Analysis of a dye-laser model including quantum noise

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An analytical study of a dye-laser model which includes quantum white noise and nonwhite pump fluctuations is presented. The extension of an earlier approximation to include quantum noise gives a unified picture of the statistical properties of the laser light for negative pump parameters. These include intensity fluctuations and discontinuous changes of the most probable intensity. An alternative approximation (obtained within the same scheme) is discussed for situations above threshold.

I. INTRODUCTION

The anomalous statistical properties of dye-laser light have been partially explained in terms of a fluctuating pump parameter.[1–9] This view is, for instance, supported by the observation of a large increase in the intensity fluctuations of a He:Ne laser in which a noise voltage is applied.[10] In addition, it is widely recognized[2–9] that a proper modeling of pump fluctuations requires the use of colored noise, that is, noise with a finite correlation time. Most theoretical studies of this system neglect spontaneous-emission noise (quantum noise). This seems to be a safe approximation above threshold. However, quantum noise is known to be important to describe intensity fluctuations well below threshold.[3] Quantum noise also plays an important role above threshold in the study of transient statistics[4,5] and in the description of the initial decay of correlation functions.[6] In this paper we focus our attention in the description of the statistical properties of dye-laser light, taking into account quantum noise. We recall that in the mathematical models in which quantum noise is neglected no stationary solution for the distribution of the light intensity exists for negative values of the pump parameter.

Experimental results for the anomalous intensity fluctuations below threshold were well reproduced by a numerical simulation of a stochastic model with colored pump noise and white quantum noise.[3] The problem of the analytical calculation of such fluctuations has been recently addressed by Fox and Roy.[8] These authors use an approximation under which the intensity (I) probability distribution has the same qualitative shape as white pump noise.[11] However, recent results of Lett *et al.*[7] show the existence of different shapes of the intensity distribution with a relative maximum at $I\neq 0$ induced by nonwhite pump noise. These experimental results are qualitatively reproduced by numerical simulations. They are explained by Lett *et al.*[7] in terms of an earlier approximation[12] devised for a situation in which quantum noise was neglected. The aim of this paper is to present an analytical calculation for negative values of the pump parameter which describes satisfactorily the anomalous intensity fluctuations results and, in addition, predicts the emergence of a relative maximum in the stationary distribution at $I\neq 0$ when the pump parameter is increased. This qualitative change in the intensity distribution is seen to occur for negative values of the pump parameter for certain values of the noise parameters. The calculation is an extension of an earlier approximation[12] including now quantum noise effects. It gives a unified picture of intensity fluctuations and changes in the intensity distribution. Unfortunately, this approximation does not seem to be quantitatively accurate in other domain of parameters. An alternative approximation, also based on the scheme of Ref. 12, gives an excellent fit to the simulations of Lett *et al.*[7] for positive values of the pump parameter but it fails below threshold. This alternative approximation is related to the one recently proposed by Jung and Hanggi[9] in which quantum noise is neglected.

We present in Sec. II the dye-laser model with quantum noise and nonwhite pump fluctuations and also the general approximation scheme for small correlation time of the noise. An extension of this approximation and the calculation of the statistical properties for negative pump parameters are presented in Sec. III. An alternative extension of the general approximation more suitable for positive pump parameters and its connection to related work is discussed in Sec. IV.

II. MODEL AND SMALL τ APPROXIMATION

We consider the usual model for a single-mode laser on resonance

$$\partial_{\bar t}\bar E=(\bar a - A\;|\bar E\,|^2)\bar E+q(\bar t)\;. \tag{1}$$

$\bar E$ is the laser complex amplitude $\bar E=\bar E_1+i\bar E_2$ and $q(\bar t)$ stands for spontaneous emission fluctuations whose real q^R and imaginary part q^I are independent Gaussian white noise of zero mean and correlation

$$\langle q^I(\bar t)q^I(\bar t\,')\rangle=\langle q^R(\bar t)q^R(\bar t\,')\rangle=2\bar D\delta(\bar t-\bar t\,')\;. \tag{2}$$

