Dynamics of spin systems with randomly asymmetric bonds: Ising spins and Glauber dynamics

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The stochastic dynamics of randomly asymmetric fully connected Ising systems is studied. We solve analytically the particularly simple case of fully asymmetric systems. We calculate the relaxation time of the autocorrelation function and show that the system remains paramagnetic even at zero temperature ($T=0$). The ferromagnetic phase is only slightly affected by the asymmetry, and the paramagnetic-to-ferromagnetic phase transition is characterized by a critical slowing down similar to second-order transition in symmetric (fully connected) systems. Monte Carlo simulations of a fully connected Ising system with random asymmetric interactions, both at finite and zero temperature, are presented. For finite T the autocorrelation function decays completely to zero for all strengths of the asymmetry. The $T=0$ behavior is more complex. In the fully asymmetric case the system is ergodic, with decaying autocorrelations, in agreement with the theoretical predictions. In the partially asymmetric case all flows terminate at fixed points (i.e., states which are stable to single spin flips). However, the typical time that it takes to converge to a fixed point grows exponentially with the size of the system. This convergence time varies from sample to sample and has a log-normal distribution in large systems. On time scales which are smaller than the convergence time, the system behaves "ergodically," and the autocorrelation function decays to zero, much like the finite-temperature case.

I. INTRODUCTION

The properties of neural networks with asymmetric synaptic coefficients ($J_{ij} \neq J_{ji}$) have been the subject of several recent studies.[1-5] We refer here specifically to systems with random asymmetry. Two types of dynamic models have been discussed. One is a system of "soft spins" (or analog neurons) obeying Langevin equations of motions. This model has been discussed in detail in a previous paper[5] (which will be referred to as paper I). A second model, which is studied in the present paper, concerns the time evolution of asymmetric networks of discrete ± 1 variables (Ising spins).

The soft-spin model has been particularly useful in analyzing the effects of adding weak random asymmetry to systems with spin-glass (SG) and ferromagnetic (FM) interactions. Using a spherical model, the SG phase has been shown to be completely suppressed by asymmetry.[1,3,5] The autocorrelation function (the time-dependent Edwards-Anderson order parameter) decays completely to zero even at low temperatures. However, its relaxation time τ_{EA} diverges when the strength of the asymmetry, denoted by k, decreases as $\tau_{\text{EA}}(k) \approx k^{-6}$, $k \rightarrow 0$. On the other hand, the spherical model predicts that as the temperature is lowered there is a slowing down, $\tau_{\text{EA}} \approx 1/T$, $T \rightarrow 0$, for all nonzero k, leading to a complete freezing of the system at zero temperature.[5]

In this paper we study analytically and numerically the stochastic dynamics of randomly asymmetric fully connected Ising systems. The analytical study focuses on the particularly simple case of full asymmetry (J_{ij} and J_{ji} being independent random variables). This model is defined in Sec. II. Section III presents the mean-field theory of the model in the static limit. The dynamic properties are studied in Sec. IV. In Sec. V we calculate the number of states which are stable to single spin flips as a function of k. The states are relevant particularly to the understanding of the dynamics at zero temperature. In Sec. VI we present and discuss the results of extensive numerical simulations made for various degrees of asymmetry at finite and zero temperature. Summary and conclusions are given in Sec. VII.

II. DYNAMIC MODEL AND MEAN-FIELD THEORY

A. Microscopic dynamics

We study an asymmetric Ising spin-glass model in the mean-field limit. The model is a system of N Ising spins ($\sigma_i = \pm 1$) which are connected by random interactions J_{ij}. For each pair of spins (i,j) the interaction J_{ij} is of the form[5]

$$J_{ij} = J_{ij}^{\text{s}} + k J_{ij}^{\text{as}}, \quad k \geq 0 , \tag{2.1}$$

where J_{ij}^{s} and J_{ij}^{as} are symmetric and antisymmetric matrices. The off-diagonal elements of J_{ij}^{s} and J_{ij}^{as} are random Gaussian variables with means J_0/N and zero, respectively. Their variance is

$$[(J_{ij}^{\text{s}} - J_0/N)^2]_J = [(J_{ij}^{\text{as}})^2]_J = \frac{J^2}{N} \frac{1}{1+k^2} . \tag{2.2}$$

Square brackets denote the "quench" average with respect to the distribution of J_{ij}. The diagonal elements J_{ii}^{s} and J_{ii}^{as} are zero. The parameter k measures the degree of asymmetry in the interactions.

The natural dynamics to study for an Ising spin model is the Glauber dynamics.[6] It is defined by a master equation for the probability $p(\sigma_1,\dots,\sigma_N;t)$ of having the configuration $\boldsymbol{\sigma}\equiv(\sigma_1,\dots,\sigma_N)$ at time t

$$\frac{d}{dt}p(\sigma_1,\dots,\sigma_N;t)=\sum_i w_i(-\sigma_i)p(\sigma_1,\dots,-\sigma_i,\dots,\sigma_N;t)-\sum_i w_i(\sigma_i)p(\sigma_1,\dots,\sigma_i,\dots,\sigma_N;t)\ , \tag{2.3}$$

where $w_i(\sigma_i)$ is the flip rate, i.e., the probability per unit time that the ith spin flips from the value σ_i to σ_i, while the others remain momentarily fixed. A local magnetic field $h_i(t)$ introduces a preference for either the $\sigma_i=+1$ or $\sigma_i=-1$ state, and as usual we choose[6,7]

$$w_i(\sigma_i)=\frac{\Gamma}{2}[1-\sigma_i\tanh(\beta h_i)]\ , \tag{2.4}$$

where

$$h_i(t)=h_i^0(t)+\sum_j J_{ij}\sigma_j(t)\ . \tag{2.5}$$

The parameter Γ sets the scale of the microscopic process, whereas $T=\beta^{-1}$ defines the "temperature" of the system, and h_i^0 is an *external* field. For convenience we will set $\Gamma=1$.

