Mean first-passage times and colored noise

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Recent work regarding stochastic motion in a bistable potential is critiqued. Emphasis is placed on mean first-passage time analysis and the effects of colored noise. Controversy has arisen regarding general theory, particular calculations, and numerical simulations. Each of these aspects of the problem is addressed in this paper.

I. INTRODUCTION

As stochastic analysis of physical problems has broadened in recent years, appreciation of the need for more realism in models has grown. One very important property which has attracted a great deal of attention has been the correlation time of the noise driving the stochastic process.[1] Historically, "white" noise has been chosen since it represents a sharp separation of time scales in the dynamics. The correlation time for white noise is zero, and this represents the observed feature that the time scale for the driving noise correlations is much shorter than the time scale for the relaxation of the driven process. Usually, two or three orders of magnitude difference in time scales makes the white-noise model very accurate. However, an increasing number of problems have arisen in which it is more realistic to model the driving noise with "colored" noise, i.e., noise with a nonzero correlation time. This reflects a not so sharp separation of time scales, and a greater precision in the analysis of stochastic processes.

In this paper, recent literature regarding the mean first-passage time (MFPT) for barrier crossing for a one-dimensional motion in a bistable potential will be critiqued. Controversy and confusion have arisen with respect to general theory, particular calculations, and numerical simulations. Having ourselves contributed to this literature,[2] and to the confusion, we seek to bring clarity to a portion of this literature with this critique.

The simple problem underlying this discussion is described by the stochastic differential equation

$$\frac{d}{dt}x = W(x)+f(t) , \tag{1}$$

in which $W(x)=ax-bx^3$ and $f(t)$ is a stochastic force with zero mean and correlation function,

$$\langle f(t)f(s)\rangle = \frac{D_0}{\tau}\exp\left[-\frac{|t-s|}{\tau}\right] . \tag{2}$$

The systematic term $W(x)$ may be regarded as the negative gradient of the bistable potential, $U(x)=-\frac{1}{2}ax^2+\frac{1}{4}bx^4$. The noise correlation exhibits a colored noise with correlation time τ and with an exponential decay. In the limit $\tau \to 0$, this noise becomes white, with correlation, $2D_0\delta(t-s)$. The potential has a local maximum at $x=0$, and two minima at $\pm\sqrt{a/b}$. The problem is to compute the MFPT to cross from one minimum to the other.

Denote the MFPT by T. For white noise, the result for T is well known and is given by[3]

$$\tau = \frac{\pi}{\sqrt{2}a}\exp\left[\frac{a^2}{4bD_0}\right] . \tag{3}$$

Controversy has arisen regarding the correct expression for the colored-noise modification of this formula. We find several alternative suggestions in the literature, putatively valid for weakly colored noise,

$$T = \frac{\pi}{\sqrt{2}a}\left[\frac{1+2a\tau}{1-a\tau}\right]^{1/2}\exp\left[\frac{a^2}{4bD_0}\right] , \tag{4}$$

$$T = \frac{\pi}{\sqrt{2}a}\exp\left[\frac{a^2}{4bD_0}(1+2a\tau)\right] , \tag{5}$$

$$T = \frac{\pi}{\sqrt{2}a}\left[\frac{1+2a\tau}{1-a\tau}\right]\exp\left[\frac{a^2}{4bD_0}(1+2a\tau)\right] , \tag{6}$$

$$T = \frac{\pi}{\sqrt{2}a}\exp\left[\frac{a^2}{4bD_0}+\frac{3}{2}a\tau\right] . \tag{7}$$