We are interested in the fluctuations of the intensity $\bar I=\bar E_1^{\,2}+\bar E_2^{\,2}$. From (1) one easily obtains a set of coupled equations for $\bar I$ and the phase $\varphi=\tan^{-1}(\bar E_2/\bar E_1)$. However, it is known[13] that these equations are stochastically equivalent to a second set of equations in which the equation for the intensity is decoupled from the phase,

$$\partial_{\bar{t}}\bar{I}=2\bar{I}(\bar{a}-A\bar{I})+2\bar{D}+2\sqrt{\bar{I}}\,q^R(\bar{t}) \ . \tag{3}$$

Fluctuations of the pump parameter $\bar{a}$ can now be introduced, replacing $\bar{a}$ in (3) by $\bar{a}+p(\bar{t})$, where $p(\bar{t})$ is a real Gaussian noise of zero mean,[14]

$$\partial_{\bar{t}}\bar{I}=2\bar{I}(\bar{a}-A\bar{I})+2\bar{D}+2\bar{I}p(\bar{t})+2\sqrt{\bar{I}}\,q^R(\bar{t}) \ . \tag{4}$$

There exists an important number of results[2-9] which support the assumption of nonwhite pump fluctuations. We then choose $p(\bar{t})$ to have a finite correlation time $\bar{\tau}$ with a correlation given by[15]

$$\langle p(\bar{t})p(\bar{t}')\rangle=\frac{\bar{Q}}{2\bar{\tau}}e^{-|\bar{t}-\bar{t}'|/\bar{\tau}} \ . \tag{5}$$

To proceed further we need an equation for the intensity probability distribution $P(\bar{I},\bar{t})$. We use here a Fokker-Planck approximation, valid for small correlation time $\bar{\tau}$, obtained along the lines of the small τ approximation of Ref. 12. A discussion of this type of approximations and a comparison with related approximations is given in Ref. 16. The presence of two sources of noise in (4) leads to a multivariable problem. Fokker-Planck approximations for multivariable non-Markovian problems based on a τ expansion have been discussed, for example, in Ref. 17. They are based on expansions on the noise parameters considered as smallness parameters. To make this statement more precise we introduce dimensionless variables $\hat{I}=(A/|\bar{a}|)\bar{I}$, $\hat{t}=|\bar{a}|\bar{t}$. In these variables the dimensionless noise intensity of the pump fluctuations and spontaneous emission are, respectively,

$$\hat{D}=\frac{A\bar{D}}{\bar{a}^2}, \quad \hat{Q}=\frac{\bar{Q}}{|\bar{a}|} \ . \tag{6}$$

From the general formulas of Ref. 17(c) (see also Ref. 18), a Fokker-Planck approximation for the stationary solution of (4) is given in these units by

$$\partial_{\hat{t}}P(\hat{I},\hat{t})=L_{\mathrm{FP}}(\hat{\tau})P(\hat{I},\hat{t}) \ , \tag{7}$$

$$L_{\mathrm{FP}}(\hat{\tau})=-2\partial_{\hat{I}}\left[\left[\frac{\bar{a}}{|\bar{a}|}-\hat{I}\right]\hat{I}+\hat{D}\right]+4\hat{D}\partial_{\hat{I}}\sqrt{\hat{I}}\,\partial_{\hat{I}}\sqrt{\hat{I}} +2\hat{Q}\partial_{\hat{I}}\hat{I}\partial_{\hat{I}}(\hat{I}-2\hat{\tau}\hat{I}^2) \ , \tag{8}$$

