Modulationally unstable ion acoustic waves

R. Bharuthram*

Department of Physics, University of Durban–Westville, Private Bag X54001, Durban 4000, South Africa

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The modulation of ion acoustic waves due to interaction with a quasistatic plasma slow response in a magnetized plasma is examined. It is found that an instability arises when the modulation is oblique to the direction of wave propagation, but for a restricted range of angles.

Nonlinear interaction of high- and low-frequency plasma phenomena has been a topic of considerable recent interest. For example, the interaction of electron plasma waves (high frequency) with ion acoustic waves (low frequency) is described by a nonlinear set of coupled equations first derived by Zakharov.[1] These equations can be used with appropriate boundary conditions to study parametric instabilities, modulational instabilities, and soliton potential structures.[2] The modulation of ion acoustic waves is the topic of interest in this paper.

In investigating the nonlinear propagation of ion acoustic waves in an unmagnetized plasma, Shimizu and Ichikawa[3] used the harmonic generation nonlinearities and stretched variables to establish a nonlinear Schrödinger type of equation. It was found that for their one-dimensional geometry (modulation parallel to wave phase velocity) the ion acoustic wave was modulationally stable. Kako and Hasegawa[4] extended their work to a magnetized plasma, and found that ion acoustic waves are modulationally unstable when the modulation is in a direct oblique to that of the wave propagation vector. On the other hand, Murtaza and Salahuddin[5] used the Krylov-Bogoliubov-Mitropolsky method[6] to investigate the modulation of ion acoustic waves in a magnetized plasma. The inclusion of the harmonic generation nonlinearities was found to produce instability in some regions of parameter space.

The nonlinear modulation of ion acoustic waves due to interaction with a quasistatic plasma slow response was first considered by Shukla.[7] For an unmagnetized plasma, the waves were found to be modulationally stable. The inclusion of an external magnetic field $\mathbf{B}_0$ was considered by Bharuthram and Shukla.[8] They restricted all motion to the x-z plane and found that waves propagating oblique to $\mathbf{B}_0$ were modulationally stable. However, it must be pointed out that the authors restricted their analysis to modulations parallel to the wave velocity vector. In this paper we extend their investigations by allowing modulations oblique to the wave velocity vector. In addition, the analysis is made more general by considering three-dimensional motion. A three-dimensional nonlinear Schrödinger equation which describes the evolution of the complex amplitude of the ion acoustic wave is derived. Analysis then shows that the ion acoustic wave is modulationally unstable for modulations in certain oblique directions to the wave propagation vector.

For low-frequency electrostatic oscillations with $\partial_t \ll \Omega_j$, where Ω_j is the gyrofrequency of the jth species [$j=e$ (i) for electrons (ions)] in the presence of an external magnetic field $\mathbf{B}_0=B_0\hat{z}$, the perpendicular (to $\mathbf{B}_0$) component of the particle fluid velocity is given by

$$\mathbf{v}_{j\perp}=\frac{c}{B_0}\hat{z}\times\nabla\phi-\frac{c}{B_0\Omega_j}\partial_t\nabla\phi\ , \tag{1}$$

where the first term on the right-hand side is the $\mathbf{E}\times\mathbf{B}$ drift and the second term the polarization drift, ϕ is the electrostatic potential, and c the speed of light. For $\partial_t \ll v_e\partial_z$, where $v_e=(T_e/m_e)^{1/2}$ is the electron thermal speed, the inertialess electrons rapidly thermalize along $\mathbf{B}_0$. Then their density perturbation associated with the ion acoustic waves in the presence of the plasma slow motion is given by[8]

$$\delta n_e/n_0=(1+\delta n_e^l/n_0)\Phi\ , \tag{2}$$

where n_0 is the equilibrium density, $\phi=e\phi/T_e$ the normalized potential, and the plasma density is written as $n_j=n_0+\delta n_j+\delta n_j^l$, where the superscript l indicates the contribution from the plasma slow motion. Then following Ref. 8, we obtain from the Eqs. (1) and (2), the ion continuity equation and Poisson's equation, a nonlinear equation for the ion acoustic wave in the presence of the plasma slow response

