Measuring filtered chaotic signals

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According to a recent prediction by Badii and Politi [in *Dimensions and Entropies in Chaotic Systems,* edited by G. Mayer-Kress (Springer-Verlag, Berlin, 1986)], low-pass filtering of chaotic signals results in systematic errors in the determination of the attractor dimension. We show that in an experimental situation with the additional complication of small sample sizes of low resolution, the prediction still holds. The entropy K_2 appears to be unaffected by the filtering.

An important step in the characterization of chaotic dynamics is the determination of the dimension of the underlying attractor. Many laboratories have adopted a method originally proposed by Grassberger and Procaccia,[1] that requires only a single-variable time series. This technique allows the extraction of the correlation dimension D_2 and the order-2 entropy K_2.[2] It can still work with noisy experimental signals, and Abraham *et al.*[3] have demonstrated that the method works even for small numbers of data points, as low as 500.

In this paper we address a problem with such analysis that was pointed out by Badii and Politi very recently.[4] A low-pass filtering of the signal, e.g., due to insufficient bandwidth of the measuring apparatus, results in a significantly skewed result for the dimension. They tested their prediction for the Duffing equation and the Baker transformation. We pursued the question how well this theory would apply to an actual experimental situation, with the additional restraints of small data sets with eight-bit resolution.

Badii and Politi consider the case of a chaotic signal $x(t)$, passed through a single-pole (first-order) low-pass filter as modeled by

$$\dot{z} = -\eta z + x \,, \tag{1}$$

with the "filter output" $z(t)$. If the chaos-generating process has a phase space of dimension d, the phase space of the combined system (including the filter) is increased to $d+1$, and so is the number of Lyapunov exponents. The additional exponent $-\eta$ is negative, and η is equal to the filter roll-off frequency. Assuming the validity of the Kaplan-Yorke conjecture,[5] η will increase the information dimension D_1 determined from signals at the filter output (i.e., from a time series of z values), provided that $0 \geq \eta > \lambda_-$. Here λ_- denotes the negative Lyapunov exponent that appears in the denominator of the Kaplan-Yorke formula. The lower the filter roll-off frequency, the more pronounced this increase. In the limit $\eta \rightarrow 0$, the filter becomes an integrator, and the increase is predicted to be as much as 1 over the unfiltered original: $D_1(\eta=0) = D_1(\eta=\infty)+1$.

Since only the negative Lyapunov exponents are affected, we expect that the positive ones and thus the entropy K_1 will remain unchanged in the filtering process. With K_1 being an upper bound for K_2,[6] it is reasonable to assume that K_2 will not depend strongly on η. This is a relevant point because from K_2 an experimentally accessible sufficient criterion for chaos can be obtained. One would prefer such an indicator to be unaffected by any sources of systematic errors.

For our experiment, we used the system described in Ref. 7 as a source of chaotic signals. It consists of an electronic circuit simulating a hybrid optical device and is described very well by the differential equation

$$\dddot{U} + a\ddot{U} + b\dot{U} + cU = d\,(U-e)^2 \,. \tag{2}$$

a,b,c,d,e are constant coefficients depending on component values and parameter choice, and—as mentioned in Ref. 7—a dimensionless time is used. Throughout this paper, we fix parameters to the same values as in Figs. 3(b), 4(b), and 6(a) of Ref. 7 (approximately $a=0.846$, $b=2.100$, $c=0.704$, $d=0.945$, and $e=3.634$); this corresponds to mildly chaotic behavior. For these conditions, one average orbit in phase space takes 4.44 time units, one unit corresponding to 12.3 μs. The phase-space contraction rate $(1/V)\dot{V}$ is constant everywhere and equal to $-a$.

An approximate description of the experimental system is given by the following two-dimensional map:

$$\begin{aligned} X_{n+1} &= A_0 + A_1 X_n + A_2 X_n^2 + A_3 X_n^4 + A_4 X_n^8 + A_5 X_n^{16} \\ &\quad + A_6 X_n^{32} - Y_n \,, \\ Y_{n+1} &= B_1 X_n \,, \end{aligned} \tag{3}$$

with one iteration step corresponding to one orbit. It resembles the Hénon map[8] with the quadratic nonlinearity being replaced with a polynomial. This polynomial allows a good least-squares fit of all experimentally determined return maps (see Ref. 7), and there is a way to smoothly vary all the A_i simultaneously such that experimentally obtained bifurcation diagrams can be simulated. For our present parameter set, $A_0=0.640$, $A_1=-0.194$, $A_2=-2.486$, $A_3=0.227$, $A_4=1.251$, $A_5=0.480$, and $A_6=-0.317$. B_1 as obtained from the fit is consistent with $B_1=\exp(-4.44a)$, so that the phase-space contraction rates of Eqs. (2) and (3) are equal; we use the latter value because it is more precise.

