# Properties of fully developed chaotic one-dimensional maps in the presence of external noise

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(Received 3 August 1987)

Noise effects on various properties of fully developed chaotic maps are studied. For the weak-noise case a perturbation method is developed. It yields the stationary density, eigenvalues, eigenfunctions, and null functions of the Frobenius-Perron operator in powers of the moments of the noise. The linear response of the noisy map to a deterministic perturbation is defined and calculated perturbatively. Correlation functions and Lyapunov exponents are also investigated. The examples are the tent map and the logistic map in the fully developed chaotic state with multiplicative white noise. The results for the Lyapunov exponents are in good agreement with those determined numerically.

## I. INTRODUCTION

There is growing interest in the properties of nonlinear dynamical systems. One of the most challenging phenomena is chaos, which involves a complex dynamical behavior and sensitivity with respect to the initial conditions. Chaotic behavior has proved to be typical in wide classes of physical, chemical, and biological systems.[1] From an information-theory point of view it can be considered as an intermediate stage between regular motion with Kolmogorov entropy $K=0$ and stochastic processes with $K=\infty$.[2] The Kolmogorov entropy of a chaotic system is finite and nonzero, although the motion is governed by strictly deterministic equations.

The one-dimensional (1D) maps[3]

$$x_t=f(x_{t-1}),\quad t=1,2,3,\ldots \tag{1}$$

can provide the simplest examples for chaotic systems. They are closely related to a number of dissipative dynamical systems and have been playing an important role in the investigation of the rise, development, and properties of the chaotic state.

Real physical systems are always subjected to the influence of external noise due to coupling to the surroundings. Noise effects can also be studied at the level of 1D maps. A noisy map is given by

$$x_t=f(\zeta_t,x_{t-1}),\quad t=1,2,3,\ldots \tag{2}$$

where $\zeta_t$'s are stochastic variables characterizing the noise. We assume in the following, that $\zeta_t$'s at different "times" $t$ are independent variables (white noise). Such maps have been examined by several authors.[4–6] The scaling properties of the period-doubling route leading to chaos in the presence of external noise was determined in Refs. 7 and 8. The Frobenius-Perron operator of noisy maps was studied in Ref. 9. A suitable formalism for the study of noisy maps (master equation, recurrence times, etc.) was introduced in Refs. 10 and 11.

In this paper we consider single-humped 1D maps of the interval [0,1] onto itself in the fully developed chaotic (FDC) state.[12] Such a map has no stable periodic orbit, but exhibits chaotic behavior, fully developed in the sense that the corresponding attractor is the whole interval [0,1]. A fully developed chaotic attractor can be observed in the logistic map $f(x)=rx(1-x)$ when the control parameter $r$ equals 4. More generally, it is the final stage of the evolution of the attractor (from fixed point through periodic orbits to chaotic attractors) of a 1D single-humped map. FDC can also be found in parameter-controlled maps at the band-merging points and also at the crisis points if one considers a suitable iterate of the original map. Consequently, our investigation is relevant for a large class of maps. In this paper we generalize the perturbation method worked out for the deterministic case in Ref. 13 for stochastic maps. Furthermore, based on it, we carry out explicit calculations for correlation and response functions and for Lyapunov exponents.

The paper is organized as follows. Section II contains general relations for noisy maps and a new expression for the dynamical response function is also derived there. In Sec. III we present a perturbation method to investigate the eigenvalues, eigenfunctions, and null functions of the Frobenius-Perron operator of noisy maps. The unperturbed maps are deterministic ones, i.e., our results are expected to be valid in the case of weak noise. We work out examples in detail. In Sec. IV we calculate the correlation and response functions in our examples using the results of Sec. III. In Sec. V the Lyapunov exponents of noisy maps are defined and calculated perturbatively. Section VI contains a comparison of our analytical and numerical results concerning the Lyapunov exponents.

 

## II. GENERAL FORMULAS

The noisy map (2) generates a Markov process.[10] The transition probability of the process is expressed as

$$P(x,t+1\,|\,y,t)=\int d\zeta\,\rho(\zeta)\delta(x-f(\zeta,y))\ . \qquad (3)$$

Here $\rho(\zeta)$ is the probability density of the noise. The Frobenius-Perron (FP) operator[11,14] of the map (2) is given by

$$\hat{H}\varphi(x)=\int dy\,\varphi(y)P(x,t+1\,|\,y,t)\ . \qquad (4)$$

The probability of finding the trajectory between $x$ and $x+dx$ after $t$ iteration is $P_t(x)dx$. One gets for the time evolution of the density $P_t(x)$

$$P_{t+1}(x)=\hat{H}P_t(x)\ . \qquad (5)$$

For the stationary density $P(x)$ one has

$$P(x)=\hat{H}P(x)\ . \qquad (6)$$

The mean value of any quantity $A(x)$ at time $t$ is defined by

$$\overline{A}_t=\int dx\,P_t(x)A(x)\ . \qquad (7)$$

Especially in the stationary state of the process we get the mean value

$$\overline{A}=\int dx\,P(x)A(x)\ . \qquad (8)$$

The correlation function of the quantities $A(x)$ and $B(x)$ is defined by

$$C_{t,t+\tau}=\int dx\int dy\,P(x,t+\tau;y,t)\times[A(x)-\overline{A}][B(y)-\overline{B}],\quad \tau\geq 0 \qquad (9)$$

where the simultaneous probability density $P(x,t_1;y,t_2)$ in the stationary state is given by

$$P(x,t+\tau;y,t)=P(x,\tau\,|\,y,0)P(y),\quad \tau\geq 0\ . \qquad (10)$$

Using (4) and the Markovian property of the process one gets for the correlation function

$$C_\tau\equiv C_{t,t+\tau}=\int dx\ A(x)\hat{H}^{\tau}\{[B(x)-\overline{B}]P(x)\},\quad \tau\geq 0 \qquad (11)$$

(for the deterministic case see Ref. 15).