We are interested in the time evolution of the average value of the spins and the time-delayed autocorrelation functions which obey the following equations of motion:[6,7]

$$\frac{d}{dt}\langle\sigma_i(t)\rangle=-\langle\sigma_i(t)\rangle+\langle\tanh[\beta h_i(t)]\rangle\ , \tag{2.6}$$

$$\frac{d}{dt}\langle\sigma_i(t)\sigma_i(t_0)\rangle=-\langle\sigma_i(t)\sigma_i(t_0)\rangle+\langle\sigma_i(t_0)\tanh[\beta h_i(t)]\rangle,\quad t\geq t_0\ . \tag{2.7}$$

The angular brackets denote the "thermal" average, i.e., average over $p(\sigma_1,\dots,\sigma_N;t)$,

$$\langle\sigma_i(t)\rangle=\sum_{\boldsymbol{\sigma}}\sigma_i p(\boldsymbol{\sigma};t)\ , \tag{2.8}$$

$$\langle\sigma_i(\tau)\sigma_i(t+\tau)\rangle=\sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}\sigma_i\sigma_i' p(\boldsymbol{\sigma};\tau)p(\boldsymbol{\sigma}';t+\tau{:}\boldsymbol{\sigma};\tau)\ , \tag{2.9}$$

where $p(\boldsymbol{\sigma}';t'{:}\boldsymbol{\sigma};t)$ is defined as the probability of having the configuration $\boldsymbol{\sigma}'$ at time t' if one has the configuration $\boldsymbol{\sigma}$ at time t.

In general, both $\langle\sigma_i(t)\rangle$ and $\langle\sigma_i(t)\sigma_i(t_0)\rangle$ depend on the specific configuration of the matrix J_{ij}. We define, therefore, the average magnetization and autocorrelation,

$$m(t)=[\langle\sigma_i(t)\rangle]_J\ , \tag{2.10}$$

$$C(t)=\lim_{\tau\to\infty}[\langle\sigma_i(t+\tau)\sigma_i(\tau)\rangle]_J\ . \tag{2.11}$$

B. Quench average and mean-field limit

The mean-field theory of Eqs. (2.6) and (2.7) for the general k case is rather complicated. However, in the particular case of a fully asymmetric J_{ij}, i.e., $k=1$, it simplifies considerably. From the mean-field theory of either the Glauber dynamics[8] or the analogous Langevin dynamics[2,5] one concludes that in the $N\to\infty$ limit and $k=1$, the local fields $h_i(t)$, Eq. (2.5), can be replaced by a time-dependent Gaussian random field with a width which is determined self-consistently. In particular, Eqs. (2.6) and (2.7) reduce to the following local equations:

$$\frac{d}{dt}\langle\sigma_i(t)\rangle=-\langle\sigma_i(t)\rangle+\tanh[\beta h_i(t)]\ , \tag{2.12}$$

$$\frac{d}{dt}\langle\sigma_i(t)\sigma_i(t_0)\rangle=-\langle\sigma_i(t)\sigma_i(t_0)\rangle+\langle\sigma_i(t_0)\rangle\tanh[\beta h_i(t)],\quad t\geq t_0\ , \tag{2.13}$$

where here $h_i(t)$ does not depend explicitly on the state of the other spins but is given by

$$h_i(t)=h_i^0(t)+J_0 m+\phi(t)\ . \tag{2.14}$$

The last term in Eq. (2.14) is a Gaussian variable with zero mean and variance

$$\langle\phi(t)\phi(t_0)\rangle_\phi=C(t-t_0)\ , \tag{2.15}$$

which reflects the effect of the random interactions J_{ij} on the dynamics of a single spin. Note that because of the asymmetry the effective field induced by J_{ij} is not a static field (even at equilibrium) as in the symmetric case.[2,5,9] The quantities m and $C(t)$ have to be determined self-consistently from Eqs. (2.12)–(2.15), i.e.,

$$m=\langle\langle\sigma_i(t)\rangle\rangle_\phi\ , \tag{2.16}$$

$$C(t)=\lim_{\tau\to\infty}\langle\langle\sigma_i(t+\tau)\sigma_1(\tau)\rangle\rangle_\phi\ , \tag{2.17}$$

where $\langle\ \rangle_\phi$ means averaging the solutions of Eqs. (2.12) and (2.13) over the Gaussian field $\phi(t)$. For fixed configuration of $\phi(t)$ the solution of Eq. (2.12) (with $h^0=0$) reads

$$\langle\sigma_i(t)\rangle=\int_{-\infty}^{t}dt'e^{(t'-t)}\tanh\{\beta[J_0 m+\phi(t')]\}\ , \tag{2.18}$$

where

$$m=\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}e^{-\phi^2/2}\tanh[\beta(J_0 m+\phi)]\ . \tag{2.19}$$

Inserting Eq. (2.18) into Eq. (2.13) and carrying our the average over $\phi(t)$ we find the following self-consistent equation for $C(t)$:

$$\frac{d}{dt}C(t)=-C(t)+\int_t^{+\infty}dt' e^{(t-t')}f(t'),\quad t\geq 0 \tag{2.20}$$

$$f(t)=\langle \tanh\{\beta[J_0m+\phi(0)]\}\tanh\{\beta[J_0m+\phi(t)]\}\rangle_\phi$$
$$=\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\left[\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\tanh\{\beta[\sqrt{1-C(t)}x+\sqrt{C(t)}z+J_0m]\}\right]^2 . \tag{2.21}$$

For a detailed derivation of Eq. (2.21) see Appendix A.

III. MEAN-FIELD EQUATIONS: THE STATIC LIMIT

In this section we study the static limit of the mean-field equations (2.19)–(2.21), appropriate for the $k=1$ case. Equating the time derivatives to zero yields the following static mean-field equation:

$$q=\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\times\left[\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\times\tanh[\beta(J_0m+\sqrt{1-q}\,x+\sqrt{q}\,z)]\right]^2 , \tag{3.1}$$

where q is the static Edwards-Anderson order parameter defined by[10]

$$q=\lim_{t\to\infty}C(t) . \tag{3.2}$$

The static mean-field equation for the average magnetization m is given by Eq. (2.19).

The static *local* susceptibility is

$$\chi=\left[\frac{\partial\langle\sigma_i\rangle}{\partial h_i^0}\right]=\beta\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}e^{-\phi^2/2}\cosh^{-2}[\beta(J_0m+\phi)] , \tag{3.3}$$

whereas the static response to an uniform field $h_i^0=h^0$ is

$$\chi_{\rm FM}=\left.\frac{\partial m}{\partial h^0}\right|_{h^0=0}=\frac{\chi}{1-J_0\chi} . \tag{3.4}$$

We first consider the possibility of a SG phase, i.e., a phase with $m=0$ and $q\neq 0$. It is straightforward to see from Eq. (3.1) that if the SG phase appears via a second-order transition, then the paramagnetic χ of Eq. (3.3) must be equal to unity at the transition. However, evaluation of Eq. (3.3) (with $m=0$) shows that $\chi<1$ for all T; see Fig. 1. To rule out a possible first-order SG transition, we have solved Eq. (3.1) numerically with $m=0$. We have found that $q=0$ is the only solution for all $T>0$.