We proceed as follows. In a first step we convince ourselves that our computer codes work right by starting with the Hénon map with an added filter equation:

$$x_{n+1}=1-AX_n^2+Y_n ,$$
$$Y_{n+1}=BX_n , \quad (4)$$
$$Z_{n+1}=\exp(-\eta)Z_n+X_n .$$

We use the standard parameters $A=1.4$ and $B=0.3$. For the moment, let us set $\eta=\infty$ (no filter). First, we determine the positive Lyapunov exponent λ_+ from a numerical computation of the divergence of trajectories.[9] Given that $\lambda_-=\ln B-\lambda_+$, we also obtain the Kaplan-Yorke dimension D_1. The Kolmogorov entropy K_1 is simply given here by $K_1=\lambda_+$.

We next iterate Eq. (4) several thousand times and store the Z values, rounded off to eight-bit integer numbers. From this file, we determine the order-2 entropy K_2 and the correlation dimension D_2 with a Grassberger-Procaccia–type program. The program runs on Atari 1040STF desktop computers; computing time is 1 h for 5000 data points and embedding dimensions up to 20. (We find that 5000 is a very reasonable number of points in terms of speed versus accuracy.) Typically, the length scales used in the evaluation (the "scaling region") range from 2^{-5} to $2^{-2.7}$.

Compared to published values[6] (see Table I), D_1, K_1, and K_2 turn out to be in excellent agreement, but $D_2=1.15$ deviates from the correct value $D_2=1.224$. This can be attributed to the low resolution intentionally used here. In a separate test, we varied the resolution from six to twelve bits without changing the scaling region. The dimension increased systematically with increasing resolution and converged to the literature value cited above (we find $D_2^{12\text{ bit}}=1.225$). Details and our explanation will be given in a forthcoming publication.

Next, we turn to the influence of the filter. For various values for η, we repeat the determination of D_2 and K_2. There is no detectable systematic variation of K_2 with η. The result for D_2 is shown in Fig. 1, in comparison with the predicted $D_1=D_1(\eta)$ (solid line). It is perfectly reasonable that all data points lie below the solid line, because (a) D_1 is an upper bound for D_2 and (b) we just stated that we underestimate D_2. However, the points clearly follow the trend of the solid line, and for $\eta\rightarrow 0$, D_2 indeed tends to $D_2(\eta=0)=D_2(\eta=\infty)+1$.

In the second step we repeat the whole procedure for the return map [Eq. (3)]. As for the Hénon map, we first compute from the trajectory divergence $\lambda_+=0.52$ bits per iteration or, in ln and time units as for Eq. (2), $\lambda_+=K_1=0.081$. We note that precisely the same value holds for the logistic map,[10] which can be obtained from Eq. (3) by truncation (X remains bounded to $|X|<0.65$) in the limit $B\rightarrow 0$. From $\lambda_++\lambda_-=-a$ we get $\lambda_-=-0.927$. Now $D_1(\eta)$ can be calculated according to Ref. 4. Again as above, we evaluate files of eight-bit numbers from an iteration of the map with simulated filters of various η. The entropy K_2 comes out independent of η again and is $K_2=0.065$ (or 0.42 bits per iteration). So, indeed, $K_2<K_1$, as it must be. Figure 2(a), similar to Fig. 1, shows $D_2=D_2(\eta)$; obviously the agreement with the prediction is very good again.

In the third step we finally turn to the time-continuous system. In order to estimate its entropy, we hold the circuit at some initial condition and then release it so that it produces chaotic output again. Due to a tiny spread in the initial condition caused by electronic noise, the subsequent trajectory is different each time the procedure is repeated. We estimate the time required until the spread in amplitudes covers the whole amplitude range of the attractor. From this "time horizon" and the measured signal-to-noise ratio we estimate $K_1=\lambda_+=0.08\pm0.02$, which agrees well with the value obtained for the return map above.