Equation (4) follows from the analysis of Hanggi, Marcheson, and Grigolini;[3] Eq. (5) follows from the analysis of Hanggi, Mroczkowski, Moss, and McClintock;[4] Eq. (6) follows from our earlier analysis;[2] and Eq. (7) follows from the analysis of Masoliver, West, and Lindenberg.[5] The controversy has been characterized as a debate regarding the form of the exponential τ dependence of T. In (5) and (6) the τ dependence includes D_0 whereas in (7) it does not. Moreover, in (4) and (6) there is a nonexponential τ-dependent prefactor, whereas in (5) and (7) there is not. Equation (5) follows from an ansatz, called Hanggi's ansatz,[2,5] the validity of which has been questioned.[5] Numerical simulations have been questioned.[5] In this paper we show that Eq. (4) is the correct, small-τ expression, i.e., there is no linear τ dependence in the exponential's argument; that application of Hanggi's ansatz may require additional conditions for validity in the weak limit, although it is known to work well at steady state;[6] that Eqs. (6) and (7) are the result of computational errors;[2,5] and that there is a strong need for additional, accurate numerical analysis with emphasis on the small-τ behavior. Each of these points is developed in Sec. II. In Sec. III we show how our earlier general theory supports the conclusion that there cannot be any linear τ dependence in the exponential factor for T.

II. CALCULATION OF THE MFPT

For small τ it is possible to characterize the colored-noise problem by an effective Fokker-Planck equation, from which the MFPT may be calculated. There are several versions of the effective Fokker-Planck equation for weakly colored noise. They all have the form

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}[W(x)P(x,t)]+\frac{\partial^2}{\partial x^2}[D(x)P(x,t)] , \tag{8}$$

in which $D(x)$ is given particular forms depending on the theory. The τ-expansion form[7] for $D(x)$ is

$$D(x)=D_0[1+\tau W'(x)] , \tag{9}$$

in which $W'(x)$ denotes the x derivative of $W(x)$. This expression was used[3] to obtain (4). The Hanggi-ansatz[4] form for $D(x)$ is

$$D(x)=D_0[1+\tau(3b\langle x^2\rangle-a)]^{-1} , \tag{10}$$

in which $\langle x^2\rangle$ denotes the steady-state value of x^2, which is a/b. This expression was used[4] to obtain (5). Fox's functional-calculus approach[2] yields

$$D(x)=D_0[1-\tau W'(x)]^{-1} , \tag{11}$$

which was used[2] to produce (6). Masoliver *et al.* used an expression which for small τ is equivalent with (11) to obtain (7). Below, we show that both (6) and (7) are incorrect results starting from (11), and that expression (4) is the correct consequence of (11).

At first sight, it might appear that for small τ expressions (9) and (11) are identical to first order in τ because $(1-s\tau)^{-1}\simeq 1+s\tau+O(\tau^2)$. However, this view overlooks the fact that $W'(x)$ in $[1-\tau W'(x)]^{-1}$ is x dependent. Any algebraic identities must apply to the entire x domain [in this case where $W(x)=ax-bx^3$, the x domain is $(-\infty,\infty)$].

For details regarding the general treatment of MFPT calculations, the reader is referred to Ref. 2. In essence, one begins with Kolomogorov's backward equation which is equivalent with (8). This ultimately leads to the MFPT equation for T,

$$W(x)\frac{\partial}{\partial x}T(x)+D(x)\frac{\partial^2}{\partial x^2}T(x)=-1 , \tag{12}$$

which is equivalent to Eq. (68) of Ref. 2. If we consider the MFPT for getting from the negative minimum at $-\sqrt{a/b}$ to the positive minimum, then the solution to (13) can be written[8]

$$T(-\sqrt{a/b})=\int_{-\sqrt{a/b}}^{0}dy\int_{-\infty}^{y}dz\frac{\psi(z)}{\psi(y)}\frac{1}{D(z)} , \tag{13}$$

in which $\psi(x)$ is defined[8] by

$$\psi(x)=\exp\left[\int_{-\infty}^{x}dx\frac{W(x)}{D(x)}\right] . \tag{14}$$

Performing the integrals in (13) is done in two stages. The y integration may be simplified because $1/\psi(y)$ is sharply peaked about $y=0$. A Gaussian approximation gives the highly accurate approximation[2,3]

$$T(-\sqrt{a/b})\cong\left[\frac{\pi D(0)}{2a}\right]^{1/2}\int_{-\infty}^{0}dz\frac{\psi(z)}{D(z)} . \tag{15}$$

