Alternate orderings: A new tool for the study of phase and photon statistics

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A new way to expand an arbitrary boson-field operator into ordered products of creation and annihilation operators is introduced: alternate ordering. A continuous order parameter s characterizes the order and takes the value $s=+1$ for normal-alternate order, $s=0$ for symmetric-alternate order, and $s=-1$ for antinormal-alternate order. All of the alternate moments of a number of well-known states of light are calculated. The expansion of density matrices and operators in terms of alternate products is briefly discussed as well as the relation with the concepts of phase and higher-order phase dependence. Finally, the relevance of the new ordering scheme in the study of photon statistics is outlined and illustrated in the derivation of moment equations for phase-sensitive light amplifiers and absorbers.

I. INTRODUCTION

The expansion of boson-field operators in series of products of creation and annihilation operators, $\hat{a}^{+}$ and $\hat{a}$, is a technique well known in quantum optics and physics. Because of the noncommutativity associated with $\hat{a}^{+}$ and $\hat{a}$, ordering prescriptions must be used to lift the ambiguity of order. Usual operator ordering prescriptions include normal ordering, symmetric ordering, and antinormal ordering.[1]

Ordering prescriptions concerning position and momentum operators $\hat{q}$ and $\hat{p}$ have appeared as early as 1927 in the context of the quantum classical correspondence for functions of operators,[2] and were later related by Moyal[3] with the operator ordering associated with the Wigner function. More exotic prescriptions have also been formulated,[4-7] but they are not widely used. With the advent of the laser, ordering prescriptions in terms of $\hat{a}^{+}$ and $\hat{a}$ were developed because they were especially well suited for the description of the harmonic oscillator and therefore, light fields.[8-11] Nowadays, these techniques are used widely, spreading into many branches of physics and mathematics.[12]

In this paper my aim is to introduce a new ordering prescription for $\hat{a}^{+}$ and $\hat{a}$ which is well adapted to the study of photon statistics and phase-dependent effects. The main reason for this is that it is often tedious to reorder the conventionally ordered products into powers of the photon-number operator.[13-15] In fact, it is not exaggerated to say that usual ordering prescriptions are not at all adapted to study the higher moments of the photon-number distribution (say, larger than two). In connection with the current interest in phase[16-18] and in phase-sensitive devices,[19,15] as well as the interest raised by the possibility of using,[20] generating,[21,22] and detecting[23] amplitude (photon-number) squeezed states, one may anticipate that the new ordering scheme proposed here will prove useful. With each operator ordering prescription one can generally associate quasiclassical distribution functions which are powerful and extensively used tools in the study of quantum-mechanical problems.[24] The problem of constructing the associated quasiclassical distribution functions will be dealt with in a future publication.

In Sec. II we will introduce the generating function for s-alternate ordered products, where s is a continuous order parameter. Various forms of this generating function are given. In Sec. III the explicit form of these alternate products is derived for particular values of s corresponding to normal-, symmetric-, and antinormal-alternate order. In Sec. IV the relationship with usual ordering prescriptions is briefly studied. We will define in Sec. V the alternate characteristic function of the density matrix and discuss briefly why the expansion of the density matrix in s-alternate products is unique (although not necessarily convergent). In Sec. VI we evaluate all the alternate moments for a number of well-known states of light and discuss in Sec. VII their relation to phase and phase operators. The concept of high-order phase dependence (or sensitivity) is introduced. Finally, to demonstrate the power of the method of s-alternate moments in the study of photon statistics in phase-sensitive devices, we recall in Sec. VIII the master equation for phase-sensitive reservoirs, and explain how to obtain equations of motion for the s-alternate characteristic function and s-alternate moments. The discussion of the solution of the equations for the low-order phase-sensitive moments brings new light on the behavior of phase-sensitive light amplifiers and absorbers.

Appendix A contains a number of useful or lengthy formulas and Appendices B–D contain a number of lengthy proofs for the benefit of the particularly interested reader.

II. THE GENERATING FUNCTION OF s-ALTERNATE PRODUCTS

We consider the creation and annihilation operators $\hat{a}^{+}$ and $\hat{a}$ of a dynamical system described by a pair of canonically conjugate Hermitian observables $\hat{q}$ and $\hat{p}$ possessing a continuous spectrum over the full range of real numbers and satisfying the commutation relation $[\hat{q},\hat{p}]=i\hbar$. We can then define $\hat{a}^{+}$ and $\hat{a}$ by

$$\hat{a}^{+}=\left[\frac{1}{2\hbar}\right]^{1/2}\left[\lambda\hat{q}-\frac{i}{\lambda}\hat{p}\right],\qquad \hat{a}=\left[\frac{1}{2\hbar}\right]^{1/2}\left[\lambda\hat{q}+\frac{i}{\lambda}\hat{p}\right]. \tag{1}$$

$\hat{a}^{+}$ and $\hat{a}$ satisfy the commutation relations

$$[\hat{a},\hat{a}^{+}]=1 . \tag{2}$$

These familiar relations obviously hold for many systems including the harmonic oscillator and boson fields. We will use the terminology relevant for the light fields of quantum optics. It is well known[10] that the algebraic properties of the operators (1) ensure that the number operator

$$\hat{N}=\hat{a}^{+}\hat{a} \tag{3}$$

has an integer spectrum associated with a complete set of eigenstates

$$|n\rangle=\frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}|0\rangle \tag{4}$$

constructed from the ground state $|0\rangle$. This is the unique vector in the kernel of $\hat{a}$:

$$\hat{a}|0\rangle=0 . \tag{5}$$

The annihilation operator $\hat{a}$ has an overcomplete set of nonorthogonal eigenstates:

$$\hat{a}|\alpha\rangle=\alpha|\alpha\rangle,\quad \alpha\in\mathcal{C} . \tag{6}$$

This state can also be generated from the vacuum:

$$|\alpha\rangle=\hat{D}(\alpha)|0\rangle , \tag{7}$$

with the displacement operator

$$\hat{D}(\alpha)=\exp(\alpha\hat{a}^{+}-\alpha^{*}\hat{a}) . \tag{8}$$

The well-known characteristic function for the moments of the photon-number operator $\hat{N}$ is[1,25]

$$G(z,t)=\mathrm{Tr}\{\rho e^{z\hat{N}}\} . \tag{9}$$

Such a characteristic function is very useful in problems where the evolution of the field density matrix are coupled along lines parallel to the diagonal in the $|n\rangle$ representation, but it does not allow study of a fully arbitrary quantum optical problem (an example is the phase-sensitive reservoir treated in Sec. VIII). In fact, we wish to find here the generalization of (9), such that, in particular, the moments of the photon-number operator can be obtained as a byproduct. To this end we propose the following generating function:

$$\hat{A}_s(\xi,\eta)=\exp\left[i\xi\hat{a}^{+}\hat{a}+i\eta\frac{\xi}{2}\frac{e^{-is\xi/2}}{\sin(\xi/2)}\hat{a}\right] . \tag{10}$$

In this expression s is a complex parameter which will be later identified with the ordering parameter. To understand better the form of $\hat{A}_s(\xi,\eta)$, let us focus for a while on the case $s=1$. Using the relations (A1)–(A3) of Appendix A, it is possible to show that $\hat{A}_1(\xi,\eta)$ can be disentangled

$$\hat{A}_1(\xi,\eta)=e^{i\xi\hat{a}\hat{a}^{+}}e^{i\eta\hat{a}} . \tag{11}$$

Therefore, multiple differentiations of this generating function will construct products of $\hat{a}^{+}$ and $\hat{a}$ containing more (or as many) annihilation operators $\hat{a}$ than creation operators $\hat{a}^{+}$ in the following way:

$$\left.\frac{\partial^{j+k}}{\partial(i\xi)^{j}\partial(i\eta)^{k}}\hat{A}_1(\xi,\eta)\right|_{\xi=\eta=0}=(\hat{a}^{+}\hat{a})^{j}\hat{a}^{k} . \tag{12}$$