Another important characteristic of the noisy map is the linear response function. In deterministic maps it has been studied in case of a static perturbation in Ref. 12 and more generally in Refs. 16 and 17. The generalization to the noisy case can be performed by applying the FP equations (4) and (5) instead of that valid in the deterministic case. Below we present a derivation, which we think to be the most direct way to get the expression of the response function. Let us examine the behavior of the average value of the smooth quantity $D(x)$. For $t<0$ the stochastic process corresponding to (2) is in its stationary state. At time $t=0$ there is a small excitation acting on it during one iteration step. This excitation can be described by the perturbed map

$$x_t=f(\zeta_t,x_{t-1})+h\delta_{0,t}F(x_{t-1})\ . \qquad (12)$$

Switching off the perturbation, the process is assumed to relax to its original stationary state. The perturbed process has the probability density $P_t(x,h)$ given by

$$P_t(x,h)=\hat{H}^{t}\hat{H}(h)P(x),\quad t\geq 0\ . \qquad (13)$$

This is just the master equation corresponding to (12). For $t>0$ (12) is identical to the original map (2), so for $t>0$ the time evolution of the probability density is governed by the same FP operator. Nevertheless, at time $t=0$ the FP operator related to (12) is expressed as

$$\hat{H}(h)\equiv\int d\zeta\,\rho(\zeta)\delta(x-f(\zeta,y)-hF(y)) \qquad (14)$$

analogously to (3) and (4). The mean value of $D(x)$ at time $t>0$ is given by

$$\overline{D}_t=\int dx\,P_t(x,h)D(x)\ . \qquad (15)$$

The change of the mean value of $D(x)$ due to the perturbation is expressed as [using (6), (8), (13), and (15)]

$$\delta\overline{D}_t=\int dx\,D(x)\hat{H}^{t}[\hat{H}(h)-\hat{H}]P(x),\quad t\geq 0\ . \qquad (16)$$

If $[\hat{H}(h)-\hat{H}]P(x)$ is of the form $S(x)h+o(h)$ [here $o(h)$ has the property $\lim_{h\to 0}o(h)/h=0$ and $S(x)$ is some smooth function], one can define the linear response as

$$\chi_t^{DF}=\lim_{h\to 0}\frac{\delta\overline{D}_t}{h}=\int dx\,D(x)H^{t}\frac{\partial\hat{H}(h)}{\partial h}P(x),\quad t\geq 0 \qquad (17)$$

or, using (14),

$$\chi_t^{DF}=\int dx\,D(x)\hat{H}^{t}\partial_x\hat{H}[-F(x)P(x)],\quad t\geq 0\ . \qquad (18)$$

This expression has the same form as that given in Ref. 17.

The coupling operator $F(x)$ of the perturbation has to fulfil the requirement that the perturbation should not change the type of the chaos.[17] Only in such circumstances can one expect linear response. An example is the fully developed chaos,[12] where both the unperturbed and the perturbed map belong to this class.

Because of the linearity of the response we can write the deviation $\delta\overline{D}_t$ for any small excitation $h_j$ $(j=0,1,2,\ldots)$ as

$$\delta\overline{D}_t=\sum_{j=0}^{t}\chi_{t-j}^{DF}h_j\ . \qquad (19)$$

For a constant perturbation, after a long time we get a new stationary state $P(x,h)$,

$$P(x,h)=\hat{H}(h)P(x,h)\ . \qquad (20)$$

We define the static response function as

$$\chi^{DF}=\lim_{h\to 0}\frac{\delta\overline{D}}{h}=\int dx\,D(x)\frac{\partial P(x,h)}{\partial h}\bigg|_{h=0}\ . \qquad (21)$$

Of course, it is meaningless if the first derivative of $P(x,h)$ with respect to $h$ does not exist at $h=0$ or if the integral in (21) diverges. The static response function can be expressed in terms of the dynamic one [using (19)]

$$\chi^{DF}=\lim_{\epsilon\to+0}\sum_{t=0}^{\infty}e^{-\epsilon t}\chi_t^{DF} . \tag{22}$$

If the static response exists, we have from (20)

$$\left.\frac{\partial P(x,h)}{\partial h}\right|_{h=0}=\left.\frac{\partial\hat{H}(h)}{\partial h}\right|_{h=0}P(x)+\hat{H}\left.\frac{\partial P(x,h)}{\partial h}\right|_{h=0} . \tag{23}$$

As a result, we get a new expression for the dynamic response function [using (23) and (17)]:

$$\chi_t^{DF}=\int dx\, D(x)(\hat{H}^{\,t}-\hat{H}^{\,t+1})\left.\frac{\partial P(x,h)}{\partial h}\right|_{h=0} . \tag{24}$$

## III. EIGENVALUES, EIGENFUNCTIONS, AND NULL FUNCTIONS OF THE FROBENIUS-PERRON OPERATOR. PERTURBATION THEORY IN THE WEAK-NOISE LIMIT

The eigenvalues and the eigenfunctions of the FP operator are the solutions of the equation

$$\hat{H}\varphi(x)=r\varphi(x) , \tag{25}$$

where all the eigenfunctions $\varphi(x)$ are $L^1$-integrable functions of $x$ in (0,1). We consider the point spectrum of $\hat{H}$. The set of eigenfunctions corresponding to the eigenvalue $r=0$ constitutes the null space $N_0$ of the FP operator. It gives rise to null spaces of higher indices through the relation

$$\hat{H}N_{j+1}=N_j,\quad j=0,1,2,\ldots . \tag{26}$$

We shall denote the $k$th element of the $j$th null space by $N_{j,k}(x)$.

Now we develop a perturbation theory to calculate the above quantities in the weak-noise limit. In the following, the term "unperturbed map" means a deterministic map assumed to have known properties and being given by (2) with $\zeta=0$. "Perturbed map" means the noisy map (2) itself. Let us assume that the noise is weak, i.e., the moments of $\zeta$ are small. The $j$th moment is defined by

$$m_j=\int d\zeta\rho(\zeta)\zeta^j,\quad j=1,2,3,\ldots . \tag{27}$$

We assume that there exist convergent expansions of the quantities (25) and (26) in the form

$$r_i=\sum_{\beta}r_i(\boldsymbol{\beta})\prod_{l=1}^{\infty}(m_l)^{\beta_l},\quad \beta_l=0,1,2,\ldots \tag{28}$$

$$\varphi_i(x)=\sum_{\beta}\varphi_i(\boldsymbol{\beta},x)\prod_{l=1}^{\infty}(m_l)^{\beta_l} , \tag{29}$$

$$N_{j,k}(x)=\sum_{\beta}N_{j,k}(\boldsymbol{\beta},x)\prod_{l=1}^{\infty}(m_l)^{\beta_l} , \tag{30}$$

$$\hat{H}=\sum_{\beta}\hat{H}(\boldsymbol{\beta})\prod_{l=1}^{\infty}(m_l)^{\beta_l} . \tag{31}$$

