well-analyzed techniques of the molecular-orbital calculations, self-consistent-field and adiabatic approximations have been developed to produce good starting function sets to enhance the convergence. Still, at the end, one has to set up and diagonalize large matrices to obtain reliable highly excited states. The quality of such calculations surely depnds on the ability of the chosen basis to represent adequately the nonseparable part of the potential. The properties of such a system, analyzed on the basis of approximately determined eigenstates, are therefore related to both the numerical accuracy and to the structure of the Hamiltonian. In this work we explicitly exclude the second influence. In order to mimic a seemingly nonseparable problem, the coordinate systems of our model potentials have been rotated by 30° in the counterclockwise direction, so that each potential is now composed of 12 terms. The basis of the variational scheme is chosen as a set of products of harmonic-oscillator functions along the nonrotated coordinates. The basic frequencies of the corresponding oscillators are obtained from the harmonic parts of the rotated potentials. In each coordinate the same number of basis functions is chosen. It is possible to obtain better results by using different-size sets, but this procedure is not applied, as it would bring an extra optimization problem.

The matrix equations are solved on the IBM 3090 of the Technische Hochschule Darmstadt Computing Center by a modified version of the Ortega diagonalization[64] which uses Householder tridiagonalization and Sturm sequences. A typical run time for a matrix of 1225×1225 is 16 min of 64-bit double precision utilizing the vector processor.

The use of basis sets with a varying number of basis functions is equivalent to the construction of different Hamiltonians H_n [see Eq. (10)], i.e., the diagonalization of the corresponding variational matrices gives the lowest n eigenvalues of these operators with the accuracy of the diagonalization procedure. $P(S)$ statistical analyses (see Sec. II) are now carried out for different energy regions. In order to apply a reasonable test, the eigenvalue spectrum is deconvoluted to uniform density of states. For this reason the number-of-states function $N(E)$,

$$N(E)=\int_0^E \rho(E')dE' , \tag{17}$$

with the density of states,

$$\rho(E)=\sum_{i=1}^{\infty}\delta(E'-E_i) , \tag{18}$$

is approximated by a smooth function

$$\bar{N}(E)=\sum_{p=0}^{k} a_p E^p . \tag{19}$$

The order of the polynomial is usually the number of degrees of freedom (here 2). The smooth density of states results as

$$\bar{\rho}(E)=d\bar{N}/dE , \tag{20}$$

and the average spacing is obtained as $\bar{\Delta}(E)=1/\bar{\rho}(E)$, and the normalized spacings become

$$S_i=\frac{E_{i+1}-E_i}{\bar{\Delta}[(E_{i+1}+E_i)/2]} . \tag{21}$$

For the model systems described above, a polynomial of second order is used to fit the smoothed number-of-states function $\bar{N}(E)$ of Eq. (19). Higher-degree polynomials produce better fits for the step function but do not change the quantitative results of $P(S)$ statistics. However, due to the extra nodes of the higher-order polynomials, their use may be misleading for long-range statistical analyses.

As mentioned above, the continuous change from the "regular" to the "irregular" part of the spectrum can be observed by monitoring the change of the parameter q of Berry and Robnik[46,55,60] or Brody.[47,63] In both cases the number of spacings in a given interval is approximated by a continuous function. We fit the nonlinear parameters in a nonstandard way to numerically determined histograms of the number of spacings versus spacing. The fit is performed by matching the incremental integrals of the distribution function to be determined to the bars of the histograms. In the standard point-by-point fitting procedure the choice of the points in the abscissa is not unique since the histograms consist of bars of finite thickness. Even though this arbitrariness may not pose an important problem, the fitting to the integral is still a more sound technique and can be applied to histograms of any thickness. We start from the optimization function

$$\epsilon=\sum_{i=0}^{K-1}\left[(DM_i)-M\int_{S_i}^{S_{i+1}}P(S)dS\right]^2 , \tag{22}$$

with the width D of a single block of the histogram, the number of blocks K, the number of spacings M_i in the histogram block i, the total number M of spacings in the analysis, and the variable S_i of the continuous function

$P(S)$ at the beginning of block i. For the Brody distribution the integral of $P(S)$ from S_i to S_{i+1} is simply

$$\int_{S_i}^{S_{i+1}} \alpha S^q \exp(-\beta S^{1+q}) dS = \exp(-\beta S^{1+q}) \Big|_{S_i}^{S_{i+1}} . \tag{23}$$

The parameter q can easily be computed by forced minimization of the optimization function ϵ.

