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Linear canonical transformations of coherent and squeezed states in the Wigner phase space

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It is shown that classical linear canonical transformations are possible in the Wigner phase space. Coherent and squeezed states are shown to be linear canonical transforms of the ground-state harmonic oscillator. It is therefore possible to evaluate the Wigner functions for coherent and squeezed states from that for the harmonic oscillator. Since the group of linear canonical transformations has a subgroup whose algebraic property is the same as that of the (2+1)-dimensional Lorentz group, it may be possible to test certain properties of the Lorentz group using optical devices. A possible experiment to measure the Wigner rotation angle is discussed.

I. INTRODUCTION

Coherent and squeezed states now form the basic language for quantum optics.[1-3] They preserve the minimum-uncertainty product in the phase space consisting of phase and intensity. The Wigner phase space, which was initially formulated in 1932,[3,4] is also becoming the standard scientific language in many branches of physics, including quantum optics.[5,6] It is therefore of interest to formulate the coherent and squeezed states within the framework of the Wigner phase-space representation.

The Wigner distribution function for the coherent states has been discussed in the literature.[7] The Wigner function for the squeezed states has also been studied recently by Schleich and Wheeler for the deformation along the "x" or "p" axis caused by real or purely imaginary parameters.[5] However, the deformation in phase space of squeezed states with complex parameters has not been systematically studied.

In this paper we shall study the squeezed states with complex parameters. It will be shown that for a complex value of the squeeze parameter, the deformation is along the direction of the phase angle of the squeeze parameter. We shall achieve this purpose not by performing a direct calculation but by studying transformation properties in phase space.

Classical mechanics can be effectively formulated in terms of the Poisson brackets and canonical transformations.[8] Although the Poisson brackets become Heisenberg's uncertainty relations in quantum mechanics, it is cumbersome to use canonical transformations in quantum mechanics because the translation operators in phase space, which are x and p, do not commute with each other.[9,10]

The basic advantage of the Wigner function is that these operators commute with each other in phase space. In this paper we study coherent and squeezed states in the Wigner phase space. We shall show that these states are canonically transformed states of the ground-state harmonic oscillator. A subset of these transformations form a group whose algebraic properties are identical to that of the (2 + 1)-dimensional Lorentz group. It may therefore be possible to design an optical experiment to test the properties of the Lorentz group.

In Sec. II we briefly review the linear canonical transformations in classical mechanics. In Sec. III we discuss the canonical transformations of the Wigner distribution function in phase space. The canonical transformation of the Wigner function is much simpler than the conventional Weyl transformation applicable to the Schrödinger picture. In Sec. IV the Wigner phase-space formalism is discussed in detail for the harmonic oscillators.

In Sec. V we discuss coherent and squeezed states in terms of canonical transformations in phase space. It is possible from this formalism to determine the Wigner function for the squeezed state with a complex parameter. It is noted in Sec. VI that the algebra of squeezed and coherent states is the same as that for the (2 + 1)-dimensional Lorentz group. This enables us to discuss a possible experiment to measure the Wigner rotation angle using optical devices.

II. LINEAR CANONICAL TRANSFORMATIONS IN CLASSICAL MECHANICS

The group of linear canonical transformations consists of translations, rotations, and squeezes in phase space.[8,10,11] These operations preserve the area element in phase space. We present in this section a short formalism which will be useful for studying coherent and squeezed states in quantum optics.

In order to define the word "squeeze" in phase space, let us consider a circle around the origin in the coordinate system of x and p. If we elongate the x axis by multiplying it by a real number greater than 1 and contract the p axis by dividing it by the same real number, the circle becomes an ellipse. The area of the ellipse remains the same as that of the circle. This is precisely an act of squeeze. If we combine this operation with rotation around the origin, the squeezing can be done in every possible direction in phase space.

The coordinate transformation representing translations,

$$x'=x+u, \quad p'=p+v , \tag{1}$$

can be written as

$$\begin{bmatrix} x' \\ p' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ p \\ 1 \end{bmatrix} . \tag{2}$$

The matrix performing the rotation around the origin by $\theta/2$ takes the form

$$R(\theta)= \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{3}$$

The matrix which squeezes along the x axis is

$$S_x(\eta)= \begin{bmatrix} e^{\eta/2} & 0 & 0 \\ 0 & e^{-\eta/2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{4}$$

The elongation along the x axis is necessarily the contraction along the p axis.