with $\hat{\tau}=|\bar{a}|\bar{\tau}$. This equation is hopefully valid in the limit $\hat{\tau}\ll 1$, $\hat{D}\ll 1$, since it neglects terms proportional to $\hat{Q}^n\hat{\tau}^m$ (with $m\geq n$ and $m>1$) and also cross-fluctuation terms which involve the product $\hat{Q}\hat{D}$ and thus couple the two sources of noise.[19] The first of the neglected terms is proportional to $\hat{Q}\hat{D}\hat{\tau}$ and would break the Fokker-Planck form of (8). In this approximation the diffusion part of the Fokker-Planck operator (8) contains two terms. The first one, proportional to $\hat{D}$, is the one which appears in the absence of pump fluctuations. The second one, proportional to $\hat{Q}$, is the one obtained in the one-variable small $\hat{\tau}$ approximation to (4) when quantum noise is neglected.[12] In the limit $\hat{D}=0$ no stationary solution exists for $\bar{a}<0$.

In the dimensionless variables in which (4) is usually studied, the parameter measuring quantum noise intensity does not appear explicitly. These variables are

$$I=\left[\frac{\bar{D}}{A}\right]^{-1/2}\bar{I}, \quad t=(\bar{D}A)^{1/2}\bar{t} \ , \tag{9}$$

and the Fokker-Planck operator (8) becomes

$$L_{\mathrm{FP}}(\tau)=-2\partial_I[2+aI-I^2+QI(1-2\tau I)] +4\partial^2_{I^2}\left[I+\frac{Q}{2}I^2(1-2\tau I)\right] , \tag{10}$$

where $a=(\bar{D}A)^{-1/2}\bar{a}$, $\tau=(\bar{D}A)^{1/2}\bar{\tau}$, $Q=(\bar{D}A)^{-1/2}\bar{Q}$, and the approximation makes sense whenever $|a|\gg 1$ and $\tau|a|\ll 1$. In the development below we will also use the fact that in these units usually $Q\gg 1$.

The stationary solution of (10) is defined in the interval in which the diffusion coefficient $D(I)=I+(Q/2)I^2(1-2\tau I)$ is positive. This gives a spurious boundary introduced by the approximation at $I_1=(2\tau)^{-1}+2Q^{-1}+O(\tau)$. The stationary solution

$$P_{\mathrm{st}}(I)=N\left|1+\frac{Q}{2}I(1-2\tau I)\right|^{(1-4\tau Q)/4\tau Q}\left|\frac{-4\tau QI+Q-(Q^2+16\tau Q)^{1/2}}{-4\tau QI+Q+(Q^2+16\tau Q)^{1/2}}\right|^{(4a\tau-1)/[4\tau(Q^2+16\tau Q)^{1/2}]} , \tag{11}$$

where N is a normalization constant, makes sense whenever P_{st} does not diverge at $I=I_1$. The requirement for this condition is, for small τ and large Q,

$$2\tau(Q+a)<1 \ . \tag{12}$$

In the white-noise limit ($\tau=0$), (11) becomes

$$P_0(I)=N(2Q^{-1}+I)^{(a/Q-1+2Q^{-2})}e^{-I/Q} \ . \tag{13}$$

This distribution has a single maximum which changes from $I=0$ to $I\neq 0$ when $a/Q=1-2Q^{-2}$. For sufficiently small τ, I_1 becomes very large and (12) is always satisfied. Under these conditions (11) has the same qualitative behavior as (13). In other words, in the domain of parameters in which (11) makes sense, it does not exhibit the emergence of a relative maximum at $I_0\neq 0$, $I_0<I_1$, when increasing τ, as found in the simulations and experiments of Lett *et al.*[7] In addition, for the parameter values ($Q=300$, $\tau=0.2$) used to fit experimental curves of the relative-intensity fluctuations below threshold,[3] (12) is not satisfied in the whole ap-

propriate range of values of the pump parameter (up to $a \cong -200$). It is then necessary to extend our discussion beyond the strict small- τ approximation as we do next.

III. STATISTICAL PROPERTIES FOR NEGATIVE PUMP PARAMETER

An extension of the small-τ approximation is carried out here following the same scheme which predicted the occurrence of a relative maximum of the intensity stationary distribution in the model in which spontaneous-emission noise is neglected.[12] This scheme allows the exploration of larger values of $\bar{\tau}$ and it has proved itself able to describe the main features of the stationary solution (for $\bar{D}=0$) which have been reobtained by other methods.[2(d),7,9] We will see that the consideration of spontaneous-emission noise has important consequences.