The $T=0$ limit is special. In this case, Eq. (3.1) reduces to

$$q=\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\left[\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\times{\rm sgn}(\sqrt{1-q}\,x+\sqrt{q}\,z)\right]^2 , \tag{3.5}$$

which has both solutions $q=0$ and $q=1$. However, it will be shown in Sec. IV that the $q=1$ solution is unstable leading to the conclusion that the system does not have a SG phase even at $T=0$.

The instability of the PM phase to FM order occurs when

$$J_0\chi=1 , \tag{3.6}$$

where χ is the PM local susceptibility (shown in Fig. 1), as is evident from Eq. (3.4). Equation (3.6) defines a critical line $J_c(T)=\chi^{-1}(T)$ in the (T,J_0) plane separating the PM and the FM phases; see Fig. 2. Note that $\chi(T=0)=(2/\pi)^{1/2}$ so that

$$J_c(T=0)=\sqrt{\pi/2} . \tag{3.7}$$

The solution of Eqs. (2.19) and (3.1) near the FM transition temperature $T_c=T(J_c)$ can be obtained expanding both equations in powers of m. Since in the FM phase q must be an even function of m and $q(m=0)=0$, we expect that $q\approx m^2+O(m^4)$. Indeed, expanding Eq. (3.1) for small m we find

$$q=\frac{J_0^2\chi^2}{1-\chi^2}m^2+O(m^4) . \tag{3.8}$$

The same procedure applied to Eq. (2.19) yields

$$m^2=A\left[\frac{J_0}{J_c(T)}-1\right],\quad A>0 . \tag{3.9}$$

In the $T=0$ limit Eq. (2.19) reduces to

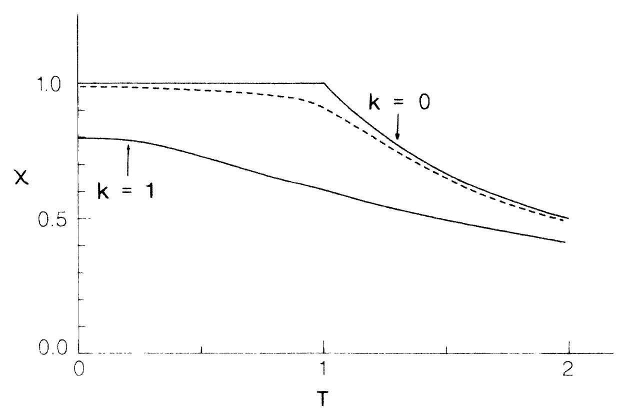

FIG. 1. Static susceptibility χ of the asymmetric Ising SG model vs T for different values of k. The $k=1$ case is given by Eq. (3.3) with $m=0$. The dashed line shows the qualitative shape of χ for a fixed small value of k.

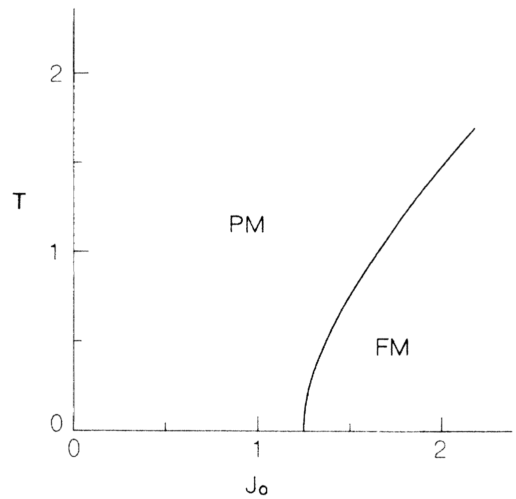

FIG. 2. Phase diagram of the fully asymmetric Ising SG model. The line separating the FM and PM phases is given by $J_c(T)$.

$$m=\mathrm{erf}\left[\frac{J_0 m}{\sqrt{2}}\right], \quad T=0 . \tag{3.10}$$

One can easily check that Eq. (3.10) does not have solutions $m\neq 0$ for $J_0<(\pi/2)^{1/2}$. In the same limit Eq. (3.1) becomes

$$q=\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\left[\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\times\mathrm{sgn}(J_0 m+\sqrt{1-q}\,x+\sqrt{q}\,z)\right]^2 . \tag{3.11}$$

If $J_0\gg 1$ the solution of Eqs. (3.10) and (3.11) reads

$$m\approx 1-\left[\frac{2}{\pi}\right]^{1/2}\frac{e^{-J_0^2/2}}{J_0}, \quad J_0\gg 1, \quad T=0 \tag{3.12}$$

$$q\approx 1-\frac{8}{\pi^2}e^{-J_0^2}, \quad J_0\gg 1, \quad T=0 . \tag{3.13}$$

The results (3.12) and (3.13) are in agreement with three results of Feigelman and Ioffe.[2] They consider asymmetric dilution of the Hopfield model in the limit $\alpha\to 0$, α being the ratio between the number of memories and the number of neurons. In the limit $\alpha\to 0$ the model is equivalent to an asymmetric SG model (2.1) and (2.2) with $J_0=1/\sqrt{\alpha}\to\infty$. In this case J_0 represents the tendency to condense in each one of the memories and m stands for the macroscopic overlap with the condensed memory.

IV. MEAN-FIELD EQUATIONS: DYNAMICS IN THE PM PHASE

In this paragraph we investigate the relaxation of the average autocorrelation function $C(t)$, given by Eq. (2.20), in the PM phase. In this phase there are no time-persistent correlations, i.e., $C(t)\approx 0$ for $t\gg 1$, so that we can expand the right-hand side of Eq. (2.21) in powers of $C(t)$, obtaining as the leading order

$$f(t)\approx\chi^2 C(t), \quad t\gg 1 , \tag{4.1}$$

with χ given by Eq. (3.3) ($m=0$). Inserting Eq. (4.1) into Eq. (2.20) and differentiating with respect to t one obtains

$$\frac{d^2C(t)}{dt^2}=(1-\chi^2)C(t), \quad t\gg 1 . \tag{4.2}$$

The solution of Eq. (4.2) which satisfies the condition $C(t\to\infty)=0$ is $C(t)\approx\exp(-t/\tau_{\mathrm{EA}})$, where the Edwards-Anderson relaxation time τ_{EA} is given by

$$\tau_{\mathrm{EA}}=\frac{1}{(1-\chi^2)^{1/2}} . \tag{4.3}$$

Since $\chi<1$ for all $T\geq 0$, the relaxational time τ_{EA} is finite at all $T\geq 0$ and no critical slowing down occurs even as $T\to 0$. The zero-temperature limit of τ_{EA} is $\tau_{\mathrm{EA}}=1/(1-2/\pi)^{1/2}\approx 1.66$; see Fig. 1.