The remaining integral is the source of subsequent confusion in the literature. Hanggi *et al.*[3] applied the method of steepest descent to its evaluation and obtained (4). As we will ultimately see below, this result is indeed correct, and the method is also valid. Nevertheless, Hanggi *et al.*[4] later introduced the Hanggi ansatz (10) which permits easy evaluation of (15) yielding (5) instead. This was said to be in agreement with numerical simulations and is clearly not in agreement with (4). Fox then evaluated (15) using (11). When he applied the method of steepest descent,[2] he also got (4). However, the work of Hanggi *et al.*[4] suggested that this result was wrong, so Fox proceeded to evaluate (15) exactly, using parabolic cylinder functions. To be precise, he pulled the $D(z)$ denominator in (15) out of the integral because it is slowly varying with z, and evaluated the resulting expression,

$$T(-\sqrt{a/b})=\left[\frac{\pi D(0)}{2a}\right]^{1/2}\frac{1+2a\tau}{D_0}\times\int_{-\infty}^{0}dz\exp\left[\int_0^z dy\frac{W(y)}{D_0}\times[1-\tau W'(y)]\right] , \tag{16}$$

in which $D(0)=D_0/(1-a\tau)$. The integral, called I, may be evaluated in closed form[2] for $W(x)=ax-bx^3$. The result is[2]

$$I=\frac{1}{2}\left[\frac{2D_0}{b-4ab\tau}\right]^{1/4}\times\sum_{n=0}^{\infty}\left[\frac{-b^2\tau}{2D_0}\right]^n\frac{1}{n!}\left[\frac{2D_0}{b-4ab\tau}\right]^{3n/2}\times\pi\exp(\tfrac{1}{4}x^2)V(3n,x) , \tag{17}$$

where $x=[(a-a^2\tau)/2D_0][2D_0/(b-4ab\tau)]^{1/2}$. The asymptotic expansion for the parabolic cylinder function[9] $V(m,x)$ is

$$V(m,x)\cong\sqrt{2/\pi}\exp(\tfrac{1}{4}x^2)x^{m-1/2}\times\left[1+\frac{(m-\frac{1}{2})(m-\frac{3}{2})}{2x^2}+\cdots\right]. \tag{18}$$

At this stage of the argument,[2] the error was made of keeping only the $m=0$ term. Doing so in (17) converts (16) into (5), in tantalizing agreement with the result of Hanggi's ansatz. However, if all values of m ($=3n$) are kept in (17), one in fact again obtains (4). This is an example of a correct theory having produced an incorrect conclusion through a faulty calculation.

It is instructive to explore this situation in some detail. The integral in (16) may be expressed as

$$\begin{aligned}I&=\int_{-\infty}^{0}dz\exp\left[\int_0^z dy\frac{W(y)}{D_0}[1-\tau W'(y)]\right]\\&=\int_{-\infty}^{0}dz\exp\left[-\frac{1}{D_0}[U(z)+\tfrac{1}{2}\tau W^2(z)]\right]\\&=\int_{-\infty}^{0}dz\exp(\alpha z^2-\beta z^4-\gamma z^6)\,,\end{aligned}\tag{19}$$

in which $(d/dx)U(x)=-W(x)$, and

$$\alpha\equiv\frac{a}{2D_0}(1-a\tau),\quad \beta\equiv\frac{b}{4D_0}(1-4a\tau),\quad \gamma\equiv\frac{b^2\tau}{2D_0}. \tag{20}$$

Applying the method of steepest descent to the last expression in (19) produces the result

$$\begin{aligned}I_{\rm SD}=&\left[\frac{2\pi}{8\alpha+\frac{8}{3}\frac{\beta^2}{\gamma}-4\beta\left[\frac{4}{9}\frac{\beta^2}{\gamma^2}+\frac{4}{3}\frac{\alpha}{\gamma}\right]^{1/2}}\right]^{1/2}\\&\times\exp\left[-\frac{1}{3}\frac{\alpha\beta}{\gamma}+\frac{1}{3}\alpha\left[\frac{4}{9}\frac{\beta^2}{\gamma^2}+\frac{4}{3}\frac{\alpha}{\gamma}\right]^{1/2}+\frac{1}{9}\frac{\beta^2}{\gamma}\left[\frac{4}{9}\frac{\beta^2}{\gamma^2}+\frac{4}{3}\frac{\alpha}{\gamma}\right]^{1/2}-\frac{2}{27}\frac{\beta^3}{\gamma^2}\right].\end{aligned}\tag{21}$$