Consistently with the τ expansion leading to (10), we look for a solution of the form[12]

$$P_{\rm st}(I)=P_0(I)+\tau P_1(I)+O(\tau^2)\ , \tag{14}$$

where $P_0(I)$ is given in (13) and the normalization of $P_{\rm st}(I)$ is guaranteed requiring that $\int_0^\infty P_1(I)dI=0$. The contribution $P_1(I)$ can be factorized as

$$P_1(I)=P_0(I)f(I)\ . \tag{15}$$

For $f(I)$ we find

$$\begin{aligned} f(I)=2I+2Q^{-1}&\left[-\frac{I^2}{2}+aI+\frac{2}{1+\frac{Q}{2}I}\right]\\ &+4Q^{-2}\left[2I-2a\ln\left[1+\frac{Q}{2}I\right]-\frac{a}{1+\frac{Q}{2}I}\right]\\ &-8Q^{-3}\left[3\ln\left[1+\frac{Q}{2}I\right]+\frac{1}{1+\frac{Q}{2}I}\right]+N'\ , \end{aligned} \tag{16}$$

where N' is a normalization constant. The anomalous boundaries found in (11) can be avoided in this scheme[12] by an exponentiation of (14) and (15) leading to

$$P_{\rm st}(I)=P_0(I)e^{\tau f(I)}\ , \tag{17}$$

which for $Q \gg 1$ gives

$$P_{\rm st}(I)=N(2Q^{-1}+I)^{(a/Q-1)}\times\exp\left[-\frac{I}{Q}+\tau\left[-\frac{I^2}{Q}+2I+2\frac{a}{Q}I\right]\right]\ . \tag{18}$$

We use this form of the stationary solution to compute the intensity fluctuations and to discuss the emergence of a relative maximum. The intensity fluctuations $\lambda(0)$ are defined by

$$\lambda(0)=\frac{\langle I^2\rangle-\langle I\rangle^2}{\langle I\rangle^2}\ . \tag{19}$$

In the absence of quantum noise, $\lambda(0)$ diverges as $\langle I\rangle\to 0$. Figure 1 shows the result of a numerical calculation of $\lambda(0)$ from (18), which is compared with the white-noise result and a direct simulation of the Langevin equation.[3] It is seen that the large reduction of the peak of $\lambda(0)$ caused by the introduction of a correlation time $\tau\neq 0$ is well described by the approximation (18). This reduction of the peak of $\lambda(0)$ can be understood in terms of the probability distributions shown in Fig. 2: The tail of the white-noise distribution is suppressed in the colored-noise case, reducing the intensity fluctuations. Although Eq. (18) does not give a very precise fit of the simulation results, it gives a decent

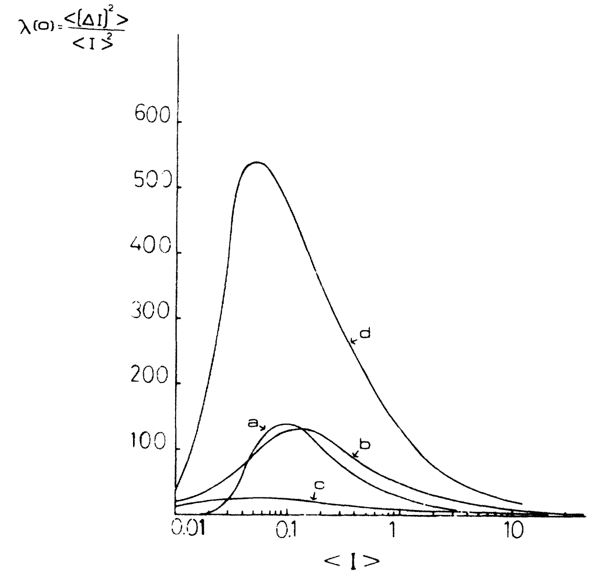

FIG. 1. Intensity fluctuations vs intensity mean value calculated from different approximations to the stationary distribution: (a) simulation results of Ref. 3 ($Q=300$, $\tau=0.2$); (b) calculation from (18) with $Q=300$, $\tau=0.2$; (c) calculation from (22) with $Q=300$, $\tau=0.2$ (see Sec. IV); (d) white-noise limit (13) with $Q=300$.