Since a canonical transformation followed by another one is a canonical transformation, the most general form of the transformation matrix is a product of the above three forms of matrices. We can simplify these mathematics by using the generators of the transformation matrices. If we use $T(u,v)$ for the translation matrix given in Eq. (2), it can be written as

$$T(u,v)=e^{-i(uN_1+vN_2)} , \tag{5}$$

where

$$N_1= \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_2= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} . \tag{6}$$

The rotation matrix is generated by

$$L= \begin{bmatrix} 0 & -i/2 & 0 \\ i/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{7}$$

and

$$R(\theta)=e^{-i\theta L} . \tag{8}$$

The squeeze matrix can be written as

$$S_x=e^{-i\eta B_1} , \tag{9}$$

where

$$B_1= \begin{bmatrix} i/2 & 0 & 0 \\ 0 & -i/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{10}$$

In addition, if we introduce the matrix B_2 defined as

$$B_2= \begin{bmatrix} 0 & i/2 & 0 \\ i/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{11}$$

which generates the squeeze along the direction which makes 45° with the x axis, then the matrices L, B_1, and B_2 satisfy the following commutation relations:

$$[B_1,B_2]=-iL, \quad [B_1,L]=-iB_2, \quad [B_2,L]=iB_1 . \tag{12}$$

This set of commutation relations is identical to that for the generators of the (2 + 1)-dimensional Lorentz group.[12] The group generated by the above three operators is known also as the symplectic group Sp(2),[13,14] and its connection with the Lorentz group has been extensively discussed in the literature.[15]

If we take into account the translation operators, the commutation relations become

$$\begin{aligned} &[B_1,N_1]=(i/2)N_1, \quad [B_1,N_2]=(-i/2)N_2 , \\ &[B_2,N_1]=(i/2)N_2, \quad [B_2,N_2]=(i/2)N_1 , \\ &[N_1,L]=(i/2)N_2, \quad [N_1,L]=(-i/2)N_1 , \\ &[N_1,N_2]=0 . \end{aligned} \tag{13}$$

These commutators, together with those of Eq. (12), form the set of closed commutation relations (or Lie algebra) of the group of canonical transformations. This group is the inhomogeneous symplectic group in the two-dimensional space or ISp(2).[11]

The translations form an Abelian subgroup generated by N_1 and N_2. Since their commutation relations with all the generators result in N_1, N_2, or 0, the translation subgroup is an invariant subgroup. The translations and the rotation form the two-dimensional Euclidean group generated by N_1,N_2, and L, which have closed commutation relations. This group also has been extensively discussed recently in connection with the internal space-time symmetries of massless particles.[16,17]

Indeed, it is of interest to see how the representations of the Lorentz group can be useful in optical sciences. It is also of interest to see how the experimental resources in optical science can be helpful in understanding some of the "abstract" mathematical identities in group theory.

III. LINEAR CANONICAL TRANSFORMATIONS IN THE WIGNER PHASE SPACE

If $\psi(x)$ is a solution of the Schrödinger equation, the Wigner distribution function in phase space is defined as

$$W(x,p)=(1/\pi)\int \psi^*(x+y)\psi(x-y)e^{2ipy}dy \ . \tag{14}$$

This is a function of x and p which are c numbers. This function is real but is not necessarily positive everywhere in phase space. The properties of this function have been extensively discussed in the literature.[4-7]

When we make linear canonical transformations of this function in phase space, the infinitesimal generators are

$$\begin{aligned} &N_1=-i\frac{\partial}{\partial x},\quad N_2=-i\frac{\partial}{\partial p}\ ,\\ &L=-\frac{i}{2}\left[x\frac{\partial}{\partial p}-p\frac{\partial}{\partial x}\right],\\ &B_1=\frac{i}{2}\left[x\frac{\partial}{\partial x}-p\frac{\partial}{\partial p}\right],\quad B_2=\frac{i}{2}\left[x\frac{\partial}{\partial p}+p\frac{\partial}{\partial x}\right]. \end{aligned} \tag{15}$$

These operators satisfy the commutation relations given in Eqs. (12) and (13). We can therefore derive the algebraic relations involving the above differential forms using the matrix representation discussed in Sec. II.