For both potential models, various size matrices are set up to compute eigenvalues of the corresponding Hamiltonian H_n. The smallest matrix is 225×225, which corresponds to 15 basis functions in each direction. The largest two-dimensional calculation is done with 35 functions in x and y directions, leading to a variational matrix with 1225 rows and columns. Calculations with larger matrices were not carried out, since convergence to the "exact" results (eigenvalues which have been obtained by the procedure described above) is achieved for the regions of interest. The eigenvalues of the Hamiltonians H_n with the original potential of model 2 are shown in Figs. 3 and 4. While Fig. 3 shows the eigenvalues on the original energy scale, Fig. 4 shows the spectrum after the deconvolution procedure, i.e., a spectrum of uniform average density. For the exact spectrum of a separable Hamiltonian H_0, even for the high-energy regime, there is no random behavior of eigenvalues. This is easily observed by the generic structure at the right-hand side of Figs. 3 and 4. In contrast, the other extreme, which corresponds to a very crude approximation of the secular problem, intuitively suggests a high degree of randomness.

For both series of computations (with models 1 and 2 as basic Hamiltonians H_0, respectively) a detailed analysis of the spacing distribution is performed. Therein, the eigenvalues which fall into the energy range 30 GEU $< E <$ 40 GEU for the first potential are explicitly considered, while for the second potential the corresponding region is chosen as 18 GEU $< E <$ 28 GEU. The number of spacings versus spacings histograms (in units of the constant average density) are plotted in Figs. 5 and 6 for the models 1 and 2, respectively, for different values of the truncation parameter n. In both model systems the qualitative picture of a transition from a Wigner- to a Poisson-type distribution can be observed.

TABLE I. Brody parameter q as a function of the dimension n of the variational matrix. (a) Model 1. (b) Model 2.

N[a]	n	q
	(a) Energy range 30 GEU $\leq E \leq$ 40 GEU	
22	484	0.888
25	625	0.635
27	729	0.630
28	784	0.553
29	841	0.329
30	900	0.111
32	1024	0.000
	(b) Energy range 18 GEU $\leq E \leq$ 28 GEU	
15	225	0.622
17	289	0.506
18	324	0.351
20	400	0.293
21	441	0.219
22	484	0.222
23	529	0.098
25	625	0.055
26	676	0.027
28	784	0.000

[a] N is the number of basis functions in one direction only. $n = N^2$.