$$\{[1-\rho_s^2(\partial_x^2+\partial_y^2)-\lambda_D^2\nabla^2]\partial_t^2-c_s^2\partial_z^2\}\Phi + \{[1-\rho_s^2(\partial_x^2+\partial_y^2)]\partial_t^2-c_s^2\partial_z^2\}(\delta n_e^l/n_0)\Phi=0\ , \tag{3}$$

where $c_s=(T_e/m_i)^{1/2}$ is the ion sound speed, $\rho_s=c_s/\Omega_i$ is the ion Larmor radius at the electron temperature, and $\lambda_D=(T_e/4\pi n_0 e^2)^{1/2}$ the electron Debye length. In arriving at (3) we have used $T_e \gg T_i$, $V_j^l=0$ for the quasistatic modulations, and the quasineutrality condition $\delta n_e^l=\delta n_i^l$. In the linear limit, (3) reduces to the dispersion relation[9] $\omega=k_z c_s/\alpha^{1/2}$, where $\alpha=1+k_\perp^2\rho_s^2+k^2\lambda_D^2$, with k_z ($k_\perp$) the component of $\mathbf{k}$ along (perpendicular to) $\mathbf{B}_0$.

For a nonlinear interaction which causes a slow amplitude modulation we introduce two time and space scales. Then by the WKB approximation,[10] we may write

$$\Phi=\Phi(\xi,\tau)\exp(-i\omega t+i\mathbf{k}\cdot\mathbf{r})+\text{c.c.}\ ,$$

$$\partial_t\to\partial_\tau-i\omega-\mathbf{V}_g\cdot\nabla_\xi\ , \tag{4}$$

$$\nabla_r \to \nabla_\xi + i\mathbf{k} ,$$

where the group velocity of the wave packet $\mathbf{V}_g = V_{gx}\hat{\mathbf{x}} + V_{gy}\hat{\mathbf{y}} + V_{gz}\hat{\mathbf{z}}$, with

$$\begin{aligned} V_{gx} &= \partial\omega/\partial k_x = -\omega k_x \rho^2/\alpha , \\ V_{gy} &= \partial\omega/\partial k_y = -\omega k_y \rho^2/\alpha , \\ V_{gz} &= \partial\omega/\partial k_z = c_s(1+k_\perp^2\rho^2)/\alpha^{3/2} , \\ \rho^2 &= \rho_s^2 + \lambda_0^2 . \end{aligned} \tag{5}$$

The substitution of Eqs. (4) into (3) and the ordering

$$\lambda_D \nabla_\xi \sim \rho_s \nabla_\xi \sim O(\epsilon^{3/2}) , \quad \omega^{-1}\partial_\tau \sim O(\epsilon^3) ,$$

$$(k\lambda_D)^2 \sim O(\epsilon) , \quad \Phi \sim O(\epsilon) , \quad \delta n_e^l/n_0 \sim O(\epsilon^2) ,$$

yields a three-dimensional nonlinear Schrödinger equation for ϕ,

$$\begin{aligned} i\partial_\tau\Phi + \tfrac{1}{2}[&\partial^2_{k_x}\omega\partial^2_{\xi_x} + \partial^2_{k_y}\omega\partial^2_{\xi_y} + \partial^2_{k_z}\omega\partial^2_{\xi_z} \\ &+ 2(\partial^2_{k_x k_y}\omega\partial^2_{\xi_x\xi_y} + \partial^2_{k_x k_z}\omega\partial^2_{\xi_x\xi_z} \\ &+ \partial^2_{k_y k_z}\omega\partial^2_{\xi_y\xi_z})]\Phi - \frac{\omega k^2\lambda_D^2}{2\alpha}(\delta n_e^l/n_0)\Phi = 0 , \end{aligned} \tag{6}$$

which may be rewritten as

$$i\partial_\tau\Phi + \tfrac{1}{2}\mathcal{L}\Phi + Q\,|\Phi|^2\Phi = 0 , \tag{7}$$

where

$$\mathcal{L} \equiv \sum_{i,j}(\partial^2_{k_i k_j}\omega)\partial^2_{\xi_i\xi_j} , \tag{8a}$$

$$Q = \omega k^2\lambda_D^2(1+k^2\lambda_D^2)/4\alpha(1+\sigma) , \tag{8b}$$

with $\sigma = T_i/T_e$. In arriving at (7) we have used the result

$$\delta n_e^l/n_0 \simeq -(1+k^2\lambda_D^2)\,|\Phi|^2\,|\,2(1+\sigma) ,$$

which has been obtained by averaging the ion and electron equations of motion parallel to $\mathbf{B}_0$ over the (fast) ion acoustic wave period,[8] and using (1) for the particle speeds in the plane perpendicular to $\mathbf{B}_0$.