The rotation of the translation operators takes the form

$$\begin{aligned} R(\theta)N_1R(-\theta)&=\left[\cos\frac{\theta}{2}\right]N_1-\left[\sin\frac{\theta}{2}\right]N_2\ ,\\ R(\theta)N_2R(-\theta)&=\left[\sin\frac{\theta}{2}\right]N_1+\left[\cos\frac{\theta}{2}\right]N_2\ . \end{aligned} \tag{16}$$

Under the same rotation, the squeeze generators become

$$\begin{aligned} R(\theta)B_1R(-\theta)&=(\cos\theta)B_1+(\sin\theta)B_2\ ,\\ R(\theta)B_2R(-\theta)&=-(\sin\theta)B_1+(\cos\theta)B_2\ . \end{aligned} \tag{17}$$

Likewise, we can derive all the algebraic relations using matrix algebra. The important point is that the group of canonical transformations in the Wigner phase space is identical to that for classical mechanics.

Next, let us consider the above transformations in terms of operators applicable to the Schrödinger wave function. From the expression of Eq. (14) it is quite clear that the operation e^{-ivx} on the wave function leads to a translation along the p axis by v. The operation of $\exp[-u(\partial/\partial x)]$ on the wave function leads to a translation of the above distribution function along the x axis by u.

Likewise, the operation in the Wigner phase space of $ix(\partial/\partial p)$ and $ip(\partial/\partial x)$ become $x^2/2$ and $\frac{1}{2}(\partial/\partial x)^2$, respectively. Thus, the transformations in phase space can be generated from the operators applicable to the wave function. The generators applicable to the wave function are

$$\begin{aligned} &\tilde{N}_1=-i\frac{\partial}{\partial x},\quad \tilde{N}_2=x\ ,\\ &\tilde{L}=\frac{1}{4}\left[\left[\frac{\partial}{\partial x}\right]^2-x^2\right],\\ &\tilde{B}_1=-i\left[\frac{x}{2}\right]\frac{\partial}{\partial x},\quad \tilde{B}_2=\frac{1}{4}\left[x^2+\left[\frac{\partial}{\partial x}\right]^2\right]. \end{aligned} \tag{18}$$

These operators satisfy the commutation relations given in Eqs. (12) and (13), except the last one. The operators $\tilde{N}_1$ and $\tilde{N}_2$ do not commute with each other, and

$$[\tilde{N}_1,\tilde{N}_2]=-i\ . \tag{19}$$

Therefore, it appears that the operators applicable to the Schrödinger wave function do not satisfy the same set of commutation relations as that for classical phase space.[9,10]

Let us consider the translation along the x axis followed by the translation along the p axis, and the operation in the opposite order. From the Baker-Campbell-Hausdorff formula for two operators,[9,10]

$$(e^{-iuN_1})(e^{-ivN_2})=(e^{iuv})(e^{-ivN_2})(e^{-iuN_1})\ . \tag{20}$$

The interchange of the above two translations results in a multiplication of the wave function by a constant factor of unit modulus.

However, this factor disappears when the Wigner function W is constructed according to the definition of Eq. (14). Therefore, the translation along the x direction and the translation along the p direction commute with each other in the Wigner phase space. This means that the commutation relation $[N_1,N_2]=0$ in the Wigner phase space and the Heisenberg relation $[N_1,N_2]=-i$ are perfectly consistent with each other. The basic advantage of the Wigner phase-space representation is that its canonical transformation property is the same as that of classical mechanics.

We now have three sets of operators. The first set consists of the three-by-three matrices in Eqs. (6), (7), (10), and (11), and this set is for classical mechanics. The differential operators in two-dimensional phase space form the second set, and they are for the Wigner function. The third set consists of the differential operators of Eq. (18) applicable to the Schrödinger wave function. The first and second sets are the same. While both the second set of double-variable operators and the third set of single-variable operators are extensively used in the literature,[11,14,18] it is interesting to see that the connection between these two sets can be established through the Wigner function.