We call this ordered product a normally alternate ordered product, alternate because there is always an operator $\hat{a}$ between $\hat{a}^{+}$ operators, normal-alternate because the latter are placed, as much as possible, on the left in analogy with standard normally ordered products.[1] Our definition (10) looks asymmetric in the creation and annihilation operators. However, we will see later that this asymmetry corresponds simply to the choice of a direction for positive anglelike variables. Obviously, it is easy to form normal-alternate products containing more $\hat{a}^{+}$ than $\hat{a}$ with the adjoint generating functions

$$\hat{A}_s^{+}(\xi,\eta)=[\hat{A}_s(\xi,\eta)]^{\dagger} . \tag{13}$$

Let us now return to the definition (10) of the generating function. It can be cast in various forms using the formulas (A1)–(A3) given in Appendix A:

$$\hat{A}_s(\xi,\eta)=\exp\left[i\eta\frac{e^{-is\xi/2}}{2\cos(\xi/2)}\hat{a}\right]\exp(i\xi\hat{a}^{+}\hat{a})\times\exp\left[i\eta\frac{e^{-is\xi/2}}{2\cos(\xi/2)}\hat{a}\right] \tag{14}$$

$$=\exp(i\xi\hat{a}^{+}\hat{a})\exp(i\eta e^{-i(s-1)\xi/2}\hat{a}) \tag{15}$$

$$=\exp(i\eta e^{-i(s+1)\xi/2}\hat{a})\exp(i\xi\hat{a}^{+}\hat{a}) . \tag{16}$$

Every form has been found useful. Equations (14)–(16) can be summarized with the following general disentangled form:

$$\hat{A}_s(\xi,\eta)=e^{iz_L\hat{a}}e^{i\xi\hat{a}^{+}\hat{a}}e^{iz_R\hat{a}} , \tag{17}$$

where z_L and z_R are tight by the constraint

$$e^{-i\xi/2}z_L+e^{i\xi/2}z_R=\eta e^{-is\xi/2} . \tag{18}$$

Let us define now the *s-alternate ordered product* of $\hat{a}^{+j}$ and $\hat{a}^{j+k}$ using the analogy with Eq. (12)

$$\{(\hat{a}^{+}\hat{a})^{j}\hat{a}^{k}\}_{sA}=\left.\frac{\partial^{j+k}}{\partial(i\xi)^{j}\partial(i\eta)^{k}}\hat{A}_s(\xi,\eta)\right|_{\xi=\eta=0} . \tag{19}$$

This abstract definition does not show the simple form which alternate products may take for special values of the ordering parameter s. This question is studied in detail below.

III. EXPLICIT EXPRESSIONS FOR s-ALTERNATE ORDERED PRODUCTS

In the case $s=1$, we have already given the rather simple explicit expression of the alternate products, Eq. (12). For the equally simple case $s=-1$ we find [by differentiating the form (16) of the generating function]

$$\{(\hat{a}^+\hat{a})^j\hat{a}^k\}_{-1A}=\hat{a}^k(\hat{a}^+\hat{a})^j . \tag{20}$$

The alternate order corresponding to $s=0$ will be called the *symmetric-alternate* order. A simple explicit form has been also found but can be established only at the end of this section.

Let us first express an s-alternate ordered product as a sum of s'-alternate ordered products: with the help of the definition (10) and (19) we can write

$$\{(\hat{a}^+\hat{a})^j\hat{a}^k\}_{sA}=\frac{\partial^{j+k}}{\partial(i\xi)^j\partial(i\eta)^k}\hat{A}_{s'}(\xi,\eta e^{-i(s-s')\xi/2})\Big|_{\xi=\eta=0}$$
$$=\left[\frac{\partial}{\partial(i\xi)}\right]^j\left[e^{-ik(s-s')\xi/2}\left[\frac{\partial}{\partial(i\eta)}\right]^k\hat{A}_{s'}(\xi,n')\right]\Bigg|_{\xi=\eta'=0}, \tag{21}$$

where $\eta'=\eta\exp[-i(s-s')\xi/2]$. Using Leibniz rule (A4), we find the following general decomposition of an s-alternate ordered product in a sum of s'-ordered products:

$$\{(\hat{a}^+\hat{a})^j\hat{a}^k\}_{sA}=\sum_{m=0}^{j}\begin{bmatrix} j\\ m\end{bmatrix}\left[\frac{(s'-s)k}{2}\right]^{j-m}\times\{(\hat{a}^+\hat{a})^m\hat{a}^k\}_{s'A} . \tag{22}$$

Writing down this expression for $s'=\pm1$, two closed forms for the s-alternate ordered product emerge:

$$\{(\hat{a}^+\hat{a})^j\hat{a}^k\}_{sA}=\left[\hat{a}^+\hat{a}+\left[\frac{1-s}{2}\right]k\right]^j\hat{a}^k \tag{23}$$
$$=\hat{a}^k\left[\hat{a}^+\hat{a}-\left[\frac{1+s}{2}\right]k\right]^j . \tag{24}$$

We recognize on the right-hand side the normal- and antinormal-alternate ordered products, respectively. The inverse relationships can also be easily obtained with the general expression (22).

Having obtained the explicit expressions (23) and (24) for the s-alternate ordered products, we seek now simple explicit expressions for the *symmetric-alternate* order ($s=0$). This can be achieved by recurrence. Let us calculate

$$\hat{a}\{(\hat{a}^+\hat{a})^j\hat{a}^k\}_{0A}\hat{a}=\hat{a}\left[\hat{a}^+\hat{a}+\frac{k}{2}\right]^j\hat{a}^{k+1} . \tag{25}$$

To simplify the right-hand side we use the formula

$$[\hat{a},(\hat{a}^+\hat{a})^j]=(\hat{a}^+\hat{a}+1)^j\hat{a}-(\hat{a}^+\hat{a})^j\hat{a}$$
$$=\sum_{m=0}^{j-1}\begin{bmatrix} j\\ m\end{bmatrix}(\hat{a}^+\hat{a})^m\hat{a} , \tag{26}$$

which is easily proved with the help of the basic commutator (2), and find

$$\hat{a}\{(\hat{a}^+\hat{a})^j\hat{a}^k\}_{0A}\hat{a}=(\hat{a}^+\hat{a}+\tfrac{1}{2}k+1)^j\hat{a}^{k+2}$$
$$=\{(\hat{a}^+\hat{a})^j\hat{a}^{k+2}\}_{0A} . \tag{27}$$

This recurrence relation allows us to prove easily by induction that

$$\{(\hat{a}^+\hat{a})^j\hat{a}^{2k}\}_{0A}=\hat{a}^k(\hat{a}^+\hat{a})^j\hat{a}^k \tag{28}$$

$$\{(\hat{a}^+\hat{a})^j\hat{a}^{2k+1}\}_{0A}=\hat{a}^k(\hat{a}^+\hat{a}+\tfrac{1}{2})^j\hat{a}^{k+1}$$
$$=\hat{a}^{k+1}(\hat{a}^+\hat{a}-\tfrac{1}{2})^j\hat{a}^k . \tag{29}$$

The expression (28) for the symmetrical products with even k is particularly simple. In the case of odd k, the alternate products do not take a so much appealing form, except when $j=1$:

$$\{\hat{a}^+\hat{a}^{2k+1}\}_{0A}=\hat{a}^{k+1}\hat{a}^+\hat{a}^{k+1} . \tag{30}$$

IV. RELATION BETWEEN s-ALTERNATE ORDERING SCHEMES AND STANDARD ORDERING SCHEMES

The generalization of standard ordering schemes with a continuous order parameter was introduced in 1969 by Cahill and Glauber.[10] They show that s ordering reduces to standard normal ordering when $s=1$, standard symmetric ordering when $s=0$, and standard antinormal ordering when $s=-1$. These ordered products, called here simply "s-ordered products," are generated by

$$\{(\hat{a}^+)^n\hat{a}^m\}_s=\frac{\partial^{n+m}}{\partial(-\alpha)^n\partial(\alpha^*)^m}\hat{D}_s(\alpha)\Big|_{\alpha=\alpha^*=0} , \tag{31}$$

where the generalized displacement operator is

$$\hat{D}_s(\alpha)=e^{s|\alpha|^2/2}\hat{D}(\alpha) . \tag{32}$$

In Appendix B we show that the generating function $\hat{A}_s(\xi,\eta)$ can be expressed as the following sum of s'-ordered products:

$$\hat{A}_s(\xi,\eta)=\left[\frac{1-s'}{2}(e^{i\xi}-1)+1\right]^{-1}\left\{\exp\left[\frac{(e^{i\xi}-1)\hat{a}^{+}\hat{a}+i\eta e^{-i(1-s)\xi/2}\hat{a}}{\frac{1-s'}{2}(e^{i\xi}-1)+1}\right]\right\}_{s'} . \tag{33}$$

Through multiple differentiations of this formula we may express an s-alternate product as a sum of s'-ordered products. Let us obtain, for example, an expression for the normally ordered expansion of an s-alternate product:

$$\{(\hat{a}^{+}\hat{a})^j\hat{a}^{k}\}_{sA}=\frac{\partial^{j+k}}{\partial(i\xi)^j\partial(i\eta)^k}\{\exp[(e^{i\xi}-1)\hat{a}^{+}\hat{a}+i\eta e^{-i(1-s)\xi/2}\hat{a}]\}_1\Big|_{\xi=\eta=0} . \tag{34}$$