If we now expand the amplitude and the argument of the exponential to first order in τ, we obtain

$$\begin{aligned}I_{\rm SD}&\cong\left[\frac{\pi D_0}{a(1+2a\tau)}\right]^{1/2}\exp\left[\frac{1}{4}\frac{\alpha^2}{\beta}-\frac{1}{8}\frac{\alpha^3}{\beta^3}\gamma\right]\\&=\left[\frac{\pi D_0}{a(1+2a\tau)}\right]^{1/2}\exp\left[\frac{a^2}{4bD_0}+O(\tau^2)\right].\end{aligned}\tag{22}$$

To get this result, we have used

$$\begin{aligned}\left[\frac{4}{9}\frac{\beta^2}{\gamma^2}+\frac{4}{3}\frac{\alpha}{\gamma}\right]^{1/2}&=\frac{2}{3}\frac{\beta}{\gamma}\left[1+3\frac{\alpha\gamma}{\beta^2}\right]^{1/2}\\&\cong\frac{2}{3}\frac{\beta}{\gamma}\left[1+\frac{3}{2}\frac{\alpha\gamma}{\beta^2}-\frac{9}{8}\left[\frac{\alpha\gamma}{\beta^2}\right]^2+\frac{27}{16}\left[\frac{\alpha\gamma}{\beta^2}\right]^3+\cdots\right]\end{aligned}\tag{23}$$

and have had to keep terms up to order $(\alpha\gamma/\beta^2)^3$ in order not to omit contributions that turn out to be of first order in τ. Two features of the result are noteworthy: (1) the amplitude contains $(1+2a\tau)^{-1/2}$ rather than $(1-a\tau)^{-1/2}$, and (2) the argument of the exponential contains no linear τ term because of a cancellation. The second feature is no accident, as we shall see later. Applying the parabolic cylinder function approach[2] to the last expression in (19) produces the result

$$I_{\rm PC}=\frac{1}{2}\left[\frac{1}{2\beta}\right]^{1/4}\sum_{n=0}^{\infty}\frac{1}{n!}(-\gamma)^n\left[\frac{1}{2\beta}\right]^{3n/2}\times\pi\exp\left[\frac{1}{4}\frac{\alpha^2}{2\beta}\right]V\left[3n,\frac{\alpha}{\sqrt{2\beta}}\right]. \tag{24}$$

This result is exact. Further evaluation requires approximate treatment of the parabolic cylinder functions $V(3n,\alpha/\sqrt{2\beta})$. They have the asymptotic expansion

$$V\left[3n,\frac{\alpha}{\sqrt{2\beta}}\right]\cong\sqrt{2/\pi}\exp\left[\frac{1}{4}\frac{\alpha^2}{2\beta}\right]x^{3n-1/2}\times\left[1+\frac{(3n-\frac{1}{2})(3n-\frac{3}{2})}{2x^2}+\cdots\right], \tag{25}$$

in which $x\equiv\alpha/\sqrt{2\beta}$ in agreement with its meaning in (17) and (18) above. The factor $(3n-\frac{1}{2})(3n-\frac{3}{2})$ is usefully rewritten as $9n(n-1)+3n+\frac{3}{4}$. In order to get a valid expression to first order in τ it is necessary to include the contributions which arise from the $(1/2x^2)[9n(n-1)+3n+\frac{3}{4}]$ terms. Insertion of (25) into (24) leads to

$$I_{\rm PC}\cong\sqrt{\pi/2\alpha}\exp\left[\frac{\alpha^2}{4\beta}-\frac{1}{8}\frac{\alpha^3}{\beta^3}\gamma\right]\times\left[1+\frac{9}{64}\frac{\alpha^4}{\beta^5}\gamma^2-\frac{3}{8}\frac{\alpha\gamma}{\beta^2}+\frac{3}{4}\frac{\beta}{\alpha^2}\right]. \tag{26}$$

Expanding everything to first order in τ yields

$$I_{\rm PC}\cong\sqrt{\pi/2\alpha}\exp\left[\frac{a^2}{4bD_0}+O(\tau^2)\right]\times\left[1-\frac{3}{2}a\tau+\frac{3}{4}\frac{bD_0}{a^2}(1-2a\tau)\right]. \tag{27}$$