We also calculate λ_+ directly from Eq. (2) and, within our precision, λ_+ is the same for the flow as for the return map. We thus use the same values for the reference curve in Figs. 2(b) and 2(c) as in Fig. 2(a), except for a vertical shift by 1.

Now we turn to the actual measurement. We pass the output signal U of the electronic circuit through a low-pass filter of selectable RC time constant. At the filter output, we measure with a LeCroy 9400 transient digitizer which has eight-bit resolution and can store up to 32 000 data points in a row. Strictly speaking, its bandwidth of 125 MHz contributes a second pole to the overall frequency response, with $\eta\approx 10^4$. This value is high enough to be safely considered infinite. The intrinsic noise in the circuit (-85 dB) is negligible at eight-bit resolution.

From the acquired data, D_2 and K_2 are evaluated the

TABLE I. Overview of generalized dimensions and entropies used in this work. D_1, information dimension; D_2, correlation dimension; K_1, Kolmogorov entropy; K_2, correlation entropy; η, filter rolloff frequency. The asterisk denotes values underestimated due to low resolution (see text). The dagger denotes no η dependence.

	Hénon map, Eq. (4) Ref. 6	This work	Return map Eq. (3)	Measured signal
D_1	1.258 26	1.26	1.09	2.09
D_2	1.224	1.15*	0.98	1.90*
$D_2(\eta)$		Fig. 1	Fig. 2(a)	Fig. 2(b)
K_1	0.4192	0.41	0.081	0.081
K_2	0.318	0.32	0.065	0.067
$K_2(\eta)$		0.32†	0.065†	0.067†

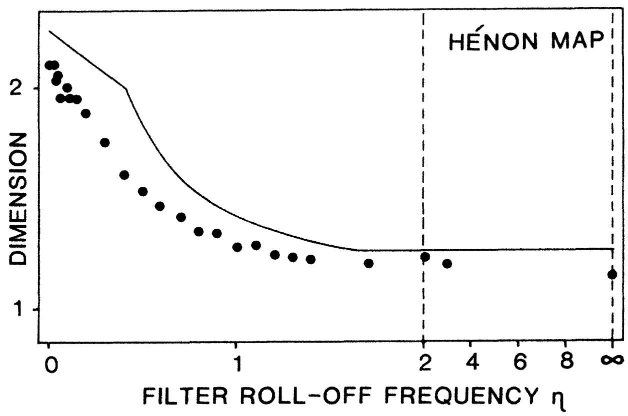

FIG. 1. D_2 values for simulated low-pass filtering with different rolloff frequencies η for the Hénon map. The solid line shows the predicted $D_1(\eta)$. Note change of scale at the dashed lines.

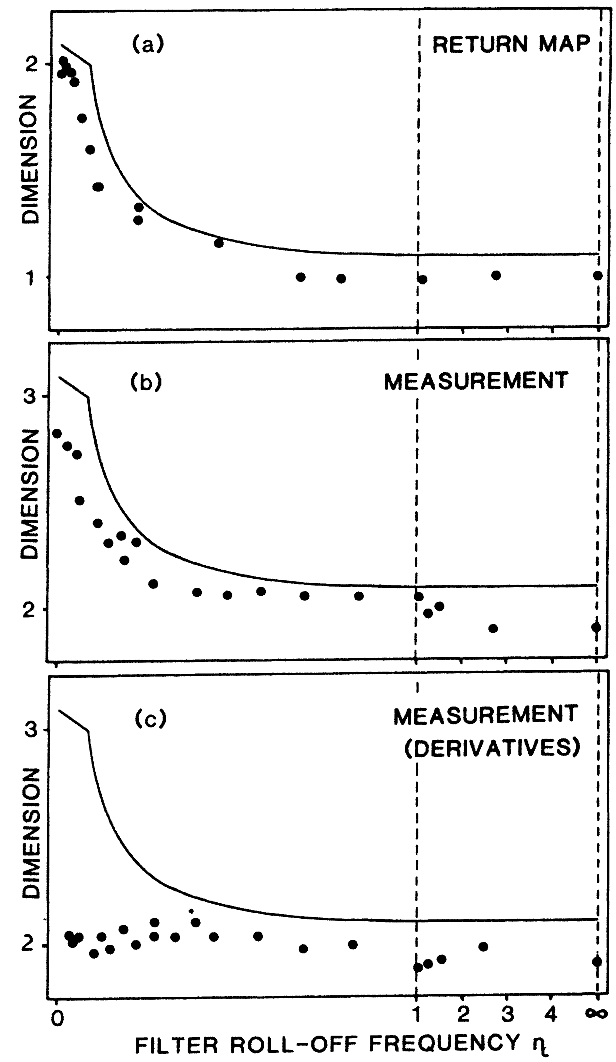

FIG. 2. Same as Fig. 1, but (a) for the return map, Eq. (3), (b) from experimental data of the low-pass filtered signal, and (c) from the derivative of the low-pass filtered signal.