Now each order of the above expansions is labeled by an index vector $\boldsymbol{\beta}$. These vectors are of infinite dimensionality, but in each term of the expansions (28)–(31) they have only a finite number of nonzero components, e.g., if we have

$$r_1=\tfrac{1}{4}-\tfrac{3}{4}m_1+m_2+\tfrac{5}{4}m_1m_2+\cdots ,$$

then

| $\boldsymbol{\beta}$ | $r_1(\boldsymbol{\beta})$ |
|---|---|
| (0,0,0,0, . . . ) | $\frac{1}{4}$ |
| (1,0,0,0, . . . ) | $-\frac{3}{4}$ |
| (2,0,0,0, . . . ) | 0 |
| (0,1,0,0, . . . ) | 1 |
| (1,1,0,0, . . . ) | $\frac{5}{4}$ |

We shall use the notation $|\boldsymbol{\beta}|$ for the sum of the components of $\boldsymbol{\beta}$. For a sharp probability density around some nonzero mean value the terms with the same value of $|\boldsymbol{\beta}|$ are approximately of the same magnitude.

From (3) and (4) we find that

$$\hat{H}(\boldsymbol{\beta})=\begin{cases}\hat{H}_j, \ \text{if } \beta_l=\delta_{jl}\\ (j=0,1,2,\ldots;\ l=1,2,2,\ldots)\\ 0,\ \text{otherwise} .\end{cases} \tag{32}$$

We give the explicit expression of the operators $\hat{H}_j$ in that case when

$$f(\zeta,x)=f_0(x)+\zeta f_1(x) . \tag{33}$$

Then the operators $\hat{H}_j$ are expressed as

$$\hat{H}_j\equiv\frac{(f_1(y))^j}{j!}\delta^{(j)}(f_0(y)-x),\quad j=0,1,2,\ldots . \tag{34}$$

From the expansion of the equation (25) we get in order $\boldsymbol{\alpha}$

$$[r_i(0)-\hat{H}_0]\varphi_i(\boldsymbol{\alpha},x)=\sum_{\beta}{}'[\hat{H}(\boldsymbol{\beta})-r_i(\boldsymbol{\beta})]\varphi_i(\boldsymbol{\alpha}-\boldsymbol{\beta},x) , \tag{35}$$

where the summation constraints are

$$|\boldsymbol{\beta}|\ge 1;\quad \alpha_l\ge\beta_l,\quad l=1,2,3,\ldots .$$

We assume the existence of a projection formalism which allows for expansions in terms of the unperturbed eigenfunctions. Let us define the projection to the $i$th eigenfunction by

$$\mathcal{P}_i\left[\sum_{j=0}^{\infty}a_j\varphi_j(0,x)\right]=a_i . \tag{36}$$

Expanding the right-hand side of (35) in terms of the unperturbed eigenfunctions, the term proportional to $\varphi_i(0,x)$ must not appear. From this requirement we get for $r_i(\boldsymbol{\alpha})$ the following expression:

$$r_i(\boldsymbol{\alpha})=\mathcal{P}_i\left[\sum_{n=1}^{|\alpha|}\sum_{\beta_1}{}'\sum_{\beta_2}{}'\cdots\sum_{\beta_n}{}'\hat{H}(\boldsymbol{\beta}_1)[r_i(0)-\hat{H}_0]^{-1}[\hat{H}(\boldsymbol{\beta}_2)-r_i(\boldsymbol{\beta}_2)]\cdots[r_i(0)-\hat{H}_0]^{-1}[\hat{H}(\boldsymbol{\beta}_n)-r_i(\boldsymbol{\beta}_n)]\varphi_i(0,x)\right], \quad (37)$$

where the summation constraints are

$$|\boldsymbol{\beta}_j|\geq 1,\quad j=1,2,3,\ldots,\sum_{j=1}^{n}\boldsymbol{\beta}_j=\boldsymbol{\alpha}.$$

To avoid ambiguities in the determination of $\varphi_i(\boldsymbol{\alpha},x)$ let us prescribe the condition

$$\mathcal{P}_i(\varphi_i(\boldsymbol{\alpha},x))=0. \quad (38)$$

So we get for the corrections of the eigenfunctions

$$\varphi_i(\boldsymbol{\alpha},x)=\sum_{n=1}^{|\alpha|}\sum_{\beta_1}{}'\sum_{\beta_2}{}'\cdots\sum_{\beta_n}{}'[r_i(0)-\hat{H}_0]^{-1}[\hat{H}(\boldsymbol{\beta}_1)-r_i(\boldsymbol{\beta}_1)]\cdots[r_i(0)-\hat{H}_0]^{-1}[\hat{H}(\boldsymbol{\beta}_n)-r_i(\boldsymbol{\beta}_n)]\varphi_i(0,x), \quad (39)$$

where the summation constraints are

$$|\boldsymbol{\beta}_j|\geq 1,\quad j=1,2,3,\ldots,\sum_{j=1}^{n}\boldsymbol{\beta}_j=\boldsymbol{\alpha}.$$

The expressions (37)–(39) are the analogs of those in the deterministic case.[13]

Let us apply the above method to the case of the bilinear (BL) map. The BL map is a perturbed tent map having the form[18]

$$f(\zeta,x)=1-|1-2x|+\zeta|1-2x|(1-|1-2x|),\qquad -\tfrac{1}{2}<\zeta<1. \quad (40)$$

Now the eigenvalues and the eigenfunctions of $\hat{H}_0$ are those of the tent map, i.e.,[13,19,20]

$$r_i(0)=4^{-i},\quad \varphi_i(0,x)=B_{2i}\left[\frac{x}{2}\right],\quad i=0,1,2,\ldots$$
$$r=0,\quad N_{0,k}(0,x)=B_{2k-1}(x),\quad k=1,2,3,\ldots \quad (41)$$

where $B_j(x)$ is the $j$th Bernoulli polynomial. It is easy to see that in this case the right-hand side of (35) is a polynomial, which can be expressed as a finite linear combination of the unperturbed eigenfunctions. The projection (36) is given by

$$\mathcal{P}_i(\varphi(x))=\frac{4^i}{(2i)!}\int_0^1\varphi^{(2i)}(x)dx,\quad i=0,1,2,\ldots. \quad (42)$$