In addition to imposing the small-τ limit on our result, each of the earlier studies[2–5] also involved the small D_0 regime such that bD_0/α^2, which is dimensionless, is much less than 1 (in Ref. 2 it was 0.05 and 0.1 for two cases). So we may drop this term in (27) as well. Now notice that

$$\frac{1}{\sqrt{\alpha}}(1-\tfrac{3}{2}\alpha\tau)=\sqrt{2D_0/a}\,\frac{1-\frac{3}{2}\alpha\tau}{\sqrt{1-\alpha\tau}}$$
$$\simeq\sqrt{2D_0/a}\,(1-\alpha\tau)+O(\tau^2)\ , \qquad (28)$$

which agrees, to first order in τ, with the result of steepest descents in (22), i.e.,

$$\frac{1}{\sqrt{1+2\alpha\tau}}\simeq 1-\alpha\tau+O(\tau^2)\ . \qquad (29)$$

Thus, we find that the method of steepest descent[3] is valid and accurate, and that proper use of the parabolic cylinder function approach requires the inclusion of $V(3n,\alpha/\sqrt{2\beta})$ for *all* n, and requires inclusion of the second term in the asymptotic expansion in order to get a clear indication that the amplitude factor is $(1+2\alpha\tau)^{-1/2}$ and not merely $(1-\alpha\tau)^{-1/2}$.

These results, which confirm Eq. (4), cannot be in agreement with Eq. (5), the result of Hanggi's ansatz. We have shown elsewhere[6] that Hanggi's ansatz is extremely natural in the steady-state situation, provided that D_0 is not too large. In particular, we showed that the Hanggi ansatz sits in the functional-calculus treatment[2] of stochastic differential equations in an especially transparent way. Hanggi[10] has argued that the ansatz enjoys a wider domain of validity than this, an issue that remains to be settled. All of this underscores the importance of reliable numerical simulations for this problem since Eqs. (5) and (6) were originally championed because of their agreement with putatively accurate numerical simulations. We have addressed the problem of colored-noise and white-noise numerical simulation algorithms elsewhere,[11] and will apply our conclusions regarding those to this problem in the near future.

Equation (7) is the result of a different type of error.[5] First of all, there is no necessity to factor out the slowly varying denominator $D(z)$ in (15) and then to proceed with either the method of steepest descent or with the parabolic cylinder functions. In fact, the parabolic cylinder function approach produces the exact result

$$J=\int_{-\omega}^{0}dx\,(A+Bz^2)e^{\alpha z^2-\beta z^4-\gamma z^6}$$
$$=\frac{1}{2}\left[\frac{1}{2\beta}\right]^{1/4}\sum_{n=0}^{\infty}\frac{1}{n!}(-\gamma)^n\left[A\left[\frac{1}{2\beta}\right]^{3n/2}V(3n,x)+B\left[\frac{1}{2\beta}\right]^{(3n+1)/2}V(3n+1,x)\right]\pi\exp(\tfrac{1}{4}x^2)\ , \qquad (30)$$

where once again $x\equiv\alpha/\sqrt{2\beta}$. The method of steepest descent yields an equation for the maximum which is cubic, rather than quadratic as it was in the case of (19). Nevertheless, both approaches produce Eq. (4) when carried through to first order in τ.

Now suppose we write

$$T(0)\equiv\frac{\pi}{\sqrt{2}a}\exp\left[\frac{a^2}{4bD_0}\right] \qquad (31)$$

for $\tau=0$, and want $T(\tau)$ for $\tau>0$. On the basis of all of the preceding analysis we may conclude that

$$\left.\frac{d}{d\tau}T(\tau)\right|_{\tau=0}=\tfrac{3}{2}aT(0) \qquad (32)$$

because in (4) we may write

$$\left[\frac{1+2\alpha\tau}{1-\alpha\tau}\right]^{1/2}\simeq 1+\tfrac{3}{2}\alpha\tau\ . \qquad (33)$$

In Ref. 5, (32) was used to conclude (erroneously) that

$$T(\tau)=T(0)\exp(\tfrac{3}{2}\alpha\tau)\ . \qquad (34)$$

Obviously, one must look at all higher-order derivatives of $T(\tau)$ to decide that the result is (34), which it is not.