Expanding the exponential and its argument, and remembering that $(e^{i\xi}-1)^m$ is the generating function of the Stirling numbers of the second kind $\mathcal{S}_n^{(m)}$ (Ref. 25) we find, eventually,

$$\{(\hat{a}^{+}\hat{a})^j\hat{a}^{k}\}_{sA}=\sum_{m=0}^{j}\sum_{n=0}^{m}\begin{bmatrix} j\\ m\end{bmatrix}\left[\left[\frac{1-s}{2}\right]k\right]^{j-m}\mathcal{S}_m^{(n)}\hat{a}^{+m}\hat{a}^{m+k} . \tag{35}$$

This formula can be obtained more directly from Eq. (23) with the help of the well-known expansion of $(\hat{a}^{+}\hat{a})^m$ in normally ordered products.[13,14]

V. THE DENSITY MATRIX AND THE s-ALTERNATE CHARACTERISTIC FUNCTION

Following the usual practice[1] the s-alternate characteristic function of any density matrix ρ can be defined as

$$\chi_{sA}(\xi,\eta)=\mathrm{Tr}[\rho\hat{A}_s(\xi,\eta)] \tag{36}$$

and forms a generalization of Eq. (9). Since ρ is Hermitian and since it is generally possible to expand the density matrix as an ordered power series in $\hat{a}^{+}$ and $\hat{a}$, we can write

$$\rho=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}r^{jk}(s)\{(\hat{a}^{+}\hat{a})^j\hat{a}^{k}\}_{sA}+[r^{jk}(s)]^*[\{(\hat{a}^{+}\hat{a})^j\hat{a}^{k}\}_{sA}]^\dagger , \tag{37}$$

where we have chosen of course the alternate ordering prescription. In (37) the coefficients $r^{jk}(s)$ are uniquely defined. Although the definition (36) is not symmetrical in $\hat{a}^{+}$ and $\hat{a}$, it is easy to see that the characteristic function depends on and only on all the coefficients $r^{jk}(s)$ and therefore constitutes a unique characterization of ρ. The convergence of the ordered expansion (37) will not be discussed here in general. In Sec. VII we will see that the Susskind and Glogower phase operator does not possess a convergent expansion. Such problems are difficult because many useful operators do not have a convergent expansion in a standardly ordered power series already. One must resort to (and be content with) weak convergence over a limited set of states like coherent states, for example.[10]

We note that the expansion (37) has a form similar to a Fourier expansion or a multipolar expansion in cylindrical coordinates. Indeed, for the sake of completeness it is necessary to mention that the combination of a discrete Fourier transform (with respect to ξ) and an integral transform (with respect to η) of Eq. (36) can lead to useful "quasiclassical distribution functions," noted $P(n,\alpha)$, which generalize the usual photon-number distribution $P(n)=\langle n\,|\,\rho\,|\,n\rangle$. Such distributions are quasiclassical in the sense that n and $\alpha=e^{i\theta}$ can play the role of action-angle-like variables. However this extension is beyond the scope of the present paper and will be reported elsewhere.

Often it is possible to find simple equations of motion for the characteristic function (36) directly. Then, one can straightforwardly obtain the equations of motion for the s-alternate moments

$$M_{sA}^{j,k}=\langle\{(\hat{a}^{+}\hat{a})^j\hat{a}^{k}\}_{sA}\rangle , \tag{38}$$

with the help of the relation

$$M_{sA}^{j,k}=\frac{\partial^{j+k}}{\partial(i\xi)^j\partial(i\eta)^k}\chi_{sA}(\xi,\eta)\Big|_{\xi=\eta=0} . \tag{39}$$

This program is carried out in Sec. VIII on the instructive example of the phase-sensitive reservoir.

All "diagonal" moments $M_{sA}^{j,0}$ contain information on the photon statistics of the state ρ since

$$M_{sA}^{j,0}=\langle N^j\rangle=\sum_{n=0}^{\infty}n^jP(n) , \tag{40}$$

where $P(n)$ represents as above the probability of finding n photons in the field. The new "off-diagonal" moments, $M_{sA}^{j,k}(k>0)$, contain information on the phase of the field state ρ. In order to gain some intuition on the role of these moments, we calculate them for a selected set of states which are frequently met in quantum optics.

VI. s-ALTERNATE MOMENTS OF SOME WELL-KNOWN LIGHT STATES

The first states encountered in our study were the pure Fock states $|\,n\,\rangle$. The evaluation of Eq. (38) is straightforward in this case and we find

$$M_{sA}^{j,k}\,|_{\text{number}}=\delta_{k,0}n^j . \tag{41}$$

Of course the "off-diagonal" moments vanish since the state is *totally* phase insensitive.

The second kind of state we encountered was the coherent state $|\,\alpha\,\rangle$. The evaluation of its alternate moments can be carried out directly with the help of the normally ordered expansion (35):

$$M_{sA}^{j,k}\mid_{\rm coh}=\alpha^k\sum_{m=0}^{j}\sum_{n=0}^{m}\begin{bmatrix}j\\m\end{bmatrix}\left[\left[\frac{1-s}{2}\right]k\right]^{j-m}\mathcal{S}_m^{(n)}\,|\alpha|^{2m}\ . \tag{42}$$

As expected, the "off-diagonal" moments do not vanish since the coherent state is a phase-dependent state. In fact much more precise statements, illuminating the physical meaning carried out by some of the low-order alternate moments, can be issued at this stage: since the mean

$$M_{sA}^{0,1}\mid_{\rm coh}=\langle\hat a\rangle=\alpha\neq 0\ , \tag{43}$$

we can infer that the state has a coherent (phase-dependent) amplitude. Moreover, introducing the complex parameter

$$\mathcal{P}=M_{sA}^{0,2}-(M_{sA}^{0,1})^2=\langle\hat a^2\rangle-\langle\hat a\rangle^2\ , \tag{44}$$

which bears some analogy with the Q parameter of Mandel,[26] a straightforward calculation shows that when $\mathcal{P}$ vanishes, the state possesses a quadrature-phase-insensitive (or independent) *noise* ($\Delta\hat p=\Delta\hat q$). Substituting Eq. (42) we can infer that the coherent state has indeed a quadrature-phase insensitive noise ($\mathcal{P}_{\rm coh}=0$). Finally the highly excited limit behavior of the alternate moments of the coherent state can be checked

$$M_{sA}^{j,k}\mid_{\rm coh}\longrightarrow|\alpha|^{2j}\alpha^k\ \text{ as }\ |\alpha|^2\to\infty\ . \tag{45}$$

As it should for a classical limit, this expression is independent of the ordering parameter s. We note the similarity of this expression with the limit expectation value of the generalized phase operators defined by Paul.[27]

A thermal state is represented by the density matrix

$$\rho_{\rm th}=(1-e^{\beta\omega})\exp(-\beta\omega\hat a^{+}\hat a)\ , \tag{46}$$

where $\beta=1/k_BT$ is the temperature and ω the photon frequency. In the $|n\rangle$ representation, the characteristic function is easy to evaluate:

$$\chi_{sA}^{\rm th}(\xi,\eta)=(1-e^{-\beta\omega})\sum_{n=0}^{\infty}e^{(i\xi-\beta\omega)n}\ , \tag{47}$$

leading to

$$M_{sA}^{j,k}\mid_{\rm th}=\delta_{k,0}(1-e^{-\beta\omega})\sum_{m=0}^{\infty}m^je^{-\beta\omega m}\ . \tag{48}$$

Using the relation

$$\sum_{m=0}^{\infty}m^je^{-\beta\omega m}=\left[\frac{d}{d(-\beta\omega)}\right]^j\left[\frac{1}{1-e^{-\beta\omega}}\right] \tag{49}$$

and the formula (A10) of Appendix A we find an elegant closed form formula for the moments of the thermal state

$$M_{sA}^{j,k}\mid_{\rm th}=\delta_{k,0}\sum_{m=0}^{j}\mathcal{S}_j^{(m)}m!(e^{\beta\omega}-1)^{-m}\ . \tag{50}$$

The "off-diagonal" alternate moments vanish as expected.