We get the results from Eqs. (37)–(39). The eigenfunction $\varphi_0(x)$ corresponding to $r_0=1$ is just $P(x)$, i.e., the stationary density. The perturbation does not affect the eigenvalue $r_0$. The stationary density is given by

$$\begin{aligned}P_{\mathrm{BL}}x&=1+(-1+2x)m_1+(\tfrac{7}{3}-10x+8x^2)m_2\\&\quad+(-\tfrac{8}{3}+\tfrac{44}{3}x-20x^2+8x^3)m_1m_2\\&\quad+(-1+12x-30x^2+20x^3)m_3+\cdots.\end{aligned} \quad (43)$$

The results for some eigenvalues are expressed as

$$\begin{aligned}r_1&=\tfrac{1}{4}-\tfrac{3}{4}m_1+m_2+\tfrac{5}{4}m_1m_2-\tfrac{5}{2}m_3+\tfrac{5}{6}m_1^2m_2-\tfrac{1}{4}m_2^2\\&\quad-\tfrac{35}{12}m_1m_3+\tfrac{25}{12}m_4+\cdots,\\r_2&=\tfrac{1}{16}-\tfrac{5}{8}m_1+\tfrac{5}{4}m_2+\tfrac{15}{2}m_1m_2-\tfrac{105}{16}m_3+\cdots,\\r_3&=\tfrac{1}{64}-\tfrac{21}{64}m_1+\tfrac{7}{8}m_2+\cdots.\end{aligned} \quad (44)$$

The corresponding eigenfunctions can be found in the Appendix.

Now we turn to the case of the null spaces. Using the expansions (30) and (31) we get from (26) in order $\boldsymbol{\alpha}$ the following equation:

$$\hat{H}_0N_{j+1,k}(\boldsymbol{\alpha},x)=N_{j,k}(\boldsymbol{\alpha},x)-\sum_{\beta}{}'\hat{H}(\boldsymbol{\beta})N_{j+1,k}(\boldsymbol{\alpha}-\boldsymbol{\beta},x), \quad (45)$$

where the summation constraints are

$$|\boldsymbol{\beta}|\geq 1,\quad \alpha_l\geq\beta_l,\quad l=1,2,3,\ldots.$$

To avoid ambiguities when solving (45) we prescribe the condition

$$\hat{H}_0^{-1}N_{j,k}(0,x)=N_{j+1,k}(0,x). \quad (46)$$

As an example we consider the case of the biquadratic (BQ) map. The BQ map is a perturbed logistic map having the form[12]

$$f(\zeta,x)=4x(1-x)+\zeta 4x(1-x)(1-2x)^2,\qquad -\tfrac{3}{4}<\zeta<1. \quad (47)$$

Let us determine a basis containing analytic functions in the extended null space of the operator $\hat{H}_0$. For the tent map it is known[19] that such a basis exists and consists of sine and cosine functions. The tent map and the logistic map are smoothly conjugated to each other,[3]

$$f_T(x)=u(f_L(u^{-1}(x))),\quad f_T(x)=1-|1-2x|,$$
$$f_L(x)=4x(1-x),$$
$$u(x)=\frac{2}{\pi}\arcsin(x^{1/2}), \quad (48)$$

where $f_T(x)$ is the tent map and $f_L(x)$ is the logistic map. Using (48) one can derive the following relation between the null functions of the FP operator of the tent map and the logistic map:

$$N^L_{j,k}(x)=N^T_{j,k}(u(x))u'(x) . \tag{49}$$

As a result, we get

$$N_{j,k}(0,x)=N^L_{j,k}(x)=P_0(x)T_n[(1-x)^{1/2}] , \tag{50}$$

where $n=(2k+1)2^{j+1}$. Here $T_n(x)=\cos(n \arccos x)$ is the $n$th Chebyshev polynomial and $P_0(x)=(1/\pi)[x(1-x)]^{-1/2}$ is the stationary density of the logistic map. All of the $N_{j,k}(0,x)$ functions are of the type (polynomial)$\times P_0(x)$. Using (34) it can be shown that the $\hat{H}_j$ operations in our example do not lead out of this class of functions. [More generally, the same is true when $f_1(x)=G(f_0(x))$ and $G(x)$ has the form $x(1-x)\times$(polynomial).] We can proceed using (45) and (46). So we get the following results (for simplicity we omit the argument $x$):

$$\begin{aligned}
N_{0,k}&=N_{0,k}(0) ,\\
N_{1,0}&=N_{1,0}(0)+\tfrac{1}{2}N_{2,0}(0)m_1+[\tfrac{3}{8}N_{1,1}(0)-\tfrac{1}{8}N_{1,0}(0)]m_1^2-\tfrac{3}{8}N_{1,1}(0)m_2+[\tfrac{9}{16}N_{1,1}(0)-\tfrac{7}{16}N_{2,0}(0)]m_1^3\\
&\quad+[-\tfrac{9}{16}N_{1,1}(0)+\tfrac{1}{2}N_{2,0}(0)-\tfrac{5}{16}N_{3,0}(0)]m_1m_2+\tfrac{5}{16}N_{3,0}(0)m_3+\cdots ,\\
N_{2,0}&=N_{2,0}(0)+[\tfrac{1}{2}N_{3,0}(0)+\tfrac{3}{4}N_{1,1}(0)-\tfrac{1}{4}N_{1,0}(0)]m_1\\
&\quad+[\tfrac{3}{8}N_{2,1}(0)-N_{2,0}(0)+\tfrac{5}{8}N_{1,2}(0)+\tfrac{3}{4}N_{1,1}(0)]m_1^2+[-\tfrac{3}{8}N_{2,1}(0)-\tfrac{5}{8}N_{3,0}(0)+\tfrac{1}{4}N_{2,0}(0)]m_2+\cdots ,\\
N_{3,0}&=N_{3,0}(0)+[\tfrac{1}{2}N_{4,0}(0)+\tfrac{3}{4}N_{2,1}(0)-\tfrac{1}{4}N_{2,0}(0)+\tfrac{5}{4}N_{1,2}(0)-\tfrac{3}{4}N_{1,1}(0)]m_1+\cdots ,\\
N_{1,1}&=N_{1,1}(0)+[\tfrac{3}{2}N_{1,1}(0)-N_{2,0}(0)]m_1+\cdots .
\end{aligned} \tag{51}$$