III. GENERAL CONCLUSIONS

We have seen that colored noise produces a change in the MFPT as compared with the result for white noise. This change, for small τ, occurs in the nonexponential prefactor of the MFPT and there is no linear τ correction to the argument of the exponential factor. This is no accident. For small τ, an effective Fokker-Planck equation exists,[2] and it leads to Eq. (15). In the first part of this section we will show that the structure of (15) guarantees these features of the form of the MFPT. Numerical simulations permit exploration of the consequences of large τ values as well. An alternative theoretical analysis is required for this case and the last part of this section will be devoted to its presentation.

For small τ we scrutinize the integral in (15) and write[12]

$$\int_{-\infty}^{0}dz\frac{\psi(z)}{D(z)}\equiv\int_{-\infty}^{0}dz\frac{1}{D(z)}\exp\left[\int_0^z dy\frac{W(y)}{D(y)}\right]$$
$$=\int_{-\infty}^{0}dz\frac{1}{D(z)}\exp\left[-\frac{1}{D_0}[U(z)+\tfrac{1}{2}\tau W^2(z)]\right]\ , \qquad (35)$$

wherein we have used $U(0)=0$ and $W(0)=0$. We have seen in Sec. II that the exponential factor in the MFPT arises from the integrand of the last expression in (35) evaluated at the z value for which it is maximal. Denote this z value by z_0 when $\tau=0$, and by z^* when $\tau>0$, but small. Writing the integrand in the form

$$\frac{1}{D}\exp\left[-\frac{1}{D_0}(U+\tfrac{1}{2}\tau W^2)\right]=\exp\left[-\frac{1}{D_0}(U+\tfrac{1}{2}\tau W^2)-\ln D\right] \quad (36)$$

the equations for z^* and z_0 are

$$U'(z^*)+\tau W(z^*)W'(z^*)+\frac{1}{D(z^*)}D'(z^*)=0 \quad (37)$$

and

$$U'(z_0)+\tau W(z_0)W'(z_0)=0\ .$$

The exponential factors of the MFPT are, respectively,

$$\frac{1}{D(z^*)}\exp\left[-\frac{1}{D_0}[U(z^*)+\tfrac{1}{2}\tau W^2(z^*)]\right] \quad (38)$$

and

$$\frac{1}{D_0}\exp\left[-\frac{1}{D_0}U(z_0)\right]. \quad (39)$$

We know that $z_0=-\sqrt{a/b}$ and that $z^*-z_0=O(\tau)$. The difference between $[D(z^*)]^{-1}$ and $[D(z_0)]^{-1}$ is merely a prefactor change. The exponential factor in (38) can be expressed as a Taylor series expansion in (z^*-z_0),

$$\begin{aligned}\exp&\left[-\frac{1}{D_0}[U(z^*)+\tfrac{1}{2}\tau W^2(z^*)]\right]\\&=\exp\left[-\frac{1}{D_0}\Big([U(z_0)+\tfrac{1}{2}\tau W^2(z_0)]+(z^*-z_0)[U'(z_0)+\tau W(z_0)W'(z_0)]\right.\\&\qquad\left.+\tfrac{1}{2}(z^*-z_0)^2\{U''(z_0)+\tau W(z_0)W''(z_0)+\tau[W'(z_0)]^2\}+\cdots\Big)\right]\\&=\exp\left[-\frac{1}{D_0}\Big(U(z_0)+(z^*-z_0)[-W(z_0)+\tau W(z_0)W'(z_0)]\right.\\&\qquad\left.+\tfrac{1}{2}(z^*-z_0)^2\{U''(z_0)+\tau[W'(z_0)]^2\}+\cdots\Big)\right],\end{aligned} \quad (40)$$

because $W=-U'$ and $W(z_0)=0$ which follows from (37). But the linear τ term is the (z^*-z_0) term which has the coefficient $W(z_0)\equiv 0$.