In examining the stability of a constant amplitude pump wave to quasistatic modulations, we follow the general method of Karpman.[8,10,11] Accordingly, we let

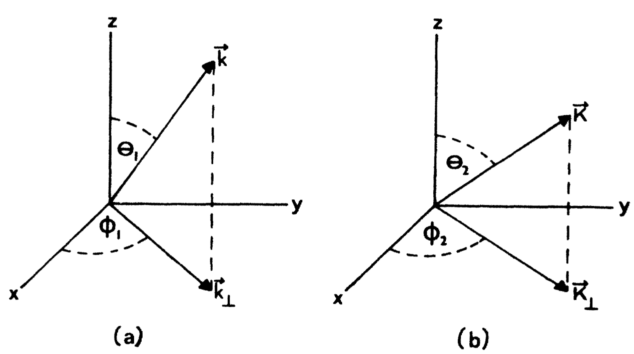

FIG. 1. Orientations of (a) $\mathbf{k}$ and $\mathbf{k}_\perp$ and (b) $\mathbf{K}$ and $\mathbf{K}_\perp$.

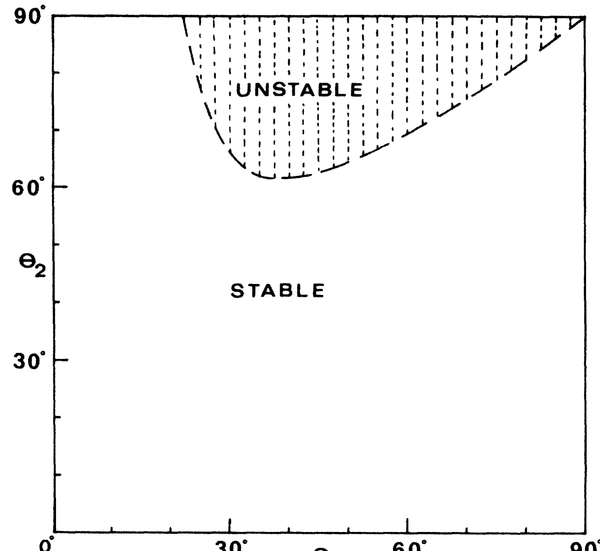

FIG. 2. Plot of the angle between $\mathbf{K}$ and $\mathbf{B}_0$ (θ_2) against the angle between $\mathbf{k}$ and $\mathbf{B}_0$ (θ_1), for $\phi_1 = \phi_2 = 0°$, $k = 2$, $\sigma = 0.1$, and $\Omega_i/\omega_{pi} = 0.25$. For a fixed θ_1, the ion acoustic waves are modulationally unstable for θ_2 values in the shaded region.

$$\Phi = [\Phi_0 + \delta\Phi(\eta)]\exp(-i\Delta\tau) ,$$

where Φ_0 is the pump amplitude, $\delta\Phi(\ll\Phi_0)$ is the perturbation, Δ a nonlinear frequency shift, and $\eta = K_x\xi_x + K_y\xi_y + K_z\xi_z - \Omega t$ (with $|\mathbf{K}| \ll |\mathbf{k}|, \Omega \ll \omega$). Equation (7) then yields $\Delta = -Q\,|\Phi_0|^2$, and an evolution equation for $\delta\Phi$ which leads to the dispersion relation