In the end we also calculate the stationary distribution of the BQ map. In this case we can expand the right-hand side of (35) in terms of the unperturbed null functions. The projections (36) are expressed as

$$\begin{aligned}
\mathcal{P}_0(\varphi(x))&=\int_0^1 \varphi(x)dx ,\\
\mathcal{P}_i(\varphi(x))&=\frac{4^i}{(2i)!}\int_0^1 \frac{d}{dx}\left[\frac{[\varphi(x)]^{(2i-1)}}{(P_0(x))^{2i}}\right]dx, \quad i=1,2,3,\dots .
\end{aligned} \tag{52}$$

Now we can use (39), where the effect of the operators $(1-\hat{H}_0)^{-1}$ on the unperturbed null functions is given by

$$(1-\hat{H}_0)^{-1}N_{j,k}(0,x)=(1+\hat{H}_0+\hat{H}_0^2+\cdots)N_{j,k}(0,x)=\sum_{l=0}^{j}N_{l,k}(0,x) . \tag{53}$$

The result is

$$\begin{aligned}
P_{\mathrm{BQ}}(x)&=P_0(x)-\tfrac{1}{2}N_{0,0}(0)m_1+\tfrac{3}{8}[N_{1,0}(0)+N_{0,0}(0)]m_2-\tfrac{3}{16}[N_{1,0}(0)+N_{0,0}(0)]m_1m_2\\
&\quad-\tfrac{5}{16}N_{0,1}(0)m_3+\tfrac{3}{32}[N_{1,0}(0)+N_{0,0}(0)]m_1^2m_2+\tfrac{9}{64}N_{0,1}(0)m_2^2\\
&\quad+\tfrac{35}{128}[N_{2,0}(0)+N_{1,0}(0)+N_{0,0}(0)]m_4+\cdots
\end{aligned} \tag{54}$$

## IV. CORRELATION AND RESPONSE FUNCTIONS

Let us consider the expression of the correlation function given by (11). The explicit time dependence of the correlation function can be evaluated if $[B(x)-\bar{B}]P(x)$ can be expanded to a convergent series in terms of the eigenfunctions and null functions of the FP operator and if the term-by-term application of the FP operator is allowed on it. We will carry out explicit calculations by choosing $B(x)=x$ in our examples.

In the case of the BL map the unperturbed eigenfunctions are polynomials, including the functions $N_{0,k}(0,x)$ [see (41)]. Let us consider the expansion of $(x-\bar{x})P(x)$ in powers of the moments of the noise. All of its terms are polynomials of $x$, and they can be expressed as finite linear combinations of the unperturbed eigenfunctions. Taking into account terms to a given order in the expansion of the perturbed eigenfunctions, there is a one-to-one correspondence between the perturbed and the unperturbed eigenfunctions (see Appendix). It makes it possible to expand $(x-\bar{x})P(x)$ in terms of the perturbed eigenfunctions, and then from (11) we get the correlation function as

$$\begin{aligned}
C_t^{\mathrm{BL}}&=\delta_{0t}\int_0^1 dx\, A(x)[N_{0,1}+2N_{0,1}m_1+(5N_{0,1}-8N_{0,2})m_1^2+(8N_{0,2}-\tfrac{2}{3}N_{0,1})m_2+(16N_{0,1}-24N_{0,2})m_1^3\\
&\qquad+(\tfrac{40}{3}N_{0,1}-24N_{0,2})m_1m_2+(-13N_{0,1}+40N_{0,2})m_3+\cdots]\\
&\quad+r_1^t(8m_1-8m_2+\tfrac{40}{3}m_1m_2+28m_3+\cdots)\int_0^1 dx\, A(x)\varphi_1(x)+r_2^t(-512m_1m_2+320m_3+\cdots)\\
&\qquad\times\int_0^1 dx\, A(x)\varphi_2(x)+\cdots .
\end{aligned} \tag{55}$$

In the case of the BQ map the terms of the expansion of $(x-\bar{x})P(x)$ can be expressed as finite linear combinations of the unperturbed null functions (50). At a given order it is also possible to express the unperturbed null functions with the perturbed ones, and then to expand $(x-\bar{x})P(x)$ in terms of the perturbed null functions. Using (11) we get the correlation function as

$$C_t^{\mathrm{BQ}}=\int_0^1 dx\; A(x)[-\tfrac{1}{2}N_{-t,0}+\tfrac{1}{8}N_{1-t,0}m_1+(-\tfrac{1}{16}N_{2-t,0}+\tfrac{1}{16}N_{-t,0})m_1^2$$
$$+(-\tfrac{3}{32}N_{1-t,0}-\tfrac{3}{32}N_{-t,0}-\tfrac{3}{32}N_{-t,1})m_2+\tfrac{1}{32}N_{3-t,0}m_1^3$$
$$+(\tfrac{3}{64}N_{2-t,0}+\tfrac{3}{64}N_{1-t,1}-\tfrac{3}{64}N_{-t,0}+\tfrac{3}{64}N_{-t,1})m_1m_2$$
$$+(\tfrac{5}{128}N_{2-t,0}+\tfrac{5}{64}N_{1-t,0})m_3+\cdots] . \tag{56}$$

Here $N_{-t,k}(x)=\hat{H}^{\,t}N_{0,k}(x)=0$ if $t>0$.