In this way we see that the exponential's argument will have τ corrections beginning with order τ^2. However, the calculation of such corrections using the effective Fokker-Planck equation is invalid since the Fokker-Planck equation is good only to order τ, not τ^2. For higher-order τ corrections, no effective Fokker-Planck equation is possible[2] and we need another approach.

Another approach for larger τ is in fact available. Merely replace the one-dimensional description given by (1) and (2) by the two-dimensional description of coupled equations,

$$\frac{d}{dt}x=W(t)+\epsilon\ , \quad (41)$$

$$\frac{d}{dt}\epsilon=-\frac{1}{\tau}\epsilon+\frac{1}{\tau}\eta(t)\ , \quad (42)$$

in which $\eta(t)$ is a stochastic force with zero mean and white-noise correlation function,

$$\langle\eta(t)\eta(s)\rangle=2D_0\delta(t-s)\ . \quad (43)$$

This implies that

$$\{\langle\epsilon(t)\epsilon(s)\rangle\}=\frac{D_0}{\tau}\exp\left[-\frac{|t-s|}{\tau}\right], \quad (44)$$

in which $\langle\ \rangle$ denotes averaging with respect to η and $\{\ \}$ denotes averaging over the distribution of initial values of ϵ. Since the process is stationary, the distribution for initial values of ϵ is equal to the steady-state distribution which is

$$P_S(\epsilon)=\left[2\pi\frac{D_0}{\tau}\right]^{1/2}\exp\left[-\frac{1}{2}\frac{\epsilon^2\tau}{D_0}\right]. \quad (45)$$

The coupled equations constitute a two-dimensional Markovian process for which a bona fide Fokker-Planck equation exists. We may find it easily, using van Kampen's lemma[13] for the purpose. We get

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial x}[(W+\epsilon)P]+\frac{\partial}{\partial\epsilon}\left[\left[\frac{\epsilon}{\tau}+\frac{D}{\tau^2}\frac{\partial}{\partial\epsilon}\right]P\right], \quad (46)$$

in which $P=P(x,\epsilon,t)$. To get the MFPT we need Kolmogorov's backward equation[2] which is the adjoint of (46) given by

$$\frac{\partial}{\partial t}Q=(W+\epsilon)\frac{\partial}{\partial x}Q-\frac{\epsilon}{\tau}\frac{\partial}{\partial \epsilon}Q+\frac{D}{\tau^2}\frac{\partial^2}{\partial \epsilon^2}Q \;, \qquad (47)$$

in which $Q=Q(x,\epsilon,\tau)$. It is straightforward to show that the differential operator

$$-\frac{\epsilon}{\tau}\frac{\partial}{\partial \epsilon}+\frac{D}{\tau^2}\frac{\partial^2}{\partial \epsilon^2}$$

has Hermite polynomials as eigenfunctions with eigenvalues given by n/τ for non-negative integers n. Explicitly, we may write

$$Q_S(x,\epsilon)=\sum_{n=0}^{\infty} b_n(x)H_n(\epsilon/\sqrt{2D_0/\tau}) \;, \qquad (48)$$

in which Q_S is the steady-state solution to (47), and H_n is the nth Hermite polynomial. The analysis of the MFPT given in Ref. 2 is readily generalized to two dimensions. The analogue to Eq. (12) is

$$W(x)\frac{\partial}{\partial x}T(x,\epsilon)+\epsilon\frac{\partial}{\partial x}T(x,\epsilon) -\frac{\epsilon}{\tau}\frac{\partial}{\partial \epsilon}T(x,\epsilon)+\frac{D}{\tau^2}\frac{\partial^2}{\partial \epsilon^2}T(x_1\epsilon)=-1 \;, \qquad (49)$$

in which $T(x,\epsilon)$ denotes the MFPT as a function of starting position (x,ϵ). Clearly, in parallel with (48) we may also write

$$T(x,\epsilon)=\sum_{n=0}^{\infty} C_n(x)H_n(\epsilon/\sqrt{2D_0/\tau}) \;. \qquad (50)$$