The transformations discussed in this section constitute the basic language for coherent and squeezed states in quantum optics. The relevance of the translation in phase space to coherent states has been noted before.[1] The word squeeze comes from quantum optics. It has been also noted that its mathematics is like that of (2 + 1)-dimensional Lorentz transformations. As was emphasized in the literature,[12,17] combining translations with Lorentz transformations is not a trivial problem. We shall discuss the problem in Secs. V and VI.

IV. HARMONIC OSCILLATORS

The one-dimensional harmonic oscillator occupies a unique place in the physics of phase space. For the Hamiltonian of the form

$$H=\tfrac{1}{2}(p^2+x^2)\ , \tag{21}$$

the Wigner function is a function only of (p^2+x^2), and is thus invariant under rotations around the origin in phase space. The Wigner function for the ground-state harmonic oscillator is[7,19]

$$W(x,p)=\frac{1}{\pi}\exp[-(x^2+p^2)]\ . \tag{22}$$

This function is localized within the circular region whose boundary is defined by the equation

$$x^2+p^2=1\ . \tag{23}$$

Therefore, the study of the Wigner function for the harmonic oscillator is the same as the study of a circle on the two-dimensional plane. The canonical transformation consists of rotations, translations, and area-preserving elliptic deformations of this circle. These transformations are straightforward.

Under the translation by r along the x axis, the above circle becomes

$$(x-r)^2+p^2=1\ . \tag{24}$$

This circle is centered around the point $(r,0)$. We can rotate the above circle around the origin. Then the resulting Wigner function is

$$R(\theta)T(r,0)W(x,p)=\frac{1}{\pi}\exp\left\{\left[\left[x-r\cos\frac{\theta}{2}\right]^2+\left[p-r\sin\frac{\theta}{2}\right]^2\right]\right\}, \tag{25}$$

where $T(r,0)$ and $R(\theta)$ are the translation and rotation operators. Because the circle of Eq. (23) is invariant under rotations around the point where $x=r$ and $p=0$, the above Wigner function is the same as the translated Wigner function,

$$T\left[r\cos\frac{\theta}{2},r\sin\frac{\theta}{2}\right]W(x,p)=R(\theta)T(r,0)W(x,p)\ . \tag{26}$$

Let us next elongate the translated circle of Eq. (24) along the x direction. The circle will be deformed into

$$e^{-\eta}(x-r')^2+e^{\eta}p^2=1\ , \tag{27}$$

where

$$r'=re^{\eta/2}\ .$$

If we rotate this ellipse, the resulting Wigner function will be

$$R(\theta)S_1(\eta)T(r,0)W(x,p)$$
$$=\frac{1}{\pi}\exp\left\{-\left[e^{-\eta}\left[x\cos\frac{\theta}{2}+p\sin\frac{\theta}{2}-r'\cos\frac{\theta}{2}\right]^2+e^{\eta}\left[x\sin\frac{\theta}{2}-p\cos\frac{\theta}{2}-r'\sin\frac{\theta}{2}\right]^2\right]\right\} \tag{28}$$

This transformation is illustrated in Fig. 1. As we shall see in Sec. V, the translated and deformed Wigner functions will be useful for studying coherent and squeezed states, respectively.

In the meantime, let us observe other useful properties of the harmonic oscillator. We noted above that, in order to study the harmonic oscillator, we can start with a circle in phase space. How does this rotational invariance manifest itself in the Schrödinger picture? The generator of rotations is

$$L=\frac{1}{4}\left[\left[\frac{\partial}{\partial x}\right]^2-x^2\right]=\tfrac{1}{2}(-H)\ . \tag{29}$$

If the wave function is a solution of the time-independent Schrödinger equation with the above Hamiltonian, the application of the rotation operator $\exp(-i\theta L)$ will only generate a constant factor of unit modulus. This is the reason why the Wigner function for the above Hamiltonian system is invariant under rotations in phase space.