Finally we wish to consider a squeezed vacuum state

$$|\zeta;0\rangle=S(\zeta)|0\rangle\ , \tag{51}$$

where $\zeta=re^{2i\phi}$ and

$$S(\zeta)=\exp[\tfrac{1}{2}(\zeta^*\hat a^2-\zeta\hat a^{+2})]\ . \tag{52}$$

The evaluation of the characteristic function

$$\begin{aligned}\chi_{sA}^{\rm sq}(\xi,\eta)&=\langle\zeta;0|\hat A_s(\xi,\eta)|\zeta;0\rangle\\&=\langle 0|S^\dagger(\zeta)\hat A_s(\xi,\eta)S(\zeta)|0\rangle\end{aligned} \tag{53}$$

(sq denotes squeezing) is very tedious and carried out in Appendix C, together with the evaluation of the s-alternate moments which read

$$M_{sA}^{j,2k+1}\mid_{\rm sq}=0\ , \tag{54}$$

$$\begin{aligned}M_{sA}^{j,2k}\mid_{\rm sq}=&\frac{(2k)!}{k!}2^{j-k}e^{2ik\phi}\sinh^kr\cosh^kr\\&\times\sum_{l=0}^{j}\begin{bmatrix}j\\l\end{bmatrix}\left[\left[\frac{1-s}{2}\right]k\right]^{j-l}\\&\times\sum_{m=0}^{l}\frac{(k+m-\frac{1}{2})!}{(k-\frac{1}{2})!}\mathcal{S}_l^{(m)}\sinh^{2m}r\ .\end{aligned} \tag{55}$$

Apart from an overall phase factor these moments do not depend on ϕ. For the first nonvanishing symmetrically ordered moments the formula (55) reads explicitely

$$M_{sA}^{0,2}\mid_{\rm sq}=e^{2i\phi}\sinh r\cosh r\ , \tag{56}$$

$$M_{sA}^{0,4}\mid_{\rm sq}=3e^{4i\phi}\sinh^2r\cosh^2r\ , \tag{57}$$

$$M_{sA}^{1,0}\mid_{\rm sq}=\sinh^2r\ , \tag{58}$$

$$M_{sA}^{1,2}\mid_{\rm sq}=e^{2i\phi}\sinh r\cosh r(1+3\sinh^2r)\ , \tag{59}$$

$$M_{sA}^{1,4}\mid_{\rm sq}=e^{4i\phi}\sinh^2r\cosh^2r(6+15\sinh^2r)\ , \tag{60}$$

$$M_{sA}^{2,0}\mid_{\rm sq}=2\sinh^2r+3\sinh^4r\ . \tag{61}$$

The squeezed vacuum ($r\neq 0$) is a state which has no coherent (phase-sensitive) amplitude

$$M_{sA}^{0,1}\mid_{\rm sq}=0\ , \tag{62}$$

but possesses, on the other hand, quadrature-phase-sensitive noise since

$$\mathcal{P}_{\rm sq}=e^{2i\phi}\sinh r\cosh r\neq 0\ . \tag{63}$$

Equation (55) can also be used to evaluate the highly squeezed limit ($|\sinh r|\gg 1$) of the moments

$$\begin{aligned}M_{sA}^{j,2k}\mid_{\rm sq}\longrightarrow&\frac{(2k)!}{k!}2^{j-k}e^{2ik\phi}\sinh^kr\cosh^kr\\&\times\frac{(k+j-\frac{1}{2})!}{(k-\frac{1}{2})!}\sinh^{2j}r\ \text{ as }r\to\pm\infty\ .\end{aligned} \tag{64}$$

Quite remarkably this limit does not depend on the ordering parameter s. We can also evaluate the weakly squeezed limit ($|\sinh r|\ll 1$) where we retain this time only the lowest order:

$$M_{sA}^{j,0}|_{\rm wsq}=2^{j-1}\sinh^2 r \quad \text{if } j>0 \ ,$$
$$M_{sA}^{j,2k}|_{\rm wsq}=\frac{(2k)!}{k!}2^{-k}e^{2ik\phi}(1-s)^j \times k^j \sinh^k r \cosh^k r \quad \text{if } k>0 \ . \tag{65}$$

The diagonal moments and the off-diagonal moments with $k=1$ dominate.

In this section we have seen that the off-diagonal alternate moments carried phaselike information of paramount importance for the state characterization and classification. In order to understand better the relationship with the concept of phase we will study in Sec. VII the relationship with the phase operators defined in the literature.

VII. RELATIONSHIP BETWEEN ALTERNATE ORDERED PRODUCTS, PHASE OPERATORS, AND HIGH-ORDER PHASE SENSITIVITY

The first definition of a phase operator is due to Dirac who constructed one from the consideration of commutation relation in his 1927 paper.[28,29] Although Dirac was aware that his operators were, to a certain extent, ill defined, it is only much later that one has widely realized the seriousness of the difficulties involved in the definition of a proper phase operator.[30] Basically, the quantum-classical (QC) correspondence imposes commutation relations for the Hermitian sine and cosine phase operators which in turn are incompatible with the desirable property that these would be functions of a *common* phase operator. Therefore one usually works with exponential (or Hermitian sine and cosine) phase operators. Lerner[31] has shown, however, that one could not even define, on the basis of the quantum-classical correspondence, a *unique* exponential phase operator. In fact, suitable sine and cosine phase operators must obey two requirements. The first one (QC correspondence)

$$(e^{\pm\phi},\hat{N})=\pm e^{\pm i\phi} \tag{66}$$

is called the Lerner criterion. The second one, which can be cast in following form:

$$\text{Spec}\begin{Bmatrix}\widehat{\sin\phi}\\ \widehat{\cos\phi}\end{Bmatrix}=[-1,1]\subset\mathbb{R} \ , \tag{67}$$

stems from the necessity to be able to interpret the result of a measurement ($\langle\widehat{\sin\phi}\rangle$ or $\langle\widehat{\cos\phi}\rangle$) as the phase of the field. Ifantis, in a series of papers,[32] has most rigorously and beautifully studied the consequences of these two requirements. On a different basis, other authors have proposed various phase operators.[27,16] The most simple and widely used exponential phase operator is due to Susskind and Glogower[33]

$$\widehat{e_S^{i\phi}}=(\hat{N}+1)^{-1/2}\hat{a} \ . \tag{68}$$

At this stage we may note that the cosine (sine) phase operator

$$\widehat{\cos\phi}=\tfrac{1}{2}[(\hat{N}+1)^{-1/2}\hat{a}+\hat{a}^+(\hat{N}+1)^{-1/2}] \tag{69}$$

naturally takes the form of a normally alternate ordered expansion if the operator $(\hat{N}+1)^{-1/2}$ can be expanded in a Taylor series. Unfortunately, the radius of convergence of the Taylor series of $(1+x)^{-1/2}$ is equal to unity and $\hat{N}$ is an unbounded operator. Thus we are led to conclude that the Susskind and Glogower cosine (sine) phase operator do not possess any convergent alternate ordered series expansion in $\hat{a}^+$ and $\hat{a}$.

The question of most crucial interest about the physical definition of a quantum phase operator is: "How does one measure the phase of a microscopic quantum field?".[34,27,35] Which operator corresponds exactly to the measurement procedure? Which operators can be measured? Barnett and Pegg have argued recently[18] that, in fact, the most natural exponential phase operator corresponding to usual experimental procedures was the unbounded operator $\hat{a}$ itself (suitably normalized by a c number). In this light it is interesting to note that our alternate products look like *natural extensions* of this unbounded exponential phase operator. Moreover, Paul[27] has argued that his generalized phase operators A_{lk}, which closely resemble ours (modulo a different ordering convention if $l\geq k$), are accessible experimentally in the case $l\geq k$ through a measurement of high-order intensity correlations in interference experiments. Certainly, this conclusion applies also directly to our alternate products which can thus be viewed as *a hierarchy of measurable generalized phase operators* connected with higher and higher-order phase- and amplitude-sensitive properties of the field. With respect to Paul's operators, they present the obvious advantage of a clear separation of the two parts and a consistent ordering prescription.

In the large-n limit, many approximation procedures have been found where the phase becomes like a classical variable.[36-38] In particular, photon-number states can be considered as classical statistical states with a given amplitude and evenly distributed phase.[36] Therefore it is not surprising that the Susskind and Glogower phase operators have proven especially useful to carry out $1/n$ expansions in that limit, although care must be taken when one approximates the unbounded exponential phase operator $\hat{a}$ by the bounded Susskind and Glogower exponential phase operator.[39] In the low-n limit the Susskind and Glogower exponential operator seems to suffer from a lack of good physical interpretation. It becomes obvious if we consider, for example, vacuum expectation values. A comparison shows that the unbounded exponential phase operator retains a sensible behavior.[18]

In Sec. VIII we demonstrate that the alternate ordering technique permits us to treat systematically a practical quantum optical problem and show that, indeed, the result can be interpreted in simple physical terms at all field intensities.

VIII. PHOTON STATISTICS IN THE PHASE-SENSITIVE RESERVOIR

Following the scheme sketched in Sec. V we will start from the master equation for the density matrix, derive an equation of motion for the characteristic function, and then, through multiple differentiations, find the evolution equations for the alternate moments.