In the $T=0$ case Eq. (2.20) also admits the solution $C(t)=1$ (which corresponds to the $q=1$ solution found in statics). However, if we write $C(t)=1-\delta C(t)$ and expand Eq. (2.20) in powers of $\delta C(t)$, we find $\delta C(t)\to\infty$, indicating that the solution $C(t)=1$ is unstable. The system, therefore, does not show freezing even at $T=0$.

To study the relaxation of the magnetization in the PM phase we assume that the initial states have nonzero magnetization. The evolution of $m(t)$, in this case, is given by Eqs. (2.12) and (2.16). Carrying out the average over $\phi(t)$ (see Appendix B) we obtain the following equation:

$$\frac{d}{dt}m(t)=-m(t)+\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\tanh\times\{\beta[h^0(t)+J_0 m(t)+x]\} . \tag{4.4}$$

In the PM phase $m(t)\approx 0$ for $t\gg 1$ (for $h^0=0$), thus we can linearize Eq. (4.4) obtaining

$$\frac{d}{dt}m(t)=-\left[1-\frac{J_0}{J_c(T)}\right]m(t), \quad t\gg 1 \tag{4.5}$$

which yields

$$m(t)\approx e^{-t/\tau_{\mathrm{FM}}}, \quad t\gg 1 \tag{4.6}$$

where the FM relaxation time is

$$\tau_{\mathrm{FM}}^{-1}=1-\frac{J_0}{J_c(T)} . \tag{4.7}$$

The equation for the average FM response function

$$G_{\mathrm{FM}}(t)=\lim_{\tau\to\infty}\left.\frac{\delta m(t+\tau)}{\delta h^0(\tau)}\right|_{h^0=0}, \quad t\geq 0 \tag{4.8}$$

is easily obtained differentiating Eq. (4.4), with respect to $h^0(t)$, i.e.,

$$\frac{d}{dt}G_{\mathrm{FM}}(t)=-\left[1-\frac{J_0}{J_c(T)}\right]G_{\mathrm{FM}}(t)+\chi\delta(t) , \tag{4.9}$$

leading to

$$G_{\mathrm{FM}}(t)=\chi e^{-t/\tau_{\mathrm{FM}}}\Theta(t) . \tag{4.10}$$

Equation (4.6) implies that the FM relaxation time diverges as $T \to T_c^+$, i.e., the system shows a critical slowing down as the FM transition is approached from above, similar to an ordinary second-order transition (in the mean-field limit).

V. NUMBER OF METASTABLE STATES OF THE ASYMMETRIC ISING SPIN-GLASS MODEL

A. General asymmetric case

In this section we calculate the average number of states which are stable, with respect to single spin flips, in the asymmetric Ising SG model. The fully asymmetric limit will be treated in more detail in Sec. V B. We use the method developed by Tanaka and Edwards[11] and Bray and Moore[12] for the SK model.[13]

A state is called metastable (at $T=0$) if it is stable, with respect to single spin flips, i.e., if all the spins are parallel to their local field

$$h_i = \sum_j J_{ij}\sigma_j \; . \tag{5.1}$$

Thus the equations of the metastable states, for a fixed configuration of J_{ij}, are

$$\sigma_i = \mathrm{sgn}\left[\sum_j J_{ij}\sigma_j\right], \quad i=1,2,\ldots,N \; . \tag{5.2}$$

Following Refs. 11 and 12, Eq. (5.4) can be written as

$$\lambda_i\sigma_i = \sum_j J_{ij}\sigma_j, \quad \lambda_i > 0 \tag{5.3}$$

so that the number of metastable states is

$$N_s = \mathrm{Tr}_{[\sigma]} \prod_i \int_0^\infty d\lambda_i \delta\left[\lambda_i\sigma_i - \sum_j J_{ij}\sigma_j\right] \; . \tag{5.4}$$

where $\mathrm{Tr}_{[\sigma]}$ is a sum over the 2^N states of the system.

We first discuss the $J_0=0$ and $0<k<1$ case. Introducing an integral representation of the δ function and carrying out the average over the configuration of J_{ij}, we find for the average number of metastable states,

$$[N_s]_J = \mathrm{Tr}_{[\sigma]} \int_0^\infty \prod_i d\lambda_i \int_{-\infty}^{+\infty} \prod_i \frac{d\phi_i}{2\pi} \exp\left[-\sum_i i\phi_i\lambda_i\sigma_i - \tfrac{1}{2}\sum_i \phi_i^2 - \frac{1-k^2}{1+k^2}\frac{1}{2N}\left[\sum_i \phi_i\sigma_i\right]^2\right]_J , \tag{5.5}$$

where we have neglected $O(1/N)$ terms. Further simplification is achieved by way of a Gaussian transformation similar to Ref. 12, which leads to

$$[N_s]_J = \left[\frac{N}{2\pi}\frac{1-k^2}{1+k^2}\right]^{1/2} \int_{-\infty}^{+\infty} dx\, e^{Ng(x;k)} , \tag{5.6}$$

$$g(x,k) = -\frac{1-k^2}{1+k^2}\frac{x^2}{2} + \ln\left[1+\mathrm{erf}\left[\frac{1-k^2}{1+k^2}\frac{x}{\sqrt{2}}\right]\right] . \tag{5.7}$$

In the limit $N\to\infty$ the integral over x may be performed by the method of steepest descents, yielding

$$[N_s]_J \approx e^{Ng(k)} , \tag{5.8}$$

where $g(k)$ is the stationary value of $g(x;k)$

$$g(k) = -\frac{1-k^2}{(1+k^2)^2}x^2 - \tfrac{1}{2}\ln\left[\frac{\pi x^2}{2}\right], \quad k \le 1 \tag{5.9}$$

and x is solution of the saddle-point equation,

$$x = \left[\frac{2}{\pi}\right]^{1/2} \frac{\exp\left[-\left[\frac{1-k^2}{1+k^2}\right]^2\frac{x^2}{2}\right]}{1+\mathrm{erf}\left[\frac{1-k^2}{1+k^2}\frac{x}{\sqrt{2}}\right]}, \quad k \le 1 \; . \tag{5.10}$$

The numerical evaluation of $g(k)$, Eqs. (5.9) and (5.10), is shown in Fig. 3. Note that $g(k=0)=0.1992$ which agrees with the value found in Refs. 11 and 12 for the SK model. For $0<k<1$, $g(k)$ is lower than $g(k=0)$ but nonzero; thus the asymmetry does not destroy, for $k<1$, the exponential growth with N of the average number of metastable states. However, it reduces the value of the coefficient of N. In the $J_0 \neq 0$ and $0 \le k < 1$ case a similar calculation shows that $g(k)$ is a positive decreasing function of J_0, so that the average number of metastable states is still exponential in N. In the $J_0 \to \infty$ limit $g(k)\to 0$ and, as expected, $[N_s]_J \to 2$.