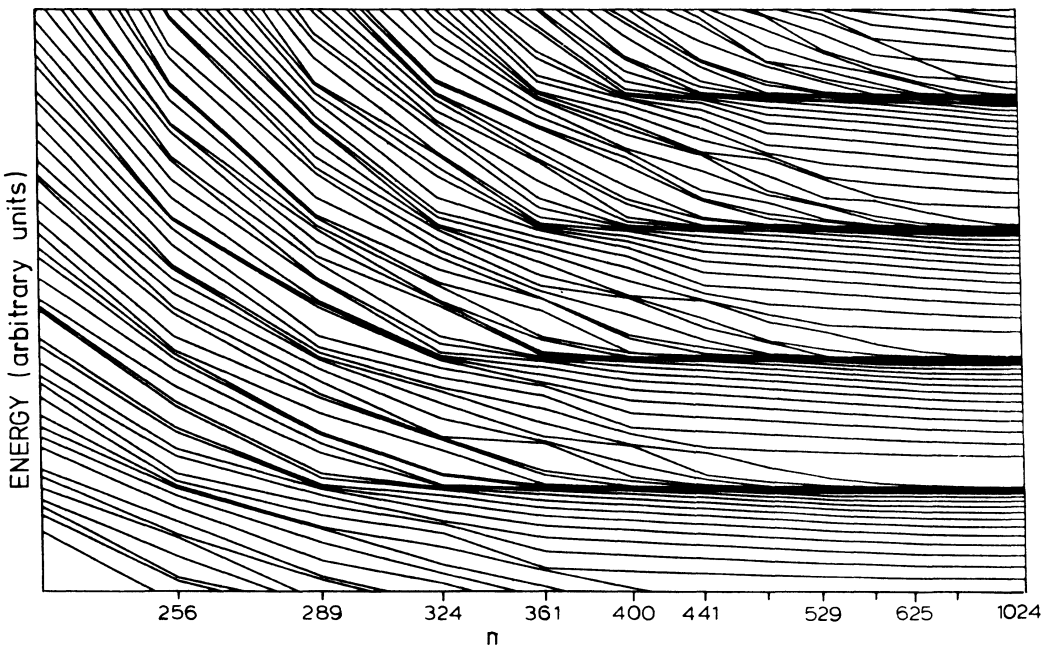

FIG. 3. Eigenvalue spectrum of H_n as a function of n. The abscissa is scaled so that it is proportional to $-1/n$.

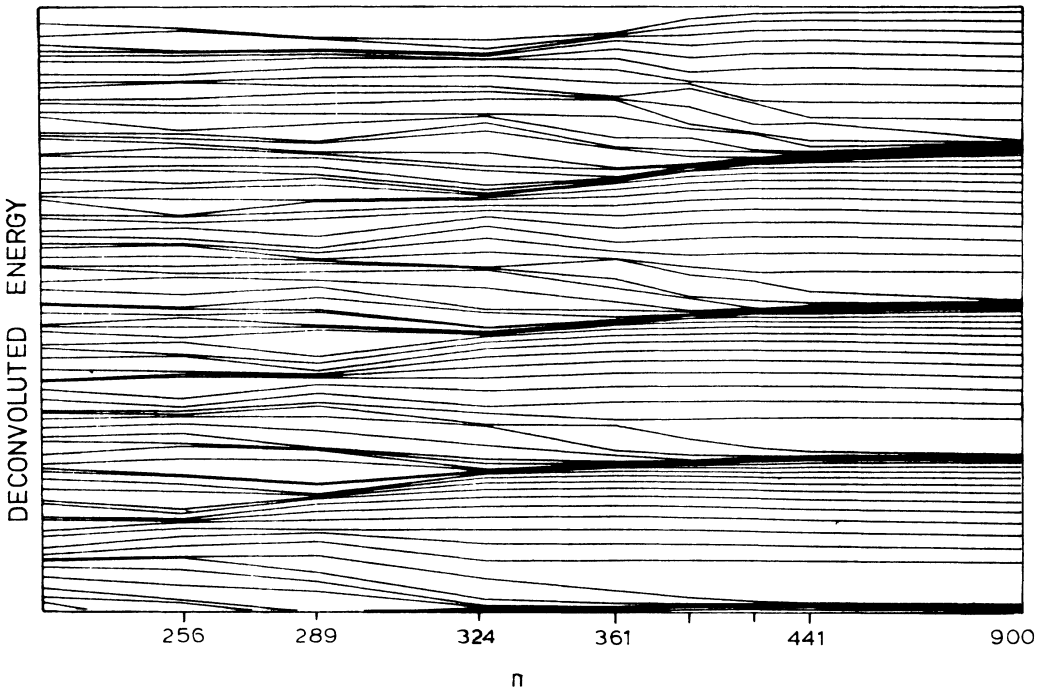

FIG. 4. Deconvoluted eigenvalue spectrum of H_n (scaled to uniform average density of states) as a function of n. The abscissa is scaled so that it is proportional to $-1/n$.

The Brody parameters q are determined for each of these calculations with the fitting procedure described above. In Table I the change in q is presented as a function of the basis size. The exact eigenvalues of the separable Hamiltonian H_0 obey the Poisson distribution as expected, with q being 0.0. This result is also reproduced in the cases of $n=1024$ and 784 for the potentials 1 and 2, respectively. The histograms from the eigenvalues of H_n have the same qualitative behavior as the ones for the spectrum of the corresponding H_0. This confirms that we were able to observe qualitative changes in the level statistics as functions of the degree of numerical accuracy. For small matrix sizes the q parameter is found to be very close to 1.0, which is supposed to identify the spectrum as Wigner-type and irregular. Upon increasing the basis size, q becomes monotonously smaller and the spectrum becomes regular.