In order to study rotations more carefully in the Schrödinger picture, let us use a and $a^\dagger$, defined in this case as

$$a=(1/\sqrt{2})\left[x+\frac{\partial}{\partial x}\right],$$
$$a^\dagger=(1/\sqrt{2})\left[x-\frac{\partial}{\partial x}\right]. \tag{30}$$

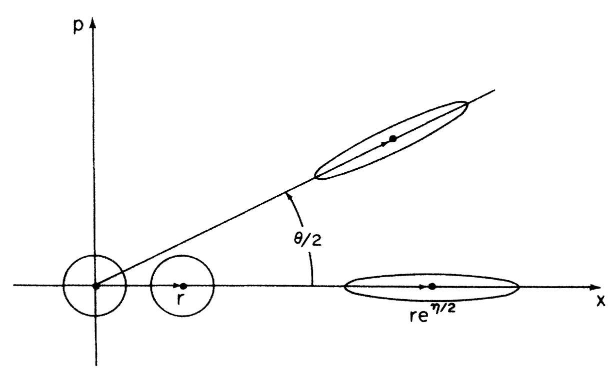

FIG. 1. Coherent and squeezed states in the Wigner phase space. The circle centered around the origin describes the ground-state harmonic oscillator. The circle around $(r,0)$ is for the coherent state. This coherent state can be squeezed to ellipse along the x axis, with a real value of the squeeze parameter. When the squeeze parameter becomes complex then the ellipse is rotated around the origin in the Wigner phase space.

These operators serve two distinct purposes in physics. They are step-up and step-down operators for the one-dimensional harmonic oscillator in nonrelativistic quantum mechanics.

On the other hand, in quantum-field theory, they serve as the annihilation and creation operators. We are here interested in the creation and annihilation of photons. Then, what is the physics of the phase space spanned by x and p variables? Indeed, the concept of creation and annihilation comes from the commutation relation

$$[a,a^\dagger]=1 \ . \tag{31}$$

This form of uncertainty relation states also that the area element in phase space cannot be smaller than Planck's constant. The area element in the Cartesian coordinate system is $(\Delta x)(\Delta p)$. It is also possible to write the area element in the polar coordinate system. If this area is described in the polar-coordinate system, the uncertainty relation is the relation between phase and intensity.[20] This is the uncertainty relation we are discussing in this paper. We are particularly interested in the minimum-uncertainty states.

In both Eq. (25) and Eq. (28) the rotation plays the essential role. Let us see how the operators a and $a^\dagger$ can be rotated. For two operators A and B, we note the relation[21]

$$e^{A}Be^{-A}=B+[A,B]+\tfrac{1}{2}[A,[A,B]] +\tfrac{1}{6}[A,[A,[A,B]]]+\cdots \ , \tag{32}$$

and

$$[L,a]=-\tfrac{1}{2}a, \quad [L,a^\dagger]=\tfrac{1}{2}a^\dagger \ . \tag{33}$$

Since $R(\theta)=e^{-i\theta L}$,

$$\begin{aligned} &R(\theta)aR(-\theta)=(e^{-i\theta/2})a \ , \\ &R(\theta)a^\dagger R(-\theta)=(e^{i\theta/2})a^\dagger \ . \end{aligned} \tag{34}$$

In terms of the a and $a^\dagger$ operators, the generators of canonical transformations take the form

$$\begin{aligned} &\tilde{N}_1=(-i/\sqrt{2})(a-a^\dagger), \quad \tilde{N}_2=(1/\sqrt{2})(a+a^\dagger) \ , \\ &\tilde{L}=\tfrac{1}{4}(aa^\dagger+a^\dagger a) \ , \\ &\tilde{B}_1=\tfrac{1}{4}(aa-a^\dagger a^\dagger), \quad \tilde{B}_2=\tfrac{1}{4}(aa+a^\dagger a^\dagger) \ . \end{aligned} \tag{35}$$

We can rotate these operators using Eq. (34). In particular, the rotations given in Eq. (17) can now be written as

$$\begin{aligned} &R(\theta)aaR(-\theta)=e^{-i\theta}aa \ , \\ &R(\theta)a^\dagger a^\dagger R(-\theta)=e^{i\theta}a^\dagger a^\dagger \ . \end{aligned} \tag{36}$$

These relations will be useful in evaluating the Wigner function for the squeezed state.

V. COHERENT STATES AND SQUEEZED STATES

In terms of the a and $a^\dagger$ operators, the coherent state is defined as

$$|\alpha\rangle=[\exp(-|\alpha|^2/2)]\sum_{n=0}^{\infty}(\alpha^n/n!)(a^\dagger)^n|0\rangle \ . \tag{37}$$

We can obtain this state by applying the translation operator to the ground state,

$$|\alpha\rangle=T(\alpha)|0\rangle \ , \tag{38}$$

where

$$T(\alpha)=\exp(\alpha a^\dagger-\alpha^* a) \ .$$

The translation operator in the phase space depends on two real parameters. In the above case, the parameter α is a complex number containing two real parameters.