In the $k>1$ case Eqs. (5.9) and (5.10) are not valid anymore; however, similar equations can be derived. We note that in this case the saddle-point value of x is pure imaginary. The evaluation of $g(k)$ at the saddle point yields $g(k)<0$, and hence $[N_s]_J = 0$ in the thermodynamic limit, for all values of $k>1$ and finite J_0. Here, again, $g(k)\to 0$ and $[N_s]_J \to 2$ in the $J_0\to\infty$ limit.

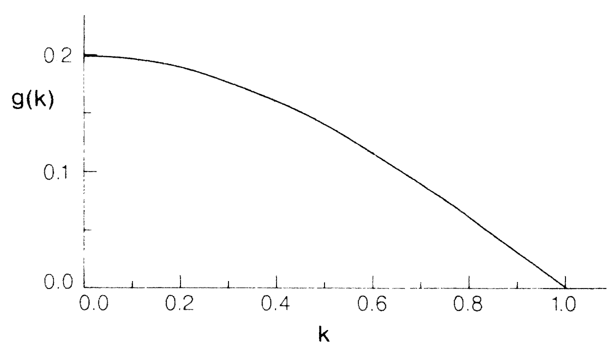

FIG. 3. Number of metastable states $g(k)$ vs k for $0 \le k \le 1$ and $J_0=0$. Note that $g(k=0)=0.1992$ (SK model) and $g(k=1)=0$ (fully asymmetric Ising SG model). In the $k>1$ case, $g(k)$ is negative (at least for all finite values of J_0) and $[N_s]_J=0$ in the thermodynamic limit.

B. Fully asymmetric case

In the fully asymmetric limit, i.e., $k=1$, Eq. (5.10) yields

$$x=\sqrt{2/\pi}\ , \tag{5.11}$$

which when substituted into Eq. (5.9) yields $g(k=1)=0$ leading to $[N_s]_J=1$. We note that this result is true for all N and not just in the thermodynamic limit. In fact, averaging Eq. (5.4) yields for $k=1$ (and $J_0=0$)

$$[N_s]_J=\mathrm{tr}_{[\sigma]}\int_0^{+\infty}\prod_i d\lambda_i\int_{-\infty}^{+\infty}\prod_i\frac{d\phi_i}{2\pi}\times\exp\left[-\frac{N-1}{2N}\sum_i\phi_i^2-\sum_i i\phi_i\lambda_i\sigma_i\right] . \tag{5.12}$$

A straightforward evaluation of the integral yields

$$[N_s]_J=1^N=1 \quad \text{for all } N\ . \tag{5.13}$$

This result can be obtained by the following simple argument. Let us choose a fixed configuration of spins $\boldsymbol{\sigma}$, and ask, what is the probability that this configuration is a metastable state? The conditions for stability are

$$\sigma_i=\mathrm{sgn}\left[\sum_j \bar{J}_{ij}\right],\quad \bar{J}_{ij}=J_{ij}\sigma_j,\quad i=1,2,\ldots,N\ . \tag{5.14}$$

For any fixed configuration $\boldsymbol{\sigma}$ the new variables $\{\bar{J}_{ij}\}$ have the same probability distribution as the J_{ij}'s. This probability distribution is symmetric around zero, and each sum in the N equations of (5.14) is an independent random variable. This implies that each equation will be satisfied with a probability $\frac{1}{2}$ hence

$$[N_s]_J=2^N(\tfrac{1}{2})^N=1 \quad \text{for all } N\ .$$

Using this argument, the result (5.13) can be extended to other classes of fully asymmetric systems (e.g., systems with directed bonds).

In the $J_0\neq 0$ case, the integrals cannot be performed and one has to again use the steepest-descent method. After a straightfoward algebra we find

$$[N_s]_J=\frac{1}{1-\sqrt{2/\pi}J_0},\quad J_0<\sqrt{\pi/2}\ , \tag{5.15a}$$

$$[N_s]_J=\frac{1}{\sqrt{2/\pi}J_0-1}+\frac{2}{1-\sqrt{2/\pi}J_0e^{-J_0^2m^2/2}}\ ,\quad J_0>\sqrt{\pi/2}\ , \tag{5.15b}$$

where m is the solution of the saddle-point equation,

$$m=\mathrm{erf}\left[\frac{J_0 m}{\sqrt{2}}\right]\ , \tag{5.16}$$

which is identical to the static mean-field equation for the magnetization m at $T=0$ [cf. Eq. (3.10)]. Equations (5.15) imply that $[N_s]_J$ diverges at the critical point. The Ising ferromagnetic result $[N_s]_J=2$ is recovered in the limit $J_0\rightarrow\infty$.

We note that Eqs. (5.15) and (5.16) also represent the average number of metastable states of a long-range Ising ferromagnet with a Gaussian random field with mean zero and variance J_0. This agrees with our dynamic study of the fully asymmetric Ising SG where the effect of the random interactions has been replaced by a Gaussian random field.

C. Discussion

The results of Fig. 3 show that the number of metastable states is a smooth function of k (between 0 and 1). Naively, this would suggest that a SG freezing occurs at $T=0$ for all k, $0<k<1$. Indeed, the spherical model (see paper I) does predict that the EA order parameter equals unity at $T=0$. However, in that model this is true for all k, even above 1, which certainly cannot hold in the Ising case.

In relating the metastable states to the actual dynamical behavior of the system, one first has to realize that the preceding results refer only to the *average* number of metastable states. Naturally one would tend to represent the *typical* number of metastable states by $\exp[\ln N_s]_J$ rather than by $[N_s]_J$. We have not yet calculated either $[\ln N_s]_J$ or the distribution of N_s. Nevertheless, on the basis of previous studies of the symmetric case, we anticipate that for $k<1$ and large N the distribution of N_s is sharply peaked around $[N_s]_J$ (i.e., $\ln[N_s]_J\approx[\ln N_s]_J$), so that Fig. 3 also represents the number of metastable states in a *typical* configuration of J_{ij}. On the other hand, because of the $\sigma_i\rightarrow-\sigma_i$ symmetry of the system, N_s is necessarily even and therefore the result (5.13) for $k=1$ cannot represent the *typical* case. In fact, it is possible that, in the $k=1$ case and large N, the distribution of N_s has most of its weight around

$$N_s=0\ , \tag{5.17}$$

and in addition has a tail (with vanishingly small weight) extending to large values of N_s. This would imply that a *typical* large, fully asymmetric system does not have *any* metastable state. In addition, the dynamics is affected not only by the number of metastable states but also by their basins of attraction, which is hard to estimate analytically.