The combination of histograms, the optimization of the switch parameter q, and the graphic representation of the eigenvalue spectrum support the conclusion that

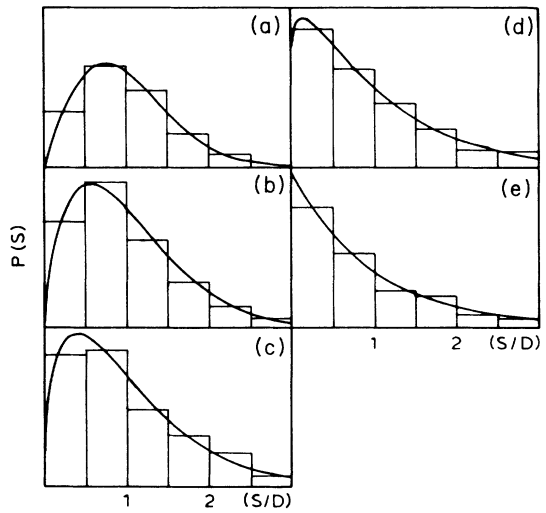

FIG. 5. Nearest-neighbor spacing histograms for different H_n for model 1 in the energy range 30 GEU $\leq E \leq$ 40 GEU. (a) $n=484$, (b) $n=784$, (c) $n=841$, (d) $n=900$, (e) $n=1024$.

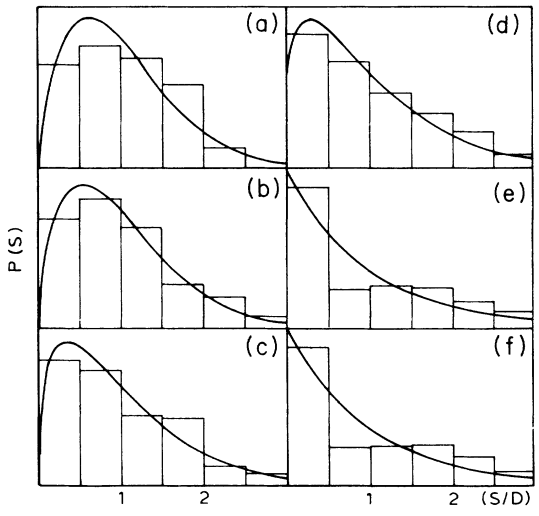

FIG. 6. Nearest-neighbor spacing histograms for different H_n for model 2 in the energy range 18 GEU $\leq E \leq$ 28 GEU. (a) $n=225$, (b) $n=289$, (c) $n=400$, (d) $n=484$, (e) $n=1024$, (f) "exact."

there is a qualitative change between regular and irregular (or chaotic) behavior in our model systems in changing the control parameter $\lambda = 1/n$, although there is no nonintegrable classical analog for these systems. The observed transition is solely a consequence of improper numerical approximations.

V. CONCLUSIONS

One of the standard procedures for the analysis of common properties of quantum states within a given energy range is the study of nearest-neighbor spacing distributions of the energy eigenvalues. It is known, from theoretical arguments, that in the semiclassical limit a Poisson distribution is expected for those energies where the corresponding classical motion is completely regular, i.e., where the trajectories migrate on toroids. For those energies where almost all trajectories move chaotically, a Wigner distribution is predicted in the semiclassical limit, while in the intermediate-energy range a mixture of both (Berry-Robnik or Brody distribution) with a mixing coefficient q is expected. Numerical studies seem to indicate that q is roughly equal to the ratio of irregular to regular trajectories in the corresponding classical energy range.[46] However, this correspondence is very weak. We demonstrate that one can generate a Wigner-type spacing distribution of an eigenvalue spectrum of a separable Hamiltonian, and it seems to be reasonable that there are many ways to construct quantum systems with any desired spectral property. With two explicit examples it is shown that all possible spacing properties in an eigenvalue spectrum can be generated just by projecting a given separable Hamiltonian to two orthogonal subspaces of the Hilbert space and solving the eigenvalue equation in both subspaces independently. If one of the subspaces has finite dimension, the eigenvalue equation reduces simply to a diagonalization of a finite matrix.

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