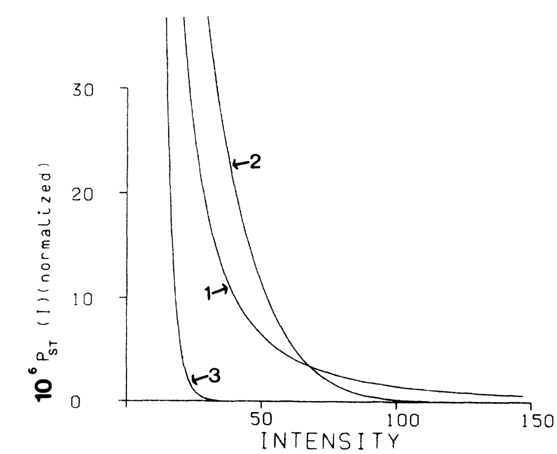

FIG. 2. Intensity stationary distributions for $\langle I\rangle$ close to the maximum of $\lambda(0)$ in Fig. 1: (1) white-noise limit (13) for $a=-260$ [$\langle I\rangle=0.12$, $\lambda(0)=453.2$]; (2) Eq. (18) for $a=-265$ [$\langle I\rangle=0.12$, $\lambda(0)=131.7$]; (3) Eq. (22) for $a=-200$ [$\langle I\rangle=0.125$, $\lambda(0)=21.7$].

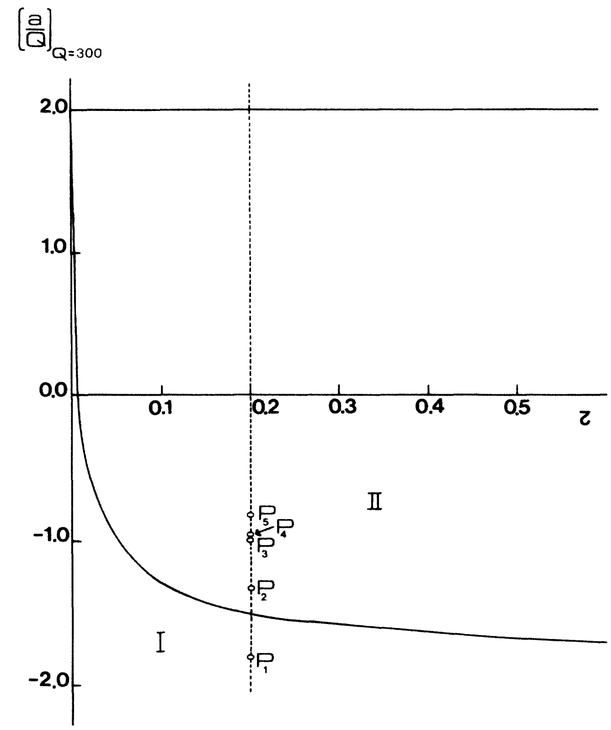

FIG. 3. Regions in parameters space according to the shape of $P_{st}(I)$. See the text.