In Sec. VI, we study the relation between the stochastic dynamics of the system and the metastable states by numerical simulations at $T=0$. We will also present simulations at finite T to check some of the predictions of paper I.

VI. NUMERICAL RESULTS

In this section we present the results of computer simulations of the dynamics of Ising systems with asymmetric interactions of the types (2.1) and (2.2). The simulations have been performed on Cray X-MP at AT&T Bell Laboratories. We have used a random sequential dynamics where at each time step a randomly chosen spin is updated according to the following rule:

$$\sigma_i(t+1)=+1 \quad \text{with probability } \{1+\exp[-2h_i(t)/T]\}^{-1}, \tag{6.1}$$
$$\sigma_i(t+1)=-1 \quad \text{with probability } \{1+\exp[+2h_i(t)/T]\}^{-1},$$

where

$$h_i(t)=\sum_j J_{ij}\sigma_j(t), \tag{6.2}$$

and T is the temperature of the system. The J_{ij}'s are chosen from a Gaussian distribution characterized by Eqs. (2.1) and (2.2) with $J_0=0$. In the $T=0$ limit, the dynamics reduces to a random sequential updating of spins according to the rule $\sigma_i(t+1)=\text{sgn}h_i(t)$.

A. Finite-T case

We have calculated the autocorrelation function

$$C(t)=\frac{1}{N}\sum_i \sigma_i(t_0)\sigma_i(t+t_0), \tag{6.3}$$

averaged over different initial states, different values of t_0, and different configurations of J_{ij}. In each trial the first few hundred steps (per spin) were excluded from the evaluation of $C(t)$. The results for $T=0.5$ are shown in Fig. 4. For comparison we also present the symmetric case $k=0$. In this case, $C(t)$ decays to a finite value $q\approx0.6$ which is close to the expected value of the EA order parameter in the SK model.[13]

The numerical results for $k\neq0$ are all consistent with the prediction that $C(t)$ decays to zero for all nonzero values of k. Figure 4 also indicates that the decay of $C(t)$ is slower when k decreases. We have chosen as a crude estimate of the relaxation time τ_{EA} of $C(t)$ the time at which $C(t)$ decays to $C(t)\approx0.1$. A log-log plot of $\tau_{EA}(k)$ is shown in Fig. 5. The small-k values fit rather nicely with the power law

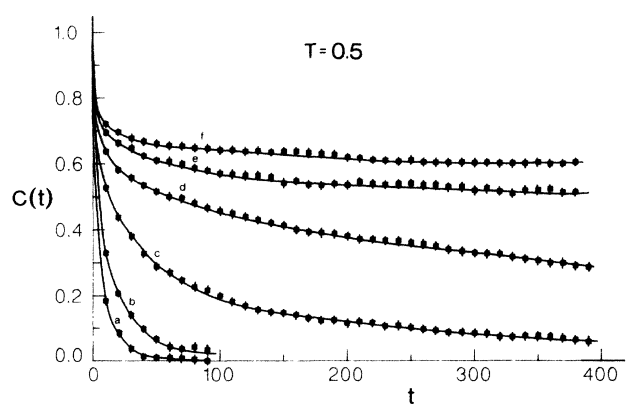

FIG. 4. Monte Carlo results of the autocorrelation function $C(t)$, Eq. (6.3), vs t for different values of k at $T=0.5$. (a) $k=0.5$, (b) $k=0.4$, (c) $k=0.3$, (d) $k=0.2$, (e) $k=0.1$, and (f) $k=0$.

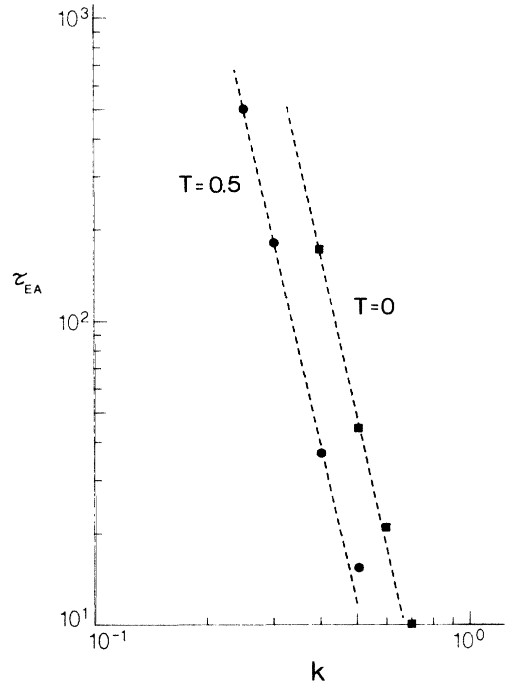

FIG. 5. Characteristic relaxation time τ_{EA}, of $C(t)$, vs k. The circles correspond to $T=0.5$ and the squares to $T=0$. τ_{EA} is estimated from the numerical results of Figs. 4 and 8.

$$\tau_{EA}(k)\approx k^{-6}, \quad k\ll 1 \tag{6.4}$$

as predicted by the results of the spherical model of paper I.

B. $T=0$ case

Extensive simulations of $T=0$ asynchronous dynamics have been performed for sizes $N=100-1200$, and k between zero and unity. Our first main observation is that for $N>100$ the system almost always converges to a metastable state, even for values of k as high as 0.6. The fraction of nonconvergent runs was less than 5%. In the $k=1$ case almost none of the flows converged, at least within our longest measuring time (which was $\approx 15\,000$ Monte Carlo steps per spin).

In order to gain more insight into the nature of the flows, we have measured the time that it takes to converge to a fixed point. We have observed large fluctuations in this convergence time τ as the sample of J_{ij} or the initial states are varied. These fluctuations suggest that the appropriate quantity for averaging is not τ but its logarithm. Figure 6 shows the distribution of $\ln\tau$ for $k=0.4$ and $N=600$ from runs of $\approx 10\,000$ samples of $\{J_{ij}\}$. As shown in the figure, the distribution of $\ln\tau$ is quite close to a Gaussian. Furthermore, comparing the histograms of different sizes (but the same value of k) shows that both the peak and the width scale linearly with N. Figure 7 shows the results for the average of $\ln\tau$ for different values of N and k. The results agree with the relation

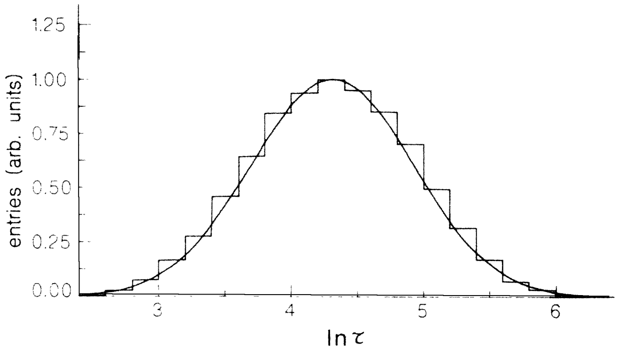

FIG. 6. Distribution of the logarithm of the convergence time τ for $k=0.4$ and $N=600$ obtained from computer runs of about 10 000 samples of $\{J_{ij}\}$. The initial states were chosen at random. The solid line is a best-fit Gaussian.