A. Equation of motion for the characteristic function

The master equation for the phase-sensitive reservoir has been obtained in Ref. 40 from the rigged reservoir response:

$$\frac{d\rho}{dt}=-\lambda^2\{C_{+-}[\hat a,[\hat a^+,\rho]]+C_{-+}[\hat a^+,[\hat a,\rho]]+C_{++}[\hat a,[\hat a,\rho]]+C_{--}[\hat a^+,[\hat a^+,\rho]]\}-\frac{i\lambda^2}{2}\{\chi_{+-}[\hat a,[\hat a^+,\rho]_+]+\chi_{-+}[\hat a^+,[\hat a,\rho]_+]\}\ , \tag{70}$$

where $C_{\pm\pm}$ and $\chi_{\pm\pm}$ are the symmetric correlation function and the linear susceptibility, respectively (λ is a coupling constant). As discussed in Refs. 40-42, we have taken into account the fact that for most cases $\chi_{++}=\chi_{--}=0$ in Eq. (70).

To obtain an equation of motion for the characteristic function $\chi_{sA}(\xi,\eta)$ we must multiply Eq. (70) by $\hat A_s(\xi,\eta)$ on the right-hand side and take the trace. For clarity, the procedure is carried out term by term. We shall treat here in detail the term $\hat a\hat a^+\rho$ to show how the procedure works. The form (15) for the generating function will be found most conveniently by calculating [again with the help of Eqs. (A1)–(A3)]

$$\begin{aligned}\mathrm{Tr}[\hat a\hat a^+\rho\hat A_s(\xi,\eta)]&=\mathrm{Tr}\{\rho\exp(i\xi\hat a^+\hat a)(\hat a\hat a^++i\eta e^{-i(s-1)\xi/2}\hat a)\exp(i\eta e^{-i(s-1)\xi/2}\hat a)\}\\&=\frac{\partial}{\partial(i\xi)}\chi_{sA}(\xi,\eta)+\left[\frac{1+s}{2}\right]i\eta\frac{\partial}{\partial(i\eta)}\chi_{sA}(\xi,\eta)+\chi_{sA}(\xi,\eta)\ .\end{aligned} \tag{71}$$

In the same way, terms like $\hat a^+\rho\hat a$ and $\hat a^2\rho$ can be translated into

$$\mathrm{Tr}[\hat a^+\rho\hat a\hat A_s(\xi,\eta)]=e^{i\xi}\frac{\partial}{\partial(i\xi)}\chi_{sA}(\xi,\eta)+e^{i\xi}\left[\frac{1+s}{2}\right]i\eta\frac{\partial}{\partial(i\eta)}\chi_{sA}(\xi,\eta)+e^{i\xi}\chi_{sA}(\xi,\eta)\ , \tag{72}$$

$$\mathrm{Tr}[\hat a^2\rho\hat A_s(\xi,\eta)]=e^{i(s-1)\xi}\left[\frac{\partial}{\partial(i\eta)}\right]^2\chi_{sA}(\xi,\eta)\ . \tag{73}$$

On the other hand, terms like $\hat a^{+2}\rho$ could not be expressed as a function of the characteristic function $\chi_{sA}(\xi,\eta)$. However, we will see later that these terms do not create a major problem: for the time being let us only transform them as a function of the two quantities:

$$\chi_{sA}^{+}(\xi,\eta)=\mathrm{Tr}[\rho\exp(i\xi\hat a^+\hat a)\hat a^+\exp(i\eta e^{-i(s-1)\xi/2}\hat a)]\ , \tag{74}$$

$$\chi_{sA}^{2+}(\xi,\eta)=\mathrm{Tr}[\rho\exp(i\xi\hat a^+\hat a)\hat a^{+2}\exp(i\eta e^{-i(s-1)\xi/2}\hat a)]\ . \tag{75}$$

These operations allow us, finally, to write the full equation of motion for the characteristic function:

$$\begin{aligned}\frac{\partial\chi_{sA}(\xi,\eta)}{\partial t}=-\lambda^2\Bigg\{&(C_{+-}+C_{-+})\left[2[1-\cos(\xi)]\frac{\partial}{\partial(i\xi)}+\{s[1-\cos(\xi)]-i\sin(\xi)\}i\eta\frac{\partial}{\partial(i\eta)}+(1-e^{i\xi})\right]\chi_{sA}(\xi,\eta)\\&+C_{++}\left[e^{is\xi}2[\cos(\xi)-1]\left[\frac{\partial}{\partial(i\eta)}\right]^2\right]\chi_{sA}(\xi,\eta)\\&+C_{--}[(1-e^{-i\xi})^2\chi_{sA}^{2+}(\xi,\eta)+2i\eta e^{i(s-1)\xi/2}(1-e^{-i\xi})\chi_{sA}^{+}(\xi,\eta)+(i\eta)^2e^{i(1-s)\xi}\chi_{sA}(\xi,\eta)]\Bigg\}\\-\frac{i\lambda^2}{2}\Bigg[&\chi_{+-}\left[-2i\sin(\xi)\frac{\partial}{\partial(i\xi)}+[1-is\sin(\xi)-\cos(\xi)]i\eta\frac{\partial}{\partial(i\eta)}+(1-e^{i\xi})\right]\chi_{sA}(\xi,\eta)\\&+\chi_{-+}\left[2i\sin(\xi)\frac{\partial}{\partial(i\xi)}+[1+is\sin(\xi)+\cos(\xi)]i\eta\frac{\partial}{\partial(i\eta)}+(1-e^{i\xi})\right]\chi_{sA}(\xi,\eta)\Bigg]\ .\end{aligned} \tag{76}$$

In fact, it is not surprising that we cannot express the two terms $\chi_{sA}^{+}(\xi,\eta)$ and $\chi_{sA}^{2+}(\xi,\eta)$ as a function of $\chi_{sA}(\xi,\eta)$: they contain a few products which have more $\hat a^+$'s than $\hat a$'s and represent a coupling to the adjoint part of the density matrix. These terms are adjoints of usual alternate products and, as it will become soon apparent, they will not hinder us to obtain *closed sets* of evolution equations for the s-alternate moments.

B. Rules for obtaining moment equations

Through multiple differentiations of Eq. (76), we can obtain the evolution equation for the moments (39). We note that this operation is greatly facilitated if all circular functions in (76) are decomposed into exponentials. Then the

Leibniz rule reduces to

$$\left[\frac{\partial}{\partial(i\xi)}\right]^j e^{ia\xi}f(\xi)\Bigg|_{\xi=0} = \sum_{m=0}^{j}\begin{bmatrix} j\\ m\end{bmatrix} a^m \left[\frac{\partial}{\partial(i\xi)}\right]^{j-m} e^{ia\xi}f(\xi)\ . \tag{77}$$

In the same way we deduce also the following useful rules:

$$\left[\frac{\partial}{\partial(i\eta)}\right]^k i\eta\frac{\partial}{\partial(i\eta)}f(\eta)\Bigg|_{\eta=0} = k\left[\frac{\partial}{\partial(i\eta)}\right]^k f(\eta)\Bigg|_{\eta=0}\ , \tag{78}$$

$$\left[\frac{\partial}{\partial(i\eta)}\right]^k i\eta f(\eta)\Bigg|_{\eta=0} = k\left[\frac{\partial}{\partial(i\eta)}\right]^{k-1} f(\eta)\Bigg|_{\eta=0}\ , \tag{79}$$

$$\left[\frac{\partial}{\partial(i\eta)}\right]^k (i\eta)^2 f(\eta)\Bigg|_{\eta=0} = k(k-1)\left[\frac{\partial}{\partial(i\eta)}\right]^{k-2} f(\eta)\Bigg|_{\eta=0}\ . \tag{80}$$

Equations (77)–(80) allow to find easily the moment equations, except for the terms involving $\chi_{sA}^{+}(\xi,\eta)$ and $\chi_{sA}^{2+}(\xi,\eta)$. Let us show briefly how to manipulate them.

If $k \geq 2$,

$$\begin{aligned}\frac{\partial^{j+k}}{\partial(i\xi)^j\partial(i\eta)^k}\chi_{sA}^{2+}(\xi,\eta)\Bigg|_{\xi=\eta=0} &= \sum_{m=0}^{j}\begin{bmatrix} j\\ m\end{bmatrix}\mathrm{Tr}\left\{\rho(\hat a^+\hat a)^{j-m}(\hat a^+)^2\left[\left[\frac{1-s}{2}\right]^k\right]^m \hat a^k\right\}\\ &= \sum_{m=0}^{j}\begin{bmatrix} j\\ m\end{bmatrix}\left[\left[\frac{1-s}{2}\right]^k\right]^{j-m}\mathrm{Tr}\{\rho[(\hat a^+\hat a)^{m+2}-(\hat a^+\hat a)^{m+1}]\hat a^{k-2}\}\ .\end{aligned} \tag{81}$$

The last term can be considered as a sum of normal-alternate moments which can be put in the form of s-alternate moments with the help of Eq. (22). $\chi_{sA}^{+}(\xi,\eta)$ can be treated in the same way if $k \geq 1$.

