PHYSICAL REVIEW A VOLUME 37, NUMBER 3 FEBRUARY 1, 1988

Exact two-body solution of the Lorentz-Dirac equation

Keith Briggs
Department of Physics, University of Adelaide, P.O. Box 498, Adelaide, South Australia 5001, Australia
(Received 21 August 1987)

We exhibit a solution to the Lorentz-Dirac equations for two classical point particles in circular orbits, each interacting with the retarded field of the other. The fields produce the exact tangential force required to balance the loss of energy by radiation.

INTRODUCTION

Previous attempts to find exact solutions to the special relativistic electromagnetic two-body problem have all made use of Fokker-type action principles and, at least partly, advanced potentials. In these theories the interaction is purely action at a distance; the fields have no physical reality and particles do not experience a self-force. Thus Schild,[1] by using time-symmetric (advanced plus retarded) potentials, found a solution in which the two particles move in circular orbits about a common center. Similarly, Bruhns and Fahnline[2] have studied a class of solutions to the time-asymmetric problem in which one particle is acted upon by the retarded field of the second, while the second is acted upon by the advanced field of the first.

All these solutions are stationary because the particles do not radiate in these theories. In contrast, the standard formulation with purely retarded fields results in the Lorentz-Dirac equation, with its nonlinear self-force term due to radiation reaction, and well-known difficulties of preacceleration and runaway solutions. It might be imagined that the Lorentz-Dirac equation would not allow stationary circular orbits because the energy of the system would be radiated away. However, the existence of the runaway solutions (in which a particle in a free-field region increases its energy without limit) should alert us to the possibility of the existence of solutions which apparently violate energy conservation. Indeed Eliezer,[3] though he does not find exact solutions, presents evidence that there are no solutions to the fixed-center Coulomb-force problem in which the particle spirals into the center.

We therefore aim here to examine the possibility that with purely retarded interactions, stationary circular solutions exist. The idea is to include radiation by using the Lorentz-Dirac equation of motion for each particle, which interacts with the full retarded field of the other. We will find that the force has a component in the direction of motion and can be used to exactly balance the radiation reaction force.

METHOD OF SOLUTION

The retarded Liénard-Wiechert potentials for a point particle lead to the following fields:[4]

$$\mathbf{E}=\frac{\sigma}{\kappa^3 R}\{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)/R+\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]/c\}\ ,\qquad \mathbf{B}=\mathbf{n}\times\mathbf{E}/c\ , \tag{1}$$

where $\sigma=e/(4\pi\epsilon_0)$ and $\kappa=1-\mathbf{n}\cdot\boldsymbol{\beta}$. Here $\boldsymbol{\beta}=\mathbf{v}/c$ and $\dot{\boldsymbol{\beta}}$ are to be evaluated at the retarded position of the particle, $\mathbf{n}$ is the unit vector from the retarded position to the observation point, and R is the retarded distance. For circular motion the electric field at the diametrically opposite point becomes

$$\mathbf{E}=\frac{\sigma}{\kappa^3 R}[(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2+2\beta^2\cos^2\phi)/R-\dot{\boldsymbol{\beta}}\kappa/c]\ , \tag{2}$$

where ϕ is the retardation angle, i.e., the angle between the retarded and actual positions of the particle. We consider now two diametrically opposite particles of opposite charges ($\sigma_1=-\sigma_2<0$) and evaluate the field of one at the position of the other. The retardation angle then satisfies

$$\phi=\beta\cos\phi \tag{3}$$

since one particle rotates through 2ϕ about the center of the circle in the time it takes for light to travel to the other particle. Thus the tangential and outward radial components of the field of particle 2 at position of particle 1 are (with r the radius of the orbit)

$$E_t=\frac{\sigma_2}{4\kappa^3 r^2\cos^2\phi}\{[\beta\cos(2\phi)-\sin\phi](1-\beta^2+2\phi^2)-2\beta^2\kappa\cos\phi\sin(2\phi)\}\ , \tag{4}$$

$$E_r=\frac{\sigma_2}{4\kappa^3 r^2\cos^2\phi}\{[\beta\sin(2\phi)+\cos\phi](1-\beta^2+2\phi^2)-2\beta^2\kappa\cos\phi\cos(2\phi)\}\ , \tag{5}$$

where we have used the following relations for the tangential and radial components of the vectors involved: $n_t=-\sin\phi$, $n_r=\cos\phi$, $\beta_t=-\beta\cos2\phi$, $\dot{\beta}_t=\dot{\beta}\sin2\phi$, $\beta_r=-\beta\sin2\phi$, $\dot{\beta}_r=\dot{\beta}\cos2\phi$, and $\dot{\beta}/c=2\beta^2\cos(\phi)/R$.