The static response function (21) exists in both cases. We have chosen $F(x)=f_1(x)$ for simplicity [see (33)]. Now

$$m_j(h)=m_j+hjm_{j-1}+o(h) . \tag{57}$$

Putting this into (43) and (54) and then using (21) we get the static response functions

$$\chi^{\mathrm{BL}}=\int_0^1 dx\; D(x)\{2N_{0,1}+[64\varphi_1(0)+12N_{0,1}]m_1+[-64\varphi_1(0)+16N_{0,2}-\tfrac{32}{3}N_{0,1}]m_1^2$$
$$+[-32\varphi_1(0)+68N_{0,2}+\tfrac{2}{3}N_{0,1}]m_2+[64\varphi_1(0)-16N_{0,2}+\tfrac{32}{3}N_{0,1}]m_1^3$$
$$+[2048\varphi_2(0)+\tfrac{704}{3}\varphi_1(0)+144N_{0,2}-\tfrac{256}{3}N_{0,1}]m_1m_2$$
$$+[\tfrac{14\,336}{3}\varphi_2(0)+\tfrac{1280}{3}\varphi_1(0)+560N_{0,2}-200N_{0,1}]m_3+\cdots\} , \tag{58}$$

$$\chi^{\mathrm{BQ}}=\int_0^1 dx\; D(x)\{-\tfrac{1}{2}N_{0,0}(0)+\tfrac{3}{4}[N_{1,0}(0)+N_{0,0}(0)]m_1-\tfrac{3}{8}[N_{1,0}+N_{0,0}(0)]m_1^2$$
$$-[\tfrac{3}{16}N_{1,0}(0)+\tfrac{3}{16}N_{0,0}(0)+\tfrac{15}{16}N_{0,1}(0)]m_2+\tfrac{3}{16}[N_{1,0}(0)+N_{0,0}(0)]m_1^3$$
$$+[\tfrac{3}{16}N_{1,0}(0)+\tfrac{3}{16}N_{0,0}(0)+\tfrac{9}{16}N_{0,1}(0)]m_1m_2$$
$$+\tfrac{35}{32}[N_{2,0}(0)+N_{1,0}(0)+N_{0,0}(0)]m_3+\cdots\} . \tag{59}$$

Since the static response functions exist, we can apply (24) to calculate the dynamic ones. To this end we have to expand $\partial_h P(x,h)\,|_{h=0}$ [which is just the quantity in the braces in (58) and (59)] in terms of the eigenfunctions and null functions of the FP operator. It can be expanded the same way, as in the case of the correlation function the quantity $(x-\bar{x})P(x)$ (see above). Putting the result into (24) we get the dynamic response functions as follows:

$$\chi_t^{\mathrm{BL}}=\delta_{0t}\int_0^1 dxD(x)[2N_{0,1}+12N_{0,1}m_1+(32N_{0,1}-48N_{0,2})m_1^2+(68N_{0,2}+\tfrac{2}{3}N_{0,1})m_2$$
$$+(96N_{0,1}-144N_{0,2})m_1^3+(\tfrac{256}{3}N_{0,1}-144N_{0,2})m_1m_2+(560N_{0,2}-200N_{0,1})m_3+\cdots]$$
$$+r_1^t(48m_1-24m_2+88m_1m_2+320m_3+\cdots)\int_0^1 dx\; D(x)\varphi_1(x)+r_2^t(-2880m_1m_2+4480m_3+\cdots)$$
$$\times\int_0^1 dx\; D(x)\varphi_2(x)+\cdots , \tag{60}$$

$$\chi_t^{\mathrm{BQ}}=\int_0^1 dx\; D(x)[-\tfrac{1}{2}N_{-t,0}+\tfrac{3}{4}N_{1-t,0}m_1-\tfrac{3}{8}N_{2-t,0}m_1^2+(-\tfrac{3}{16}N_{1-t,0}-\tfrac{15}{16}N_{-t,1})m_2$$
$$+\tfrac{3}{16}N_{3-t,0}m_1^3+(\tfrac{3}{32}N_{2-t,0}+\tfrac{3}{32}N_{1-t,0}+\tfrac{9}{32}N_{1-t,1}+\tfrac{9}{32}N_{-t,1})m_1m_2+\tfrac{35}{32}N_{2-t,0}m_3+\cdots] . \tag{61}$$

In the case of the BL map the correlation functions and the dynamic response functions decay exponentially. In the case of the BQ map the null functions play an essential role and one cannot tell, using this method, whether the decay is exponential.

## V. LYAPUNOV EXPONENTS

The Lyapunov exponents characterize the sensitivity of the motion with respect to the initial conditions. In a deterministic system this quantity can be defined essentially uniquely, since a trajectory is determined by the initial conditions (at least in principle). For a deterministic 1D system the Lyapunov exponent is given by

$$\lambda=\lim_{t\to\infty}\lim_{\epsilon\to 0}\left\{\frac{1}{t}\ln\left|\frac{1}{\epsilon}[f^t(x+\epsilon)-f^t(x)]\right|\right\} ,$$
$$f^t(x)=f(f^{t-1}(x)) . \tag{62}$$

(It shows how fast the exponential separation of two nearby trajectories is.) In contrast, in the presence of any external noise the initial conditions are not the only data determining the motion. The other factor is the realization of the stochastic process corresponding to the noise. If we want to obtain a quantity characterizing the sensitivity of the motion with respect to the initial conditions only, we have to perform an averaging over the noise. This can be done several ways; consequently, we get several definitions for the Lyapunov exponents. We shall use the following one.

Let us calculate the Lyapunov exponent for a fixed realization of the noise using the deterministic definition, and then let us average it over all the possible realizations. For a noisy 1D map this is given by

$$\lambda=\left\langle \lim_{t\to\infty}\lim_{\epsilon\to 0}\frac{1}{t}\ln\left|\frac{1}{\epsilon}[f(\zeta_1,f(\zeta_2,\ldots f(\zeta_t,x+\epsilon)\cdots))-f(\zeta_1,f(\zeta_2,\ldots f(\zeta_t,x)\cdots))]\right|\right\rangle . \tag{63}$$

Here $\langle \cdots \rangle$ means averaging over all $\zeta_n$.

The Lyapunov exponent (63) is closely related to the Kolmogorov entropy. Namely, if we take a space-time grid and calculate the coarse-grained Kolmogorov entropy, we find that as a consequence of the Khinchin axioms[21] it separates into two terms. One of them is the coarse-grained entropy of the noise. This term goes to infinity if we go on with the refinement of the grid, but the other term remains finite and goes to the Lyapunov exponent (63). In this sense we can say that the Kolmogorov entropy of a noisy mapping is the sum of the Kolmogorov entropy of the noise and the Lyapunov exponent.