If $0 \leq k \leq 1$,

$$\begin{aligned}\frac{\partial^{j+k}}{\partial(i\xi)^j\partial(i\eta)^k}\chi_{sA}^{2+}(\xi,\eta)\Bigg|_{\xi=\eta=0} &= \sum_{m=0}^{j}\begin{bmatrix} j\\ m\end{bmatrix}\mathrm{Tr}\left[\rho(\hat a^+\hat a)^{m}(\hat a^+)^2\left[\frac{1-s}{2}\right]^{j-m}\hat a\right]\\ &= \sum_{m=0}^{j}\begin{bmatrix} j\\ m\end{bmatrix}\left[\frac{1-s}{2}\right]^{j-m}\mathrm{Tr}\{\rho[(\hat a^+\hat a)^{m+1}-(\hat a^+\hat a)^{m}]\hat a^+\}\ .\end{aligned} \tag{82}$$

The last trace can be written in the form

$$\mathrm{Tr}\{\rho\hat a[(\hat a^+\hat a)^{m+1}-(\hat a^+\hat a)^m]\}^*\ . \tag{83}$$

We recognize in (83) antinormal-alternate moments which we can again express as a sum of s-alternate moments with the help of Eq. (22). The same kind of treatment applies to the case $k=0$ [for $\chi_{sA}^{+}(\xi,\eta)$ as well].

The procedure above allows us to treat the terms $\chi_{sA}^{+}(\xi,\eta)$ and $\chi_{sA}^{2+}(\xi,\eta)$ and to complete our translation of the evolution equation for the characteristic function (76) into an evolution equation for the s-alternate moments of the density matrix. The full equations are presented in Appendix D since they are quite involved. As far as the calculation length was concerned, the whole calculation was found highly competitive with respect to the Wigner function approach of Ref. 15. For the high-order moments we note that the new hierarchy of differential equations is more complicated than the Wigner hierarchy. This is the price paid for the great simplification to produce directly the coupled equations of motion for the quantities of interest: the moments of the photon number. It is impossible to obtain such equations with standard methods.[10] If we are interested only in the low-order moments we can restrict the procedure by directly averaging the equation of motion (70) with the corresponding low-order alternate products. This procedure is *very expedient* and straightforward.

Generally the moments of the photon number are also coupled to other quantities of interest, the "off-diagonal" moments, in the following way:

$M_{sA}^{j,k}$ is coupled to

$$\begin{cases} -M_{sA}^{j',k} & \text{if } j' \leq j \\ -M_{sA}^{j',k+2} & \text{if } j' \leq j-1 \\ -M_{sA}^{j',k-2} & \text{if } j' \leq j \text{ and } k \geq 2\ . \end{cases}$$

Such a coupling allows us to form closed sets of low-order moments whose exact evolution describes the evolution of the photon statistical distribution with a refinement increasing with the size of the set. The full results in Appendix D have proven useful to ensure the accuracy of our calculations in Ref. 15. The reader will also find there the discussion of the solution for the low-order diagonal moments. To end our investigations it is necessary to discuss and interpret the behavior of the low-order off-diagonal moments in the phase-sensitive light amplifier and absorber.

C. Evolution of the low-order phase sensitivity in phase-sensitive linear amplifiers and absorbers

The evolution equations for the alternate moments can be written with the help of Eqs. (D1)–(D7). For the low-order off-diagonal moments $M_{sA}^{0,1}$ and $M_{sA}^{0,2}$ they take the simple form

$$\frac{dM_{sA}^{0,1}}{dt}=-i\lambda^2\chi_{-+}M_{sA}^{0,1} , \tag{84}$$

$$\frac{dM_{sA}^{0,2}}{dt}=-2i\lambda^2\chi_{-+}M_{sA}^{0,2}-2\lambda^2C_{--} . \tag{85}$$

Equations (84) and (85) can be more straightforwardly obtained by directly averaging the master equation (70) with $\{\hat{a}\}_{sA}$ and $\{\hat{a}^2\}_{sA}$. We introduce now the notation of Ref. 15:

$$-i\lambda^2\chi_{-+}\equiv g , \tag{86}$$

$$2\lambda^2C_{--}\equiv\tfrac{1}{2}\sinh(2r)\coth(\beta\omega/2)e^{-2i\phi} . \tag{87}$$

Here g is the gain (absorption) coefficient, r the squeezing parameter, ϕ the squeezing angle, and β characterizes the rigged reservoir temperature (in the unsqueezed case). With these definitions the solution of Eqs. (84) and (85) is

$$M_{sA}^{0,1}(t)=M_{sA}^{0,1}(0)\sqrt{G} , \tag{88}$$

$$M_{sA}^{0,2}(t)=[M_{sA}^{0,2}(0)+\tfrac{1}{2}\sinh(2r)\coth(\beta\omega/2)e^{2i\phi}]G-\tfrac{1}{2}\sinh(2r)\coth(\beta\omega/2)e^{2i\phi} , \tag{89}$$

where $G=e^{2gt}$.

Let us discuss the behavior of the first moment (88): the output of the amplifier ($G>1$) retains forever an initial coherent amplitude $M_{sA}^{0,1}(0)$. On the other hand, asymptotically ($G\rightarrow 0$) the absorber destroys the coherent amplitude (it is, in fact, the definition of an absorber). In both cases we note that the device does not alter the argument of $M_{sA}^{0,1}(t)$, i.e., the *phase* of the signal. This conclusion is fully independent of the signal intensity and is a consequence of the fact that $\chi_{--}=\chi_{++}=0$ for most rigged reservoirs.[40]

In Ref. 15 the discussion of the diagonal moments was carried out as a function of the second moment $\mathcal{M}_0\equiv M_{sA}^{0,2}(0)$. However, the most relevant physical parameter depending on the second moment is the parameter $\mathcal{P}$. Equations (88) and (89) lead to

$$\mathcal{P}_{\text{out}}=M_{sA}^{0,2}(t)-[M_{sA}^{0,1}(t)]^2$$
$$=[\mathcal{P}_0+\tfrac{1}{2}\sinh(2r)\coth(\beta\omega/2)e^{2i\phi}]G-\tfrac{1}{2}\sinh(2r)\coth(\beta\omega/2)e^{2i\phi} , \tag{90}$$

which depend only on the value of the signal input parameter $\mathcal{P}_0$ and the reservoir parameters. The solution of the equation

$$\mathcal{P}_{\text{out}}=0 \tag{91}$$

gives us information on the critical value of the integrated gain (absorption) for which the output has a phase-insensitive quadrature phase noise:

(a) if $\text{Im}(\mathcal{P}_0e^{-2i\phi})\neq 0$, the device gives always an output containing phase-dependent quadrature phase noise.
(b) if $\text{Im}(\mathcal{P}_0e^{-2i\phi})=0$, the critical integrated gain (absorption) is

$$G_{\text{crit}}=\frac{\tfrac{1}{2}\sinh(2r)\coth(\beta\omega/2)}{P_0+\tfrac{1}{2}\sinh(2r)\coth(\beta\omega/2)} , \tag{92}$$

where $P_0=\mathcal{P}_0e^{-2i\phi}\in\mathbb{R}$. The conditions under which G_{crit} exists will be now discussed separately for the case of the amplifier and the absorber.

The amplifier case is characterized by $G>1,\beta>0$.[15] When (and only when) the parameters satisfy the inequality

$$r\lessgtr 0,\quad -\tfrac{1}{2}\sinh(2r)\coth(\beta\omega/2)\lessgtr P_0\lessgtr 0 \tag{93}$$

can the output of the amplifier have phase-insensitive quadrature noise for $G=G_{\text{crit}}>1$. If the reservoir is unsqueezed ($r=0$), an input with phase-insensitive quadrature noise gives rise to an output with the same property.

The absorber case is characterized by $0<G<1,\beta>0$. To have a phase-insensitive quadrature noise at the output for $G=G_{\text{crit}}\in(0,1]$, the parameters must satisfy the inequality:

$$r\lessgtr 0,\quad P_0\lessgtr 0 . \tag{94}$$

Asymptotically, if the reservoir is unsqueezed, the absorber always suppresses phase-sensitive quadrature noise $[\mathcal{P}(G=0)=0]$.