It is possible to evaluate the Wigner function from the above expression to obtain the form given in Eq. (25),[7,9] with

$$r\cos\frac{\theta}{2}=\sqrt{2}[\mathrm{Re}(\alpha)], \quad r\sin\frac{\theta}{2}=\sqrt{2}[\mathrm{Im}(\alpha)] \ . \tag{39}$$

It is also possible to obtain the Wigner function starting from a real value of α by rotation. From the rotation properties of the a and $a^\dagger$ operators given in Sec. IV, the rotation of this operator becomes

$$R(\theta)T(r)R(-\theta)=T(\alpha) \ , \tag{40}$$

with

$$\alpha=(e^{-i\theta/2})r \ .$$

This means that we can make α complex starting from a real number r by rotation.

The squeezed state $|\xi,\alpha\rangle$ is defined to be[2,3,5,18,22]

$$|\xi,\alpha\rangle=S(\xi)|\alpha\rangle=S(\xi)T(\alpha)|0\rangle \ , \tag{41}$$

where

$$S(\xi)=\exp\left[\frac{\xi}{2}a^\dagger a^\dagger-\frac{\xi^*}{2}aa\right] \ . \tag{42}$$

Here again the parameter ξ is complex and contains two real numbers for specifying the direction and the strength of the squeeze.

If ξ is real, it is possible to evaluate the Wigner function by direct evaluation of the integral. If, on the other hand, ξ is complex, the present authors were not able to manage the calculation. We can, however, overcome this difficulty by using the method of canonical transformation developed in this paper. We can make ξ complex starting from a real value of η by rotating the above squeeze operator using the rotation properties of the a and $a^\dagger$ operators.

Let us start from a real value of ξ for which the evaluation is possible.[5] For the real value η, the squeeze operator becomes

$$S(\eta)=\exp\left[-\frac{\eta}{2}\left[x\frac{\partial}{\partial x}\right]\right] \ . \tag{43}$$

This operator makes the scale change of x to $(e^{-\eta/2})x$. It is therefore possible to visualize the deformation of the circle into an ellipse in the phase space. Let us next rotate this ellipse. From Eqs. (36) and (42),

$$R(\theta)S(\eta)R(-\theta)=S(\xi) \ , \tag{44}$$

where

$$\xi=(e^{-i\theta})\eta \ .$$

The operator $S(\xi)$, when applied to the wave function, leads to the Wigner function which is elongated along the $\theta/2$ direction in the phase space. It is indeed possible to evaluate the Wigner function for the squeezed state with a complex value of ξ simply by rotating the ellipse elongated along the x direction.

Table I describes how we can determine the Wigner functions for coherent and squeezed states. Figure 1 illustrates how the above calculation can be carried out. The translated circle in phase space describes the coherent state. This circle can be elongated along the x direction. The resulting ellipse is for the squeezed state with a real parameter. This ellipse can be rotated. This rotated ellipse corresponds to the squeezed state with a complex parameter.

TABLE I. How to evaluate the Wigner function for coherent and squeezed states.

	Coherent states	Squeezed states
Direct computation	Possible	Not known
Canonical transformation	Possible	Possible

VI. POSSIBLE MEASUREMENT OF THE WIGNER ROTATION

We have noted in Sec. II that the transformation group contains the subgroup Sp(2) which is locally isomorphic to the (2 + 1)-dimensional Lorentz group. It may therefore be possible to design experiments in optics to test the mathematical identities in the Lorentz group. The Wigner rotation is a case in point. Two successive applications of Lorentz boosts in different directions is not a Lorentz boost, but is a boost preceded by a rotation which is commonly called the Wigner rotation.[23–26] This effect exhibits itself in the Thomas effect in atomic physics.[25]

Since the mathematics of squeeze is the same as that of Lorentz boost, we can discuss the possibility of measuring the effect of the Wigner rotation in optical experiments. In order to illustrate how the Wigner rotation comes into this subject, let us start with a circle of unit radius centered around the origin in the Cartesian-coordinate system with the coordinate variables x and p, whose equation is given in Eq. (23). If we squeeze this circle by elongating along the x axis, the squeeze matrix applicable to the vector (x,p) is

$$S(0,\lambda)=\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} . \tag{45}$$