As a consequence of the ergodicity, it can be shown that the quantity

$$\lim_{t\to\infty}\lim_{\epsilon\to 0}\frac{1}{t}\ln\left|\frac{1}{\epsilon}[f(\zeta_1,\ldots f(\zeta_t,x+\epsilon)\cdots) -f(\zeta_1,\ldots f(\zeta_t,x)\cdots)]\right| \tag{64}$$

has a Dirac $\delta$-like probability distribution around its mean value given by (63), so one can measure $\lambda$ in a numerical simulation without averaging over the noise,

$$\lambda=\lambda_m=\lim_{t\to\infty}\frac{1}{t}\sum_{j=0}^{t-1}\ln|f'(\zeta_{j+1},x_j)|\,,\quad x_j=f(\zeta_j,x_{j-1}) \tag{65}$$

and can calculate $\lambda$ by averaging over $x$ as well,

$$\lambda=\lambda_c=\int_0^1 dx\, P(x)\left\langle \lim_{t\to\infty}\frac{1}{t}\sum_{j=0}^{t-1}\ln|f'(\zeta_{j+1},x_j)|\right\rangle = \int_0^1 dx\, P(x)\langle \ln|f'(\zeta,x)|\rangle \,. \tag{66}$$

Here $P(x)$ is the stationary probability distribution (6). We also have used the identity

$$\int_0^1\langle\varphi(f(\zeta,x))\rangle P(x)dx=\int_0^1\varphi(x)P(x)dx \,. \tag{67}$$

In case of additive noise $f'$ becomes independent of the noise and (66) goes over to the definition of the Lyapunov exponent used in Refs. 5 and 7.

In our examples (BL and BQ map)

$$f(\zeta,x)=f_0(x)+\zeta G(f_0(x)),\quad G(x)=x(1-x) \tag{68}$$

so (66) can be written as

$$\lambda=\int_0^1 dx\,P(x)\ln|f_0'(x)| -\sum_{j=1}^{\infty}\frac{(-1)^j}{j}m_j\int_0^1 dx\,P(x)[G'(f_0(x))]^j \,. \tag{69}$$

For the BL map $P(x)$ is given by (43) and for the BQ map by (54). Putting these expressions into (69), we get, as results,

$$\lambda^{\rm BL}=\ln 2-\tfrac{1}{6}m_2+\tfrac{1}{3}m_1m_2-\tfrac{1}{3}m_1^2m_2-\tfrac{1}{45}m_2^2-\tfrac{1}{20}m_4 +\tfrac{1}{3}m_1^3m_2+\tfrac{1}{45}m_1m_2^2+\tfrac{1}{15}m_2m_3+\tfrac{8}{45}m_1m_4+\cdots\,, \tag{70}$$

$$\lambda^{\rm BQ}=\ln 2-\tfrac{1}{16}m_2+\tfrac{3}{32}m_1m_2-\tfrac{3}{64}m_1^2m_2-\tfrac{13}{512}m_4+\cdots\,.$$

## VI. COMPARISON OF NUMERICAL AND ANALYTICAL RESULTS FOR THE LYAPUNOV EXPONENT

We have also performed numerical work to check our perturbative results for the Lyapunov exponents. There is an infinite variety of the probability densities of the noise. We have chosen a simple case just to visualize our results. The probability density of the noise was

$$\rho(\zeta)=\begin{cases}\dfrac{1}{|z|}, & \text{if } 0\le\zeta\le z \ \ (z>0)\\ \text{or } z\le\zeta\le 0 & (z<0)\\ 0, & \text{otherwise}\,.\end{cases} \tag{71}$$

The quantity $z$ has been changed from $-0.4$ to $+0.9$ and from $-0.7$ to $+0.9$ in increments of 0.1 in the case of the BL map and the BQ map, respectively. We have calculated the Lyapunov exponent from a $10^4$ iteration for each value of $z$. So we have reached a precision of about $10^{-3}$. In the case of the BL map we have performed the calculation directly according to (65). The results are displayed on Fig. 1. The continuous curve is the result of the analytical calculation [see (70)]. If $|z|<0.5$, the agreement between theory and numerical measurement is good.

In the case of the BQ map a problem has arisen, since this map has a critical point at $x=\frac{1}{2}$. It implies in the numerical work that for points close enough to $\frac{1}{2}$ we get $f'(x)=0$ and, consequently, an error [see (65)]. For $10^4$ iterations this situation occurs many times. To avoid

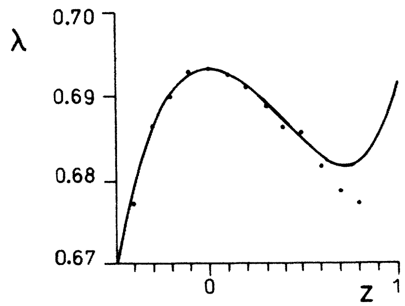


FIG. 1. Comparison of the Lyapunov exponent of the BL map obtained numerically (dots) and that obtained analytically (solid curve). The numerical data were obtained from $10^4$ iterations at each point and the accuracy is $10^{-3}$.

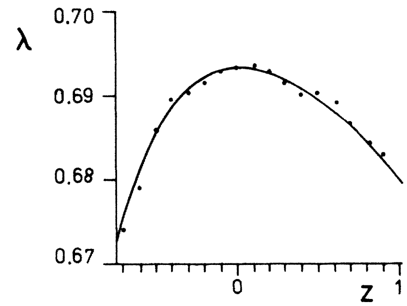


FIG. 2. Comparison of the Lyapunov exponent of the BQ map obtained numerically (dots) and that obtained analytically (solid curve). The number of iterations was $10^4$ at each point and the accuracy is $10^{-3}$.

this difficulty, we have applied a smooth conjugation. If we have

$$f(\zeta,x)=u(\tilde{f}(\zeta,u^{-1}(x))) , \tag{72}$$

where $u(x)$ is a strictly monotonous function of [0,1], one can easily derive from (65) that

$$\lambda_m=\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{t}\ln|\tilde{f}\,'(\zeta_{j+1},u^{-1}(x_j))| . \tag{73}$$

Here $x_j$'s are the points of the original iteration, i.e.,

$$x_{j+1}=f(\zeta_{j+1},x_j) . \tag{74}$$

In our case,

$$f(\zeta,x)=f_{\rm BQ}(\zeta,x) , \tag{75}$$

and we have chosen

$$u(x)=\sin^2\left[\frac{\pi x}{2}\right] . \tag{76}$$

So we obtain

$$\begin{aligned}|\tilde{f}\,'(\zeta_{j+1},u^{-1}(x_j))| &=2\{1+\zeta_{j+1}[1-8x_j(1-x_j)]\}\\ &\quad\times[1-\zeta_{j+1}4x_j(1-x_j)]^{-1/2}\\ &\quad\times[1+\zeta_{j+1}-\zeta_{j+1}4x_j(1-x_j)]^{-1/2} .\end{aligned} \tag{77}$$

This expression is definitely positive for every $x_j$. From (73) and (77) we have the results shown on Fig. 2. The differences between numerical and analytical results are within the numerical error in the whole region of $z$.