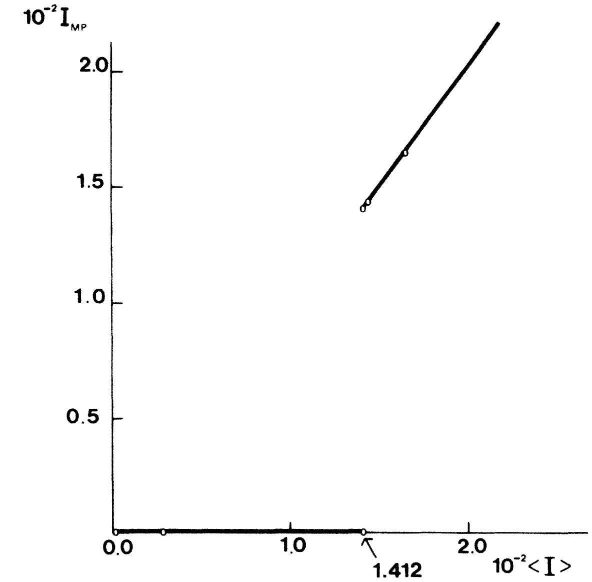

FIG. 5. Discontinuous change of the most probable intensity. The mean value $\langle I\rangle = 1.412$ corresponds to the point P_3 in Figs. 3 and 4.

account overall of them when compared with the white-noise limit, and in particular, of the main feature of the large reduction of the peak of $\lambda(0)$.[20]

The emergence of a relative maximum of $P_{st}(I)$ is visualized in Figs. 3–5. $P_{st}(I)$ given by (18) has always a finite value at $I=0$, and one or two relative extrema, or none, depending on the values of the pump and noise parameters. To leading order in Q^{-1} and also neglecting τ with respect to R, the extrema are located at the points solution of[21]

$$2\tau I^2-[2\tau(Q+a)-1]I+Q-a=0\ . \tag{20}$$

Figure 3 shows the parameter space divided in different regions according to the shape of P_{st}. The location of the curves separating the different regions depends on the actual value of Q. Region I is associated with the absence of relative extrema and in region II there exist a relative minimum and maximum at $I\neq 0$. Within this region the absolute maximum changes from $I=0$ to $I\neq 0$. For $a \gtrsim Q$ there exists a single maximum at $I\neq 0$. The relative maximum of $P_{st}(I)$ appears when crossing the boundary between regions I and II. This can be obtained increasing the correlation time τ or the pump parameter a. Figure 4 shows the emergence of this max-

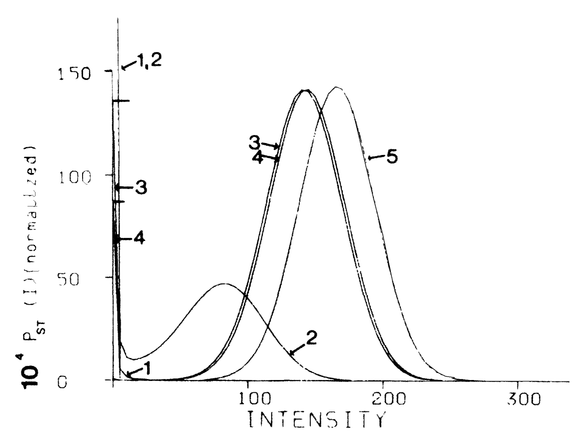

FIG. 4. Intensity stationary distributions for $Q=300$, $\tau=0.2$, calculated from Eq. (18). The five distributions correspond to the points indicated in Fig. 3. Values of $P_{st}(I)$ at the origin $I=0$ are P_1, 134.30; P_2, 62.30; P_3, 0.14×10^{-1}; P_4, 0.90×10^{-2}; P_5, 0.58×10^{-4}.

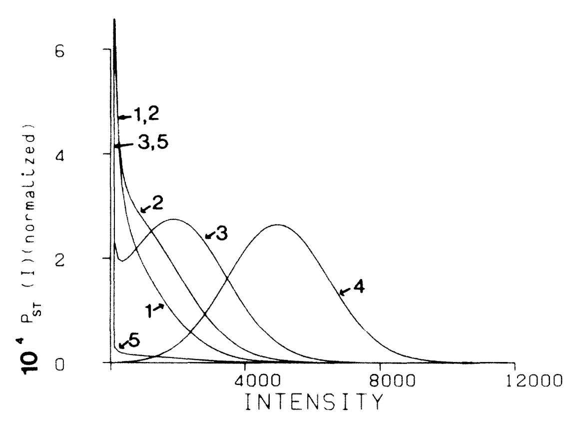

FIG. 6. Intensity stationary distributions computed from (22) for the same parameters as the simulation of Ref. 7: $Q=5000$, $\tau=10^{-3}$. Pump parameters a are (1) 400; (2) 1000; (3) 2100; (4) 5000. The distribution numbered 5 corresponds to $a=1000$ when neglecting quantum noise.