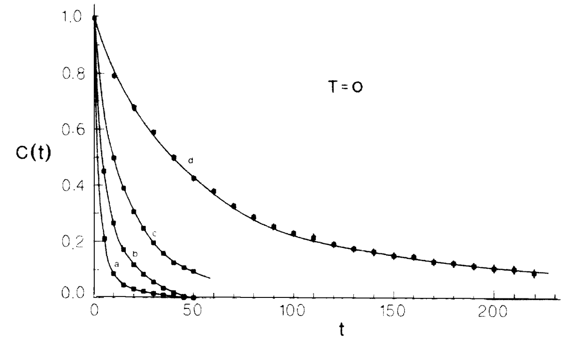

FIG. 8. Autocorrelation function $C(t)$ vs t for different values of k at $T=0$. (a) $k=0.7$, (b) $k=0.6$, (c) $k=0.5$, and (d) $k=0.4$.

$$\overline{\ln\tau(N,k)}=Na(k), \quad 0<k<1 \tag{6.5}$$

where the coefficient $a(k)\to 0$ as $k\to 0$ and $a(k)\to\infty$ as $k\to 1$.

Since the time to reach a fixed state diverges exponentially with N (at least for a not-too-small value of k), it is important to know whether a *finite*-time measurement of $C(t)$ will exhibit symptoms of freezing. To check this we have measured $C(t)$, Eq. (6.3), at $T=0$, taking into account only the data with $t_0\geq 50$ Monte Carlo steps per spin at $t+t_0\leq\tau-t_0$, where τ is the time at which the flow converged to a fixed state. The results for $0.4<k<1$ are presented in Fig. 8. Evaluation of $C(t)$ for smaller values of k requires much larger systems to ensure a sizeable convergence time.

The striking conclusion from Fig. 8 is that, although the system relaxes eventually to a fixed state, the auto correlations, measured along the flows before they terminate, decay completely to zero. In fact, comparing Figs. 8 and 4 one observes that the $T=0$ $C(t)$ behaves similarly to that of $T=0.5$. To check this point quantitatively we present in Fig. 5 the relaxation time of $C(t)$, $\tau_{EA}(k)$ extracted from the $T=0$ data [again with the criterion $C(\tau_{EA})\approx 0.1$]. Although the range of the data is limited, the k dependence of the $T=0$ results for small values of k is very similar to the $T=0.5$ results. Both agree well with the power law, Eq. (6.4). This should be contrasted with the prediction of the spherical model that for all fixed values of $k>0$, τ_{EA} diverges as T^{-1} when $T\to 0$.[5]

The above numerical results yield the following picture. In the fully asymmetric systems ($k=1$) the flows at $T=0$ and finite T have similar characteristics. The system continues to fluctuate indefinitely, causing a complete decay of autocorrelations $C(t)$ with a finite relaxation time, in agreement with the analytical results of Secs. III and IV. In the partially asymmetric case ($0<k<1$) at $T=0$ the flows wander initially "ergodically" in configuration space, again giving rise to a complete decay of $C(t)$ with a finite relaxation time. However, in any finite system (and $k<1$) the probability that the flow will "accidentally hit" one of the metastable states in a time less than t is finite if $t\approx\tau(N,k)$ and grows to unity when $t\gg\tau(N,k)$, where the characteristic convergence time τ diverges exponentially with N.

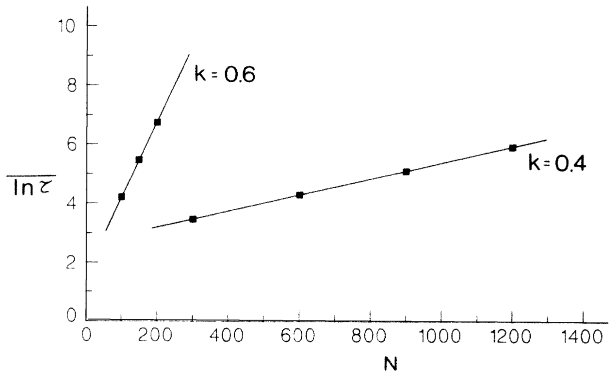

FIG. 7. Averaged logarithm of the convergence time τ at $T=0$, for $k=0.6$ and $k=0.4$, vs N. Each point was calculated with 5000–10 000 samples.

VII. SUMMARY AND CONCLUSIONS

In this paper we have compared the dynamics of randomly asymmetric Ising systems with the predictions of the asymmetric spherical model.[5] The results of the Monte Carlo simulations indicate that, at finite temperature, the autocorrelation function decays completely to zero for all values of the strength of the asymmetry, denoted by k. Thus the asymmetric system does not undergo a spin-glass freezing at any finite T. When the strength of the asymmetry decreases, the autocorrelation relaxation time τ_{EA} grows as $\tau_{EA}(k)\approx k^{-6}$, $k\to 0$, at low temperature, in agreement with the results of the spherical model.

The results of the dynamics at zero temperature are more complex. In the fully asymmetric systems ($k=1$) there are, at most, few states which are stable (to single spin flips) and the system remains unfrozen, even at $T=0$, as shown by the mean-field theory (Secs. IV and V). This suggests that the spin-glass freezing, at $T\to 0$, predicted by the spherical model for all values of $k>0$, is an artifact of the quasilinearity of that model.

As for partially asymmetric systems ($k<1$) our numerical simulations show that also these systems exhibit an extremely long-lived disordered "paramagnetic" state at $T=0$. This is despite the fact that there is an exponentially large number of stable states. The lifetime of the disordered phase, τ, fluctuates enormously from sample to sample. For large systems, it obeys a log-normal distribution, which scales with the system size N, i.e., the average and the width of $\ln\tau$ are proportional to N. At times which are long compared to τ the system converges to one of the stable states, and becomes completely frozen. The exponentially long lifetime of the disordered phase implies that the basins of attraction of the stable fixed points are extremely small. A similar phenomenon of a disordered phase which has an extremely long (but finite) lifetime has been found in a recent study of turbulence in the Kuramoto-Sivashinsky equation.[14] A disordered, phase-turbulent state has been found to be stable with a lifetime which grows exponentially with the system size. At larger time scales the system settles in a spatially ordered, "cellular" state.