This will deform the circle into the ellipse

$$(e^{-\eta})x^2+(e^{\eta})p^2=1 \ . \tag{46}$$

If we squeeze the circle centered around the origin along the $\theta/2$ direction with the deformation parameter η, the squeeze matrix is

$$S(\theta,\lambda)=\begin{pmatrix} \cosh\frac{\lambda}{2}+\left(\sinh\frac{\lambda}{2}\right)\cos\theta & \left(\sinh\frac{\lambda}{2}\right)\sin\theta \\ \left(\sinh\frac{\lambda}{2}\right)\sin\theta & \cosh\frac{\lambda}{2}-\left(\sinh\frac{\lambda}{2}\right)\cos\theta \end{pmatrix} , \tag{47}$$

and the circle is deformed into the ellipse

$$e^{-\lambda}\left[x\cos\frac{\theta}{2}+p\sin\frac{\theta}{2}\right]^2+e^{\lambda}\left[x\sin\frac{\theta}{2}-p\cos\frac{\theta}{2}\right]^2=1 \ . \tag{48}$$

In order to understand the squeeze mechanism thoroughly, we should know how to squeeze an ellipse. We can achieve this goal by studying two successive squeezing properties. Let us therefore consider the squeeze $S(\theta,\lambda)$ of the circle centered around the origin preceded by $S(0,\eta)$. This will result in another ellipse,

$$e^{-\xi}\left[x\cos\frac{\alpha}{2}+p\sin\frac{\alpha}{2}\right]^2+e^{\xi}\left[x\sin\frac{\alpha}{2}-p\cos\frac{\alpha}{2}\right]^2=1 \ . \tag{49}$$

where

$$\cosh\xi=(\cosh\eta)\cosh\lambda+(\sinh\eta)(\sinh\lambda)\cos\theta \ ,$$

and

$$\tan\alpha=\frac{(\sin\theta)[\sinh\lambda+(\tanh\eta)(\cosh\lambda-1)\cos\theta]}{(\sinh\lambda)\cos\theta+(\tanh\eta)[1+(\cosh\lambda-1)(\cos\theta)^2]} \ .$$

This is an ellipse elongated along the $\alpha/2$ direction with the parameter ξ.

The above calculation gives an indication that two successive squeezes become one squeeze. This is not true. The product of the matrices $S(\theta,\lambda)S(0,\eta)$ does not result in $S(\alpha,\xi)$. Instead, it becomes[16,17,23–25]

$$S(\theta,\lambda)S(0,\eta)=S(\alpha,\xi)R(\phi) \ , \tag{50}$$

where

$$\tan\left(\frac{\phi}{2}\right)=\frac{(\sin\theta)[\tanh(\lambda/2)][\tanh(\eta/2)]}{1+[\tanh(\lambda/2)][\tanh(\eta/2)](\cos\theta)} \ .$$

The right-hand side of the above equation is a squeeze preceded by a rotation, which may be called the Wigner rotation.[23–26] Although Eq. (49) does not show the effect of this rotation which leaves the initial circle centered around the origin invariant, we need the derivation of Eq. (49) in order to determine α, ξ, and eventually ϕ.

The study of coherent states representations requires transformations of a circle not centered around the origin. If we squeeze this circle by applying $S(0,\eta)$, the circle is transformed into the ellipse given in Eq. (27). If we squeeze this ellipse by applying $S(\theta,\lambda)$, the net effect is the squeeze $S(\alpha,\xi)$ preceded by the Wigner rotation $R(\phi)$. If we apply this rotation to the circle of Eq. (24),

$$\left[x - r\cos\frac{\phi}{2}\right]^2 + \left[p - r\sin\frac{\phi}{2}\right]^2 = 1 \ . \tag{51}$$

The effect of this rotation is illustrated in Fig. 2.