## VII. SUMMARY

We investigated the properties of fully developed chaotic maps under the influence of weak noise. We assumed that the attractor was not destroyed and, consequently, the noise was a multiplicative one, causing no changes at the critical point and at the ends of the attractor. We treated this noise as a small, random perturbation of the map and developed a perturbation method to calculate characteristic quantities. We were able to determine corrections to the eigenvalues, eigenfunctions, and null functions of the Frobenius-Perron operator in powers of the moments of the noise. We did concrete calculations on simple but nontrivial examples. We calculated the eigenvalues and eigenfunctions of the bilinear map and the null functions of the biquadratic map. These results enabled us to determine correlation functions and linear response functions for both cases. We also calculated the stationary densities and the Lyapunov exponents of the bilinear and biquadratic maps. The definition of the Lyapunov exponent is not unique in the noisy case, and we chose such a definition that relates the Lyapunov exponent to the suitably defined finite part of the Kolmogorov entropy. We also did some computer work to compare experimental and analytical results for the Lyapunov exponent.

## ACKNOWLEDGMENTS

This work was partially supported by the Hungarian Academy of Sciences through Grants No. AKA 1-3-86-324 and No. OTKA 819.

## APPENDIX

Eigenvalues and eigenfunctions of the BL map are as follows:

$$r_1=\tfrac{1}{4}-\tfrac{3}{4}m_1+m_2+\tfrac{5}{4}m_1m_2-\tfrac{5}{2}m_3+\tfrac{5}{6}m_1^2m_2-\tfrac{1}{4}m_2^2-\tfrac{35}{12}m_1m_3+\tfrac{25}{12}m_4+\cdots ,$$

$$\varphi_1=\varphi_1^0+(-4N_{0,1}+3N_{0,2})m_1+(-12N_{0,1}+9N_{0,2})m_1^2+(12\varphi_2^0-18N_{0,1}+15N_{0,2})m_2$$

$$+(-36N_{0,1}+27N_{0,2})m_1^3+(8\varphi_2^0+44N_{0,1}-\tfrac{69}{2}N_{0,2}+15N_{0,3})m_1m_2$$
$$+(-28\varphi_2^0+38N_{0,1}-30N_{0,2}+28N_{0,3})m_3+(-108N_{0,1}+81N_{0,2})m_1^4$$
$$+(\tfrac{16}{3}\varphi_2^0+\tfrac{764}{3}N_{0,1}-\tfrac{399}{2}N_{0,2}+55N_{0,3})m_1^2m_2+(16\varphi_2^0+64\varphi_3^0+558N_{0,1}-\tfrac{945}{2}N_{0,2}+105N_{0,3})m_2^2$$
$$+(-\tfrac{56}{3}\varphi_2^0-\tfrac{352}{3}N_{0,1}+\tfrac{195}{2}N_{0,2}+49N_{0,3})m_1m_3$$
$$+(\tfrac{448}{3}\varphi_2^0+96\varphi_3^0+676N_{0,1}-560N_{0,2}+252N_{0,3})m_4+\cdots,$$

$$r_2=\tfrac{1}{16}-\tfrac{5}{8}m_1+\tfrac{5}{4}m_2+\tfrac{15}{2}m_1m_2-\tfrac{105}{16}m_3+\cdots,$$

$$\varphi_2=\varphi_2^0+(-\tfrac{5}{9}\varphi_1^0+\tfrac{82}{3}N_{0,1}-\tfrac{45}{2}N_{0,2}+5N_{0,3})m_1+(-\tfrac{10}{27}\varphi_1^0+\tfrac{2540}{9}N_{0,1}-\tfrac{695}{3}N_{0,2}+50N_{0,3})m_1^2$$
$$+(\tfrac{1}{9}\varphi_1^0+\tfrac{80}{3}\varphi_3^0+162N_{0,1}-\tfrac{275}{2}N_{0,2}+35N_{0,3})m_2$$
$$+(-\tfrac{40}{81}\varphi_1^0+\tfrac{76\,360}{27}N_{0,1}-\tfrac{20\,890}{9}N_{0,2}+500N_{0,3})m_1^3$$
$$+(\tfrac{134}{9}\varphi_1^0+\tfrac{1520}{9}\varphi_3^0-\tfrac{2332}{3}N_{0,1}+\tfrac{1819}{3}N_{0,2}-\tfrac{1810}{9}N_{0,3}+\tfrac{140}{3}N_{0,4})m_1m_2$$
$$+(-\tfrac{10}{3}\varphi_1^0-160\varphi_3^0-\tfrac{2948}{3}N_{0,1}+795N_{0,2}-\tfrac{910}{3}N_{0,3}+75N_{0,4})m_3+\cdots,$$

$$r_3=\tfrac{1}{64}-\tfrac{21}{64}m_1+\tfrac{7}{8}m_2+\cdots,$$

$$\varphi_3=\varphi_3^0+(\tfrac{7}{15}\varphi_1^0-7\varphi_2^0-\tfrac{850}{3}N_{0,1}+\tfrac{469}{2}N_{0,2}-\tfrac{203}{3}N_{0,3}+7N_{0,4})m_1$$
$$+(\tfrac{889}{225}\varphi_1^0-\tfrac{133}{3}\varphi_2^0-\tfrac{33\,726}{5}N_{0,1}+\tfrac{55\,769}{10}N_{0,2}-1561N_{0,3}+147N_{0,4})m_1^2$$
$$+(-\tfrac{1}{45}\varphi_1^0+\tfrac{7}{3}\varphi_2^0+\tfrac{140}{3}\varphi_4^0-2082N_{0,1}+\tfrac{3479}{2}N_{0,2}-539N_{0,3}+63N_{0,4})m_2+\cdots.$$

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