IX. CONCLUSIONS

In this work we have advocated the use of a new way to order the creation and annihilation operators in the quantum theory of any many-boson system (photons, phonons, helium atoms, etc. . .). We expect it to be most helpful in quantum optical problems where one is interested in the photon statistics of the field and its phase-dependent properties. Although many interesting investigations remain to be done (some of them were mentioned in the course of this work), we expect to have convincingly established the following main points: (1) The problem of obtaining a quantum-mechanical multipolar-like expansion of the density matrix in successive powers of the field intensity (photon-number operator) and the field phase (annihilation or creation operator) can be solved completely using the unambiguous and consistent alternate ordering prescription for $\hat{a}$ and $\hat{a}^+$, (2) The hierarchy of alternate moments characterize the field completely and are measurable properties (expectation values of a generalized phase operator) which probe higher- and higher-order amplitude- and phase-sensitive characteristics of the field, and (3) The alternate ordering technique is a valuable and efficient tool for the calculation of the boson statistics and generalized phase operator expectation values in practical quantum-mechanical (for the present instance quantum optical) devices which works at all field intensities. As an illustration, a few new results pertinent to the behavior of phase-sensitive light amplifiers and absorbers have been obtained.

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APPENDIX A: USEFUL FORMULAS

We quote or demonstrate a number of useful formulas which are extensively used during the course of the present study.

1. Identities concerning exponential operator "sandwiches"

Let $f(z_1,z_2)$ be any analytic function of two variables near the origin. The operator-valued functional $\hat{f}(\hat{a},\hat{a}^+)$, obtained by replacing z_1 and z_2 with $\hat{a}$ and $\hat{a}^+$ in the Taylor expansion, satisfies the following identities (see, e.g., Ref. 1):

$$e^{x\hat{a}}\hat{f}(\hat{a},\hat{a}^+)e^{-x\hat{a}}=\hat{f}(\hat{a},\hat{a}^+ + x) , \tag{A1}$$

$$e^{x\hat{a}^+}\hat{f}(\hat{a},\hat{a}^+)e^{-x\hat{a}^+}=\hat{f}(\hat{a}-x,\hat{a}^+) , \tag{A2}$$

$$e^{x\hat{a}^+\hat{a}}\hat{f}(\hat{a},\hat{a}^+)e^{-x\hat{a}^+\hat{a}}=\hat{f}(\hat{a}e^{-x},\hat{a}^+e^{x}) . \tag{A3}$$

These formulas are extensively used all along our work, starting with the many different forms (14)–(16) taken by the generating function $\hat{A}_s(\xi,\eta)$.

2. Multiple differentiation of products: Leibniz rule

Let $f(x)$ and $g(x)$ be two functions of the variable x. The nth derivative of the product with respect to x is[43]

$$\frac{d^n}{dx^n}[f(x)g(x)]=\sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix}\frac{d^m}{dx^m}[f(x)]\frac{d^{n-m}}{dx^{n-m}}[g(x)] . \tag{A4}$$

This rule is necessary when using Eqs. (19) and (38).

3. Integrals issuing from the coherent state representation

We reproduce here the formula (A2) of Cahill and Glauber,[10] which helps to calculate Gaussian integrals obtained by using the coherent-state resolution of the identity

$$\int \exp(-z|\alpha|^2+\alpha x+\alpha^* y)\frac{d^2\alpha}{\pi}=z^{-1}\exp(z^{-1}xy) . \tag{A5}$$

When we take the derivative of this expression with Leibniz rule we also obtain

$$\int \alpha^n(\alpha^*)^m\exp(-z|\alpha|^2+\alpha x+\alpha^* y)\frac{d^2\alpha}{\pi}=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{m!}{(m-k)!}x^{m-k}y^{n-k}z^{n-m-k-1}\exp(z^{-1}xy) . \tag{A6}$$

These formulas are used in Appendix B.

4. A miscellaneous differentiation formula

The evaluation of the high-order alternate moments of the squeezed state (and the thermal state) requires a closed-form formula for the expression

$$\left[\frac{d}{dt}\right]^j\left[\frac{e^{at}}{(1-be^t)^k}\right] \quad \text{with } a,b,k\in\mathbb{R}^+ . \tag{A7}$$

The application of the Leibniz rule gives the result

$$\sum_{l=0}^{j}\begin{pmatrix} j \\ l \end{pmatrix}a^{j-l}e^{at}\left[\frac{d}{dt}\right]^l\left[\frac{1}{1-be^t}\right]^k . \tag{A8}$$

Now we apply the Faa di Bruno formula[43] to calculate the multiple derivatives of the composed function

$$\left[\frac{d}{dt}\right]^l\left[\frac{1}{1-be^t}\right]^k = \sum_{m=0}^{l}\frac{(k-1+m)!}{(k-1)!}\frac{1}{(1-be^t)^{k+m}}\mathscr{S}_l^{(m)}(be^t)^m . \tag{A9}$$

Inserting this into the previous expression, we get the final result

$$\left[\frac{d}{dt}\right]^j\left[\frac{e^{at}}{(1-be^t)^k}\right]=\frac{e^{at}}{(1-be^t)^k}\sum_{l=0}^{j}\begin{pmatrix} j \\ l \end{pmatrix}a^{j-l}\sum_{m=0}^{l}\frac{(k-1+m)!}{(k-1)!}\mathscr{S}_l^{(m)}\left[\frac{be^t}{1-be^t}\right]^m . \tag{A10}$$

Of course, once one knows the explicit expressions (A9) and (A10) it becomes possible to prove them by recurrence also.

APPENDIX B: STANDARD s-ORDERING OF THE GENERATING FUNCTION $\hat{A}_s(\xi,\eta)$

We proceed to derive the s-ordered expansion (33) of our generating function $\hat{A}_s(\xi,\eta)$. We draw heavily, in this appendix, on the notation and techniques used by Cahill and Glauber.[10] Their formula (5.18) allows us to calculate the coefficient

$$f_{nm}(s')=(n!m!)^{-1}\int \mathrm{Tr}[\hat{F}\hat{D}_{s'}(\alpha)](-\alpha)^n(\alpha^*)^m\frac{d^2\alpha}{\pi} \tag{B1}$$

of the expansion

$$\hat{F}=\sum_{n,m} f_{nm}(s')\{\hat{a}^{+n}\hat{a}^m\}_{s'} . \tag{B2}$$

In our specific case we have

$$\mathrm{Tr}[\hat{A}_s(\xi,\eta)\hat{D}_{-s'}(\alpha)]=\int e^{(1-s')|\alpha|^2}\exp(\alpha\beta^*-\alpha^*\beta+i\eta e^{-i(s-1)\xi/2}\beta)\langle\beta|e^{i\xi\hat{a}^+\hat{a}}|\beta\rangle\frac{d^2\beta}{\pi} . \tag{B3}$$

Using the well-known expression[1]

$$\langle\beta|e^{i\xi\hat{a}^+\hat{a}}|\beta\rangle=\exp[(e^{i\xi}-1)|\beta|^2] \tag{B4}$$

and Eq. (A5) of Appendix A, we find

$$\mathrm{Tr}[\hat{A}_s(\xi,\eta)\hat{D}_{-s'}(\alpha)] =\frac{1}{1-e^{i\xi}}\exp\left[|\alpha|^2\left[\frac{1-s'}{2}+\frac{1}{e^{i\xi}-1}\right] +i\alpha\eta\frac{e^{-i(s-1)\xi/2}}{1-e^{i\xi}}\right] . \tag{B5}$$

With the help of Eq. (A6) we can calculate now the coefficient (B1)

$$f_{nm}(s')=\frac{1}{n!(m-n)!}\frac{1}{e^{i\xi}-1}\left[\frac{1-s'}{2}+\frac{1}{e^{i\xi}-1}\right]^{-(m+1)} \times\left[\frac{i\eta e^{-i(s-1)\xi/2}}{e^{i\xi}-1}\right]^{m-n} . \tag{B6}$$

The final summation (B2) can be written as Eq. (33) of the main text.

APPENDIX C: s-ALTERNATE MOMENTS OF THE SQUEEZED VACUUM

The squeezed vacuum (51) possesses nontrivial alternate moments. To evaluate them we will apply the following strategy: (1) disentangle the squeezing operator (52) and use a disentangled form of the generating function in the expression of the characteristic function (53), (2) bring all functions of the operator $\hat{a}^+$ on the left-hand side and functions of $\hat{a}$ on the right-hand side (it is a kind of normal ordering), (3) evaluate the action of the operators between the vacuum "sandwiches", and (4) carry out the multiple differentiations required to find the moments and evaluate them in $\xi=\eta=0$. It must be noted that *in principle* the same strategy can be applied to calculate the moments corresponding to a squeezed coherent state. The actual calculations would, however, be more complicated.