Finally, let us emphasize that we have focused in this work on a dynamics which is stochastic even at $T=0$, where the order of updating is kept random. This stochasticity destroys the cyclic attractors that may exist in algorithms with fixed order of updating, as found in Ref. 15.

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APPENDIX A

In this appendix[16] we calculate averages of the type

$$f(t)=\langle F(a+\phi(0))F(a+\phi(t))\rangle_\phi\ , \tag{A1}$$

where $F(x)$ is any function of x, $-\infty<x<+\infty$, for which the Fourier transform exists, and the average over $\phi(t)$, i.e., $\langle(\cdots)\rangle_\phi$, is defined as

$$\langle(\cdots)\rangle_\phi=Z^{-1}\int D\phi(\cdots)\times\exp\left[-\frac{1}{2}\int\frac{d\omega}{2\pi}\phi(\omega)C^{-1}\times(\omega)\phi(-\omega)\right], \tag{A2}$$

where $C(\omega)$ are the Fourier components of $C(t)$ and Z is a normalization factor. Equation (2.21) is the particular case of $F(x)=\tanh(\beta x)$ and $a=J_0m$. Let us define $g(\alpha)$ as

$$F(x)=\int\frac{d\alpha}{2\pi}g(\alpha)e^{-i\alpha x}\ , \tag{A3}$$

so that $f(t)$ can be written as

$$\begin{aligned}f(t)&=Z^{-1}\int D\phi\int\frac{d\alpha}{2\pi}\frac{d\alpha'}{2\pi}g(\alpha)g(\alpha')\exp[-ia(\alpha+\alpha')]\\&\quad\times\exp\left[-\frac{1}{2}\int\frac{d\omega}{2\pi}\{\phi(\omega)C^{-1}(\omega)\phi(-\omega)+i\alpha[\phi(\omega)+\phi(-\omega)]+i\alpha'[\phi(\omega)e^{-i\omega t}+\phi(-\omega)e^{i\omega t}]\}\right]\\&=\int\frac{d\alpha}{2\pi}\frac{d\alpha'}{2\pi}g(\alpha)g(\alpha')\exp\{-\tfrac{1}{2}[\alpha^2+\alpha'^2+2\alpha\alpha'C(t)]-ia(\alpha+\alpha')\}\ .\end{aligned} \tag{A4}$$

In deriving Eq. (A4) we have used the properties $C(t=0)=1$ and $C(-t)=C(t)$. By definition we have

$$g(\alpha)=\int dx\, e^{i\alpha x}F(x)=\beta\int dx\, e^{i\alpha(x+a)}F(x+a)\ , \tag{A5}$$

which when substituted in Eq. (A4) yields

$$f(t)=\int dx\, F(x+a)\int dy\, F(y+a)\int\frac{d\alpha}{2\pi}\frac{d\alpha'}{2\pi}\exp\{-\tfrac{1}{2}[\alpha^2+\alpha'^2+2\alpha\alpha'C(t)]+i\alpha x+i\alpha' y\}\ . \tag{A6}$$

Carrying out the integrals over α and α' we obtain, after a straightforward algebra,

$$\begin{aligned}f(t)&=\frac{1}{2\pi[1-C^2(t)]^{1/2}}\int dx\, F(x+a)\int dy\, F(y+a)\exp\left[-\frac{x^2+y^2-2xyC(t)}{2[1-C^2(t)]}\right]\\&=\int\frac{dz}{\sqrt{2\pi}}e^{-z^2/2}\left[\int\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}F(\sqrt{1-C(t)}x+\sqrt{C(t)}z+a)\right]^2,\end{aligned} \tag{A7}$$

as can be easily verified performing the integration over z.

APPENDIX B

In this appendix we show how, using the method explained in Appendix A, $\langle \tanh\{\beta[J_0 m(t)+\phi(t)]\}\rangle_\phi$ can be calculated. From Eqs. (A2) and (A3) we have

$$\langle \tanh\{\beta[J_0 m(t)+\phi(t)]\}\rangle_\phi = Z^{-1}\int D\phi \int \frac{d\alpha}{2\pi} g(\alpha) e^{-i\alpha\beta J_0 m(t)} \times \exp\left[-\frac{1}{2}\frac{d\omega}{2\pi}\{\phi(\omega)C^{-1}(\omega)\phi(-\omega)+i\alpha\beta[\phi(\omega)e^{-i\omega t}+\phi(-\omega)e^{i\omega t}]\}\right]$$

$$=\int \frac{d\alpha}{2\pi} g(\alpha)\exp\left[-i\alpha\beta J_0 m(t)-\frac{\beta^2\alpha^2}{2}\right]$$

$$=\int dx\,\tanh(x)\int\frac{d\alpha}{2\pi}\exp\left[-\frac{\beta^2\alpha^2}{2}+i\alpha[x-\beta J_0 m(t)]\right]$$

$$=\int \frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\tanh\{\beta[J_0 m(t)+x]\}\ . \tag{B1}$$

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[1]A. Hertz, G. Grinstein, and S. A. Solla, in *Proceedings of the Heidelberg Colloquium on Glassy Dynamics and Optimization, Heidelberg, 1986*, edited by J. L. van Hemmen and I. Morgenstern (Springer-Verlag, Berlin, 1987).

[2]M. V. Feigelman and L. B. Ioffe, Int. J. Mod. Phys. B **1**, 51 (1987).

[3]R. Bausch, H. K. Janssen, R. Kree, and A. Zippelius, J. Phys. C **19**, L779 (1986).

[4]G. Parisi, J. Phys. A **19**, L675 (1986).

[5]A. Crisanti and H. Sompolinsy, Phys. Rev. A **36**, 4922 (1987).

[6]R. J. Glauber, J. Math. Phys. **4**, 294 (1963).

[7]M. Suzuki and R. Kubo, J. Phys. Soc. Jpn. **24**, 51 (1968).

[8]H. J. Sommers, Phys. Rev. Lett. **58**, 1268 (1987).

[9]H. Sompolinsky and A. Zippelius, Phys. Rev. B **25**, 6860 (1982).

[10]S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

[11]F. Tanaka and S. F. Edwards, J. Phys. F **10**, 2769 (1980).

[12]A. J. Bray and M. A. Moore, J. Phys. C **14**, 1313 (1981).

[13]D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975); S. Kirkpatrick and D. Sherrington, Phys. Rev. B **17**, 4384 (1978).

[14]B. I. Shraiman, Phys. Rev. Lett. **57**, 325 (1986).

[15]H. Gutfreund, J. Reger, and P. Young, J. Phys. A (to be published).

[16]This method of calculation was suggested to us by D. J. Gross.