same way as above. We find $K_2=0.067$, provided the embedding vectors are sufficiently long such as to avoid the systematic error described in Ref. 11, which is practically the same result as for the return map. The results of the dimension analysis from the filtered signal show the same trend as predicted [Fig. 2(b)]. This is certainly a confirmation of the result of Badii and Politi, but beyond that it demonstrates that the increase of dimension by filtering can be seen clearly in a realistic experimental situation with small data sets of low resolution.

In Fig. 2(c) we show points that are derived from time series of the time derivative of the filter output. For that experiment, we employed the electronic differentiator that was used in Ref. 7 for producing phase portraits. It appears that D_2 now does not appreciably depend on η, but retains similar values as $D_2(\eta=\infty)$. In other words, differentiation by and large undoes the low-pass filtering, which is plausible at least for the case of $\eta \to 0$ where the filter becomes an integrator. The entropies K_2 come out independent of η again, with the same value as above.

Our experiment demonstrates clearly that a single-pole low-pass filter leaves the entropy alone, while the dimension is increased in agreement with the prediction. The idea of an increase in dimension of a filtered signal may first appear counterintuitive. At face value, it sounds paradoxical that through the introduction of a new time constant the quantity that describes the geometry of the attractor—the dimension—should be affected, but the quantity that relates to the dynamics—the entropy—should not. This thought is misleading though.

What the filter really does is a redistribution of information on the time axis: any given time slot will contain

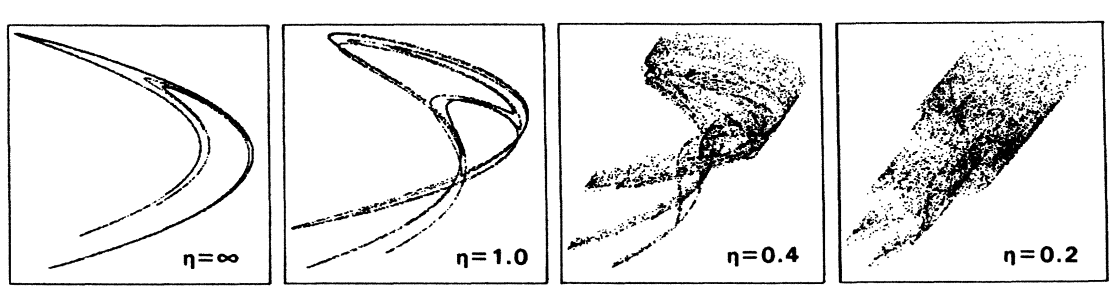

FIG. 3. Influence of degree of filtering on attractor shape. Shown are return maps (Z_{n+1} vs Z_n) of the "filtered Hénon system," Eq. (4), with eight-bit resolution. All four attractors are scaled to the same size.

more information on the past because of the filter's memory effect. The filter can neither create nor destroy information all by itself. Our understanding is therefore that, due to the "reshuffling" by the filter, the geometrical structure of the attractor is modified considerably and expands into the new phase space of larger dimension, but the rate of creation of information is unaffected.

Figure 3 illustrates how the shape of the Hénon attractor is blurred with increasing η. The corresponding figures for the attractor of Eq. (3) (not shown) look somewhat similar, and we verified that those in turn are close look alikes to their experimentally produced counterparts. The experimentally produced phase portrait, i.e., a representation of the attractor of Eq. (2), becomes very fuzzy even for moderate filtering. Obviously, when it comes to measuring chaotic signals, it is important to have sufficient bandwidth. If this is not possible, we suggest analyzing the derivative of the measured signal.

Note added in proof. Since this work was submitted, Badii and co-workers have demonstrated the increase in dimension discussed here by numerical filtering of experimental data.[12]

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