Insertion into (49) yields the equation

$$-1=\sum_{n=0}^{\infty}\left[WC_n'H_n+\epsilon C_n'H_n-\frac{n}{\tau}C_nH_n\right] \;, \qquad (51)$$

in which the prime denotes the x derivative and in which we have used the eigenfunction properties of the H_n's. Now, we do not want the MFPT for initial starting values of both x and ϵ, but only for initial x values. Thus, we should average (50) over the distribution (45),[14]

$$T(x)\equiv \int_{-\infty}^{\infty} d\epsilon\, P_S(\epsilon)T(x,\epsilon) =C_0(x) \;. \qquad (52)$$

Moreover, multiplying (51) by $P_S(\epsilon)H_m(\epsilon/\sqrt{2D_0/\tau})$ and integrating produces the system of coupled equations

$$-1=WC_0'+\sqrt{2D_0/\tau}C_1' \;, \qquad (53)$$

$$0=WC_1'+\sqrt{2D_0/\tau}(\tfrac{1}{2}C_0'+2C_2')-\frac{1}{\tau}C_1 \;, \qquad (54)$$

$$0=WC_2'+\sqrt{2D_0/\tau}(\tfrac{1}{2}C_1'+3C_3')-\frac{2}{\tau}C_2 \;, \qquad (55)$$

$$\vdots$$

$$0=WC_m'+\sqrt{2D_0/\tau}[\tfrac{1}{2}C_{m-1}'+(m+1)C_{m+1}']-\frac{m}{\tau}C_m \;, \quad m\geq 1 \;. \qquad (56)$$

It is useful to rewrite Eq. (56) in the form

$$0=W\tau C_m'+\sqrt{2D_0\tau}[\tfrac{1}{2}C_{m-1}'+(m+1)C_{m+1}']-mC_m \;, \quad m\geq 1 \;. \qquad (57)$$

Study of Eqs. (53) and (57) shows that the system has systematic solutions of the form

$$C_m=\tau^{m/2}\sum_{n=0}^{\infty}\tau^n N_{m,n} \;. \qquad (58)$$

Knowing this, it is readily seen that to zeroth order in τ, i.e., to order τ^0, Eqs. (53) and (57) imply

$$-1=WC_0'+D_0C_0'' \;, \qquad (59)$$

which is precisely (12) for $\tau=0$. [This follows from (57) because $C_1=\frac{1}{2}\sqrt{2D_0\tau}C_0'$ which is $O(\tau^0)$.] We may rewrite (57) in still another useful way by taking its x derivative and rearranging terms to get

$$C_m'=(m-\tau W')^{-1} \times\{W\tau_m''+\sqrt{2D_0\tau}[\tfrac{1}{2}C_{m-1}''+(m+1)C_{m+1}'']\} \;. \qquad (60)$$

One may be tempted to expand $(m-\tau W')^{-1}$ but while it is well defined as is, for all values of $x\in(-\infty,0)$, each term of its expansion is divergent. This reflects the fact the smallness of τ alone does not make $\tau W'$ small for all x. We addressed this point before in some detail.[2] Using (60) together with (53) and keeping terms to order $\tau^{1/2}$ in (60), while leaving $(m-\tau W')^{-1}$ as is, gives

$$C_1'\cong(1-\tau W')^{-1}\sqrt{2D_0\tau}\tfrac{1}{2}C_0'' \qquad (61)$$

and

$$-1=WC_0'+(1-\tau W')^{-1}D_0C_0'' \;, \qquad (62)$$

which is precisely (12) with $D(x)$ in (12) given by (11). Thus, we have discovered a systematic procedure for obtaining τ corrections to (59), the $\tau=0$ equation for the MFPT. However, the next order in τ produces the equation

$$-1=WC_0'+D_0(1-\tau W')^{-1}C_0''+D_0W\tau(1-\tau W')^{-2}C_0''' +D_0^2\tau(2-\tau W')^{-1}(1-\tau W')^{-2}C_0'''' \;, \qquad (63)$$

a fourth-order differential equation for C_0. This is no longer readily tractable by analysis. It does, however, formally reflect the breakdown of the effective Fokker-Planck picture, which was valid for Eq. (62), and is still only a second-order differential equation. The main difficulty resides in determining the appropriate boundary conditions for (63). The domain of ϵ integration in (52) is related to this question.

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