imum for a fixed value of τ along the line indicated in Fig. 3. It is seen that the emergent maximum rapidly dominates the stationary distribution[22] and at a well-defined point (P_3) the maximum changes from $I=0$ to $I\neq 0$. This change of the most probable intensity value is shown in Fig. 5. A qualitative analogy of this change with a first-order transition has been pointed out.[7] The discontinuity of the most probable intensity is a consequence of having $\tau\neq 0$. This phenomenon discussed in Ref. 7 in terms of experimental results and simulations is described here through an analytical calculation which takes into account quantum noise. The important consequence of the consideration of quantum noise is that the transition can occur for negative values of a depending on the noise parameters. Finally, we note that the relative minimum of $P_{\rm st}(I)$ in region II of Fig. 3 is not always apparent. For example, in the scale of Fig. 4 it is not seen for $a/Q=-0.417(P_5)$, but the numerical results show its existence.

IV. APPROXIMATION ABOVE THRESHOLD

The simulation results for the stationary distribution of Lett *et al.*[7] correspond to positive parameters of the pump parameter. These distributions are poorly reproduced (in a quantitative sense) by (18). This makes it desirable to find a different extension of the τ expansion to this other range of parameters. An approximation has been recently reported for the case in which quantum noise is neglected, which gives remarkably good results.[9] However, this approximation cannot be directly applied to our case and its direct derivation breaks down when including a white-noise source in the intensity equation. We extend here our Fokker-Planck approximation (10) in a way which in the absence of quantum noise reproduces the results of Ref. 9. This gives an alternative path to understand those results within the τ-expansion scheme. The basic idea is to extrapolate the diffusion coefficient obtained in first order in τ to larger values of τ (Ref. 23),

$$D(I)=I+\frac{Q}{2}I^2(1-2\tau I)\cong I+\frac{QI^2}{2+4\tau I}\ . \qquad (21)$$

In this case, $D(I)$ given by (21) is always positive, avoiding the appearance of spurious boundaries. The solution of the resulting Fokker-Planck equation can be approximated for small τ and large Q as

$$P_{\rm st}(I)=N(2Q^{-1}+I)^{(a/Q-1)}(1+2\tau I)\times\exp\frac{2}{Q}\left[-\frac{I}{2}+\tau\left[-\frac{I^2}{2}+aI\right]\right]\ . \qquad (22)$$

If the same calculation is done with $\bar{D}=0$ in (4), one reobtains (22) with the prefactor $(2Q^{-1}+I)^{(a/Q-1)}$ replaced by $I^{(a/Q-1)}$. This reproduces the result of Ref. 9. It is also important to note that in comparing (18) with (22) the latter can be understood as a partial exponentiation of (14). Indeed, (22) is obtained when one does not exponentiate Q-independent terms.

Figure 6 shows stationary distributions computed from (22). They reproduce with great accuracy the simulation results of Lett *et al.*[7] (no visual difference can be seen in the scale of the figure) and display the discontinuous change of the most probable intensity which occurs for these noise parameters at positive values of a. However, and unfortunately, (22) gives a very poor picture of the behavior below threshold. Intensity fluctuations computed from (22) and shown in Fig. 1 make clear this statement. The reason why $\lambda(0)$ is underestimated is that (22) does not reproduce the tail of $P_{\rm st}$ as seen in Fig. 2: For negative a, the exponential factor $e^{2\tau I}$ in (18) compensates the fast decrease of the remaining exponential. This effect is not included in