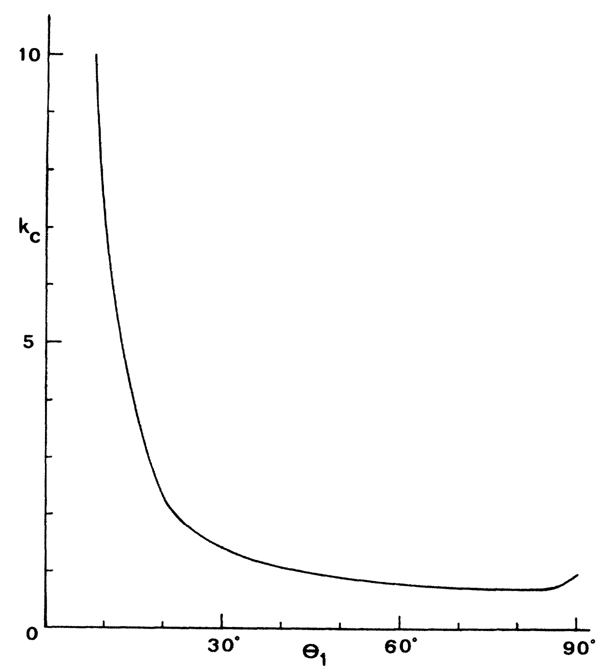

FIG. 3. Variation of the critical k value for instability, k_c, with θ_1. The other parameters are fixed as in Fig. 2.

$$(\Omega - \mathbf{K}\cdot\mathbf{V}_g)^2 = D^2/4 - QD\,|\Phi_0|^2 , \tag{9}$$

where the group velocity of the wave packet has been included, and

$$D = \partial_{k_x}^2 \omega K_x^2 + \partial_{k_y}^2 \omega K_y^2 + \partial_{k_z}^2 \omega K_z^2 + 2(\partial_{k_x k_y}^2 \omega K_x K_y + \partial_{k_x k_z}^2 \omega K_x K_z + \partial_{k_y k_z}^2 \omega K_y K_z) . \tag{10}$$

Letting $\Omega = \mathbf{K}\cdot\mathbf{V}_g + i\gamma$ in (9), we obtain the growth rate

$$\gamma = (QD\,|\Phi_0|^2 - D^2/4)^{1/2} . \tag{11}$$

Since for long-wavelength perturbations ($|\mathbf{K}| \ll |\mathbf{k}|$) the term $D^2/4$ is small, an instability occurs ($\gamma > 0$) provided $QD > 0$.

A study of the product QD was undertaken for obliquely (to $\mathbf{B}_0$) directed wave propagation ($\mathbf{k}$) and amplitude perturbation ($\mathbf{K}$). Their orientations are defined by the angles shown in Fig. 1. For $\phi_1 = \phi_2 = 0°$ (motion restricted to x-z plane), θ_1 was fixed in the range 0°–90° and θ_2 varied from 0°–90° for different $|\mathbf{k}|$ values. Instability ($QD > 0$) was found to occur for restricted values of θ_2, but always for $\theta_2 > \theta_1$. This is shown in Fig. 2 for $k = 2$. In addition, for a given θ_1, there was a critical k value, k_c, below which the ion acoustic waves were modulationally stable for all values of θ_2. The variation of k_c with θ_1, is shown in Fig. 3. It must be noted that for the large k_c values corresponding to small values of θ_1, electron Landau damping could be very strong. It is seen from Fig. 2 that for small oblique angles of wave propagation (θ_1) the perpendicular wave dispersion is too weak to produce a modulational instability. For fully three-dimensional motion ($\mathbf{k}_\perp \nparallel \mathbf{K}_\perp$), a finite angle between $\mathbf{k}_\perp$ and $\mathbf{K}_\perp (\phi_1 \neq \phi_2)$ was found to reduce the range of θ_2 values over which an instability occurred for a given θ_1. The instability was strongest for motion in the x-z plane (or y-z plane by symmetry).

To summarize, for wave propagation ($\mathbf{k}$) and long-wavelength amplitude perturbations ($|\mathbf{K}|^{-1}$) oblique to an external magnetic field $\mathbf{B}_0$ and to each other ($\mathbf{k} \nparallel \mathbf{K}$) the interaction of ion acoustic waves with a quasistatic plasma slow response is found to produce a modulational instability. This contrasts with the earlier results of Shukla[7] (for an unmagnetized plasma) and Bharuthram and Shukla[8] (magnetized plasma with oblique $\mathbf{k} \| \mathbf{K}$) who found the ion acoustic wave to be modulationally stable. Our results are in agreement with those of Kako and Hasegawa[4] who had a model similar to ours, but considered harmonic generated nonlinearities.

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*Also at Plasma Physics Research Institute, University of Natal, Durban, South Africa.

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