Using the disentangling theorem of SU(1,1) (Refs. 44 and 45), we can write the squeezing operator (52) as

$$S(\zeta)=\exp[\tfrac{1}{2}\mathcal{S}^*(\hat{a}^+)^2]\exp[-\ln(\cosh r)(\hat{a}^+\hat{a}+\tfrac{1}{2})] \times\exp(-\tfrac{1}{2}\mathcal{S}\hat{a}^2) , \tag{C1}$$

where

$$\mathcal{S}=e^{-2i\phi}\tanh r . \tag{C2}$$

The commutation to the left of all exponentials in Eq. (53) containing $\hat{a}^+$ can be made with the rules (A1)–(A3) except the following one:

$$\exp(-\tfrac{1}{2}\mathcal{S}\hat{a}^2)\exp(\tfrac{1}{2}\mathcal{S}^*e^{2i\xi}\hat{a}^2) , \tag{C3}$$

where the left (right) term issues from the squeezing operator on the left (right) in Eq. (53). This commutation cause a problem (in the case $\xi=0$) since we have been unable to carry it out with the help of the usual operator exponentiation formula and the usual disentangling theorem of SU(1,1) (Refs. 44 and 45) (because of a singularity in the coefficients). Therefore we developed a disentangling theorem of our own, using the faithful 2×2 matrix representation of SU(1,1), as suggested on very general grounds by Gilmore.[45] We only quote the result:

$$\exp(-\tfrac{1}{2}\mathcal{S}\hat{a}^2)\exp(\tfrac{1}{2}\mathcal{S}^*e^{2i\xi}\hat{a}^2) =\exp[\tfrac{1}{2}\mathcal{S}^*e^{2i\xi}(1+|\mathcal{S}|^2e^{2i\xi})^{-1}\hat{a}^{+2}] \times\exp[\tfrac{1}{2}(1+|\mathcal{S}|^2e^{2i\xi})^{-2}(\hat{a}^+\hat{a}+\tfrac{1}{2})] \times\exp[-\tfrac{1}{2}\mathcal{S}(1+|\mathcal{S}|^2e^{2i\xi})^{-1}\hat{a}^2] . \tag{C4}$$

When the action of all the experimental operator on the vacuum state is evaluated the characteristic function is expressed in closed form:

$$\chi_{sA}(\xi,\eta)=\left[\frac{1}{1+\tanh^2 r\, e^{2i\xi}}\right]^{1/2} \times \exp\left[\frac{1}{2}\left[\ln[(\cosh r)^{-2}] + \frac{i\eta e^{-i(s-1)\xi/2}e^{2i\phi}\tanh r}{1+\tanh^2 r\, e^{2i\xi}}\right]\right] . \tag{C5}$$

This is the most important intermediate step of our calculation. To obtain the s-alternate moments we derive now with respect to $i\eta$ first. Using the identity generating Hermite polynomials

$$\left[\frac{d}{dz}\right]^k e^{-z^2}=(-1)^k H_k(z)e^{-z^2} \tag{C6}$$

and

$$H_k(z=0)=(-1)^{k/2}\frac{k!}{(k/2)!} \quad \text{if } k \text{ is even ,}$$

$$H_k(z=0)=0 \quad \text{if } k \text{ is odd ,}$$

we find

$$\left[\frac{\partial}{\partial(i\eta)}\right]^{2k}\chi_{sA}^{\mathrm{sq}}(\xi,\eta)\Big|_{\eta=0}=\frac{1}{2^k}\frac{(2k)!}{k!}e^{2ik\phi}\frac{\tanh^k r}{\cosh r} \times\frac{e^{-2i\xi(s-1)k/2}}{(1-\tanh^2 r\, e^{2i\xi})^{k+1/2}} . \tag{C7}$$

The differentiation of this expression with respect to $i\xi$ can be directly carried out using the formula (A10). Then Eqs. (54) and (55) of the main text follow.

APPENDIX D: s-ALTERNATE MOMENT EQUATIONS OF THE PHASE-SENSITIVE AMPLIFIER

The s-alternate moment evolution equations have exactly the same structure as the master equation (70) and the characteristic function equation (76). We give here, therefore, only the coefficients. The coefficient of $C_{+-}+C_{-+}$ is

$$\sum_{m=1}^{j}\binom{j}{m}\left[[1+(-1)^m]M_{sA}^{j-m+1,k}+sk\left[\frac{1+(-1)^m}{2}\right]M_{sA}^{j-m,k}+k\left[\frac{1-(-1)^m}{2}\right]M_{sA}^{j-m,k}+M_{sA}^{j-m,k}\right] . \tag{D1}$$

The coefficient of χ_{+-} is

$$\sum_{m=1}^{j}\binom{j}{m}\left[[1-(-1)^m]M_{sA}^{j-m+1,k}+sk\left[\frac{1-(-1)^m}{2}\right]M_{sA}^{j-m,k}+k\left[\frac{1+(-1)^m}{2}\right]M_{sA}^{j-m,k}+M_{sA}^{j-m,k}\right] . \tag{D2}$$

The coefficient of χ_{-+} is

$$-2kM_{sA}^{j,k}+\sum_{m=1}^{j}\binom{j}{m}\left[[1-(-1)^m]M_{sA}^{j-m+1,k}+sk\left[\frac{1-(-1)^m}{2}\right]M_{sA}^{j-m,k}+k\left[\frac{1+(-1)^m}{2}\right]M_{sA}^{j-m,k}+M_{sA}^{j-m,k}\right] . \tag{D3}$$

The coefficient of C_{++} is

$$\sum_{m=1}^{j}\binom{j}{m}[2s^m-(s+1)^m-(s-1)^m]M_{sA}^{j-m,k+2} . \tag{D4}$$

The coefficient of C_{--} (case $k=0$) is

$$\sum_{m'=2}^{j}\binom{j}{m'}(-1)^{m'}(2^{m'}-2)\sum_{m=0}^{j-m'}\binom{j-m'}{m}(1+s)^{j-m'-m}(M_{sA}^{m,2})^* . \tag{D5}$$

The coefficient of C_{--} (case $k=1$) is

$$\sum_{m'=2}^{j}\binom{j}{m'}(-1)^{m'}(2^{m'}-2)\sum_{m=0}^{j-m'}\sum_{l=0}^{m+1}\binom{j-m'}{m}\binom{m+1}{l}\left[\frac{1-s}{2}\right]^{j-m'-m}\left[\frac{1+s}{2}\right]^{m+1-l}(M_{sA}^{l,1})^* -2\sum_{m'=1}^{j}\binom{j}{m'}\left[\left[\frac{1-s}{2}\right]^{m'}-\left[-\frac{1+s}{2}\right]^{m'}\right]\sum_{m=0}^{j-m'}\binom{j-m'}{m}\left[\frac{1+s}{2}\right]^{j-m'-m}(M_{sA}^{m,1})^* . \tag{D6}$$

The coefficient of C_{--} (case $k\geq 2$) is

$$\sum_{m'=2}^{j}\begin{bmatrix} j \\ m' \end{bmatrix}(-1)^{m'}(2^{m'}-2)\left\{\sum_{m=0}^{j-m'}\sum_{l=0}^{m+2}\begin{bmatrix} j-m' \\ m \end{bmatrix}\begin{bmatrix} m+2 \\ l \end{bmatrix}(-1)^{m+2-l}\left[\left[\frac{1-s}{2}\right]k\right]^{j+2-l-m'}M_{sA}^{l,k-2}\right.$$

$$\left.-\sum_{m=0}^{j-m'}\sum_{l=0}^{m+1}\begin{bmatrix} j-m' \\ m \end{bmatrix}\begin{bmatrix} m+1 \\ l \end{bmatrix}(-1)^{m+1-l}\left[\left[\frac{1-s}{2}\right]k\right]^{j+1-l-m'}M_{sA}^{l,k-2}\right\}$$

$$-2k\sum_{m'=1}^{j}\begin{bmatrix} j \\ m' \end{bmatrix}\left[\left[\frac{1-s}{2}\right]^{m'}-\left[-\frac{1+s}{2}\right]^{m'}\right]\sum_{m=0}^{j-m'}\sum_{l=0}^{m+1}\begin{bmatrix} j-m' \\ m \end{bmatrix}\begin{bmatrix} m+1 \\ l \end{bmatrix}(-1)^{m+1-l}$$

$$\times\left[\left[\frac{1-s}{2}\right](k-1)\right]^{j+1-l-m'}M_{sA}^{l,k-2}-k(k-1)\sum_{m=0}^{j}\begin{bmatrix} j \\ m \end{bmatrix}(1-s)^m M_{sA}^{j-m,k-2}\ . \quad \text{(D7)}$$

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