We now attempt to satisfy the Lorentz-Dirac equation for particle 1. In covariant form it reads[5]

$$ma_\mu=\frac{e_1}{c}F_{\mu\nu}v^\nu+\frac{2e_1\sigma_1}{3c^3}(\dot{a}_\mu-v_\mu a^2/c^2)\ , \tag{6}$$

where v and a are the four-velocity and acceleration, and $F_{\mu\nu}$ is the retarded field of particle 2. If we can satisfy (6), then by symmetry the equivalent equation for particle 2 will also be satisfied. The second term in (6) is the radiation reaction force $F_{\mu\,\text{react}}$. Its space component for circular motion is always directed oppositely to the velocity and has magnitude

$$F_{\text{react}}=\frac{2e_1\sigma_1\beta^3}{3(1-\beta^2)^2r^2} . \tag{7}$$

Thus for a circular motion to be a solution to the Lorentz-Dirac equation we need the following relations to hold:

$$e_1E_t=F_{\text{react}} , \tag{8}$$

$$e_1(E_r+vB)=-m(1-\beta^2)^{-1/2}v^2/r , \tag{9}$$

where $B=[E^2-(\mathbf{E}\cdot\mathbf{n})^2]^{1/2}/c$ is the magnitude of the magnetic field, which is directed normal to the plane of the orbit and so it produces a purely radial force.

Equation (8) ensures that the tangential force compensates for the loss of energy by radiation. With the field (4), it is independent of r and has the unique solution for $\phi\in(0,1)$ of $\phi\approx 0.368\,267$. This solution was obtained numerically with Brent's algorithm.[6] We note that this value is obtained from geometric considerations alone. To complete the solution it is merely necessary to solve Eqs. (5) and (9) for r, which requires the specification of the charge and mass. For example, for the electron-positron system we obtain $r\approx 4.42$ fm, or about two classical electron radii.

DISCUSSION

It is usually considered that the Lorentz-Dirac equation gives physically meaningful results only when the force due to radiation reaction is small compared to the external force. This is certainly not the case with this solution. One may consider the existence of the above solution with its apparent violation of energy conservation as another example of the breakdown of classical point-particle electromagnetism when extended beyond its domain of applicability. Since the system is a rotating dipole it will have a nonzero total radiation rate. This should be no more surprising than the existence of runaway solutions to the Lorentz-Dirac equation, since in this model the point particles have an infinite self-energy to draw upon. Perhaps the most surprising aspect of the above solution is its uniqueness.

We intend to discuss the unequal mass case in a future paper. However, order-of-magnitude estimates would seem to eliminate the possibility of any solution of the above type resembling the hydrogen atom. For the ground state we have $\beta\approx 0.05$, so that the retardation angle ϕ needed is also of this order; whereas by considering the position of the center of mass we can see that the maximum angle possible is of the order $m_e/m_p\approx 0.0005$.

[1]A. Schild, Phys. Rev. **131**, 2762 (1963).
[2]B. Bruhns, Phys. Rev. D **8**, 2370 (1973); D. E. Fahnline, J. Math. Phys. **20**, 1118 (1979).
[3]C. J. Eliezer, Proc. Cambridge Philos. Soc. **39**, 173 (1943).
[4]A. O. Barut, *Electrodynamics and the Classical Theory of Fields and Particles* (Dover, New York, 1980), p. 169.
[5]C. Teitelboim, D. Villaroel, and Ch. G. van Weert, Riv. Nuovo Cimento 3 (No. 9), 38 (1980).
[6]G. E. Forsythe, *Computer Methods for Mathematical Computation* (Prentice-Hall, Englewood Cliffs, 1977), p. 164.