Next, if we apply the squeeze $S(\alpha,\xi)$ to the above circle, the resulting ellipse is

$$e^{-\xi}\left[(x-a)\cos\frac{\alpha}{2} + (y-b)\sin\frac{\alpha}{2}\right]^2 + e^{\xi}\left[(x-a)\sin\frac{\alpha}{2} - (y-b)\cos\frac{\alpha}{2}\right]^2 = 1 \ , \tag{52}$$

where

$$a = r\left\{\left[\cosh\frac{\xi}{2} + \left[\sinh\frac{\xi}{2}\right]\cos\alpha\right]\cos\frac{\phi}{2} + \left[\sinh\frac{\xi}{2}\right](\sin\alpha)\sin\frac{\phi}{2}\right\} ,$$

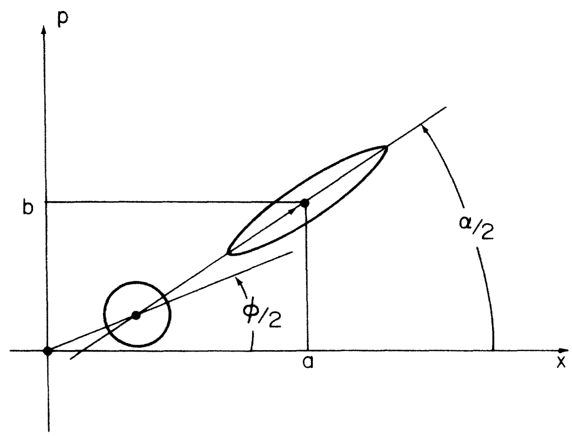

FIG. 2. Two repeated squeezes resulting in one squeeze preceded by one rotation. The circle around $(r,0)$ in Fig. 1 is rotated around the origin by $\phi/2$ and is then elongated along the $\alpha/2$ direction.

$$b = r\left\{\left[\sinh\frac{\xi}{2}\right](\sin\alpha)\cos\frac{\phi}{2} + \left[\cosh\frac{\xi}{2} - \left[\sinh\frac{\xi}{2}\right]\cos\alpha\right]\sin\frac{\phi}{2}\right\} .$$

The effect of this squeeze is also illustrated in Fig. 2.

The Wigner rotation angle ϕ can now be determined from a,b, which can be measured. In terms of these parameters,

$$\tan\left[\frac{\phi}{2}\right] = \frac{b\left[\cosh\frac{\xi}{2} + \left[\sinh\frac{\xi}{2}\right]\cos\alpha\right] - a\left[\sinh\frac{\xi}{2}\right]\sin\alpha}{a\left[\cosh\frac{\xi}{2} - \left[\sinh\frac{\xi}{2}\right]\cos\alpha\right] - b\left[\sinh\frac{\xi}{2}\right]\sin\alpha} \ . \tag{53}$$

The parameters ξ and α can be measured or determined from Eq. (49). The angle ϕ determined from the above expression can be compared with the angle calculated from η,λ, and α according to the expression given in Eq. (50).

Indeed, if the parameters of the coherent and squeezed states can be determined experimentally, the Wigner rotation can be measured in optical laboratories. The question is then whether this experiment can be carried out with the techniques available at the present time. While the analysis presented in this section is based on single-mode squeezed states, the squeezed states that have been generated to date are two-mode states.[3,18] Hence, in order to be directly applicable to experiment, the present work has to be extended to the two-mode case, unless the single-mode squeezed state can be generated in the near future. In the meantime, the present work indicates that some of optical experiments may serve as analog computers for the (2 + 1)-dimensional Lorentz group.

VII. CONCLUDING REMARKS

It is quite clear from this paper that the coherent and squeezed states can be described by circles and ellipses in the Wigner phase space. One circle or ellipse can be transformed into another by area-preserving transformations. The group governing these transformations is the inhomogeneous symplectic group ISp(2).

We studied the generators of these transformations both for phase space and for the Schrödinger representation. It has been shown that the connection between these two sets of operators can be established through the Wigner function.

We also studied in detail rotations in the Wigner phase space and their counterparts in the Schrödinger representation. It is now possible to evaluate the Wigner function for a squeezed state with a complex parameter.

The correspondence (local isomorphism) between Sp(2) and the (2 + 1)-dimensional Lorentz group allows us to study quantum optics using the established language of the Lorentz group. At the same time it allows us to look into possible experiments in optical science to study some of mathematical formulas in group theory. It may be possible to measure the Wigner rotation angle in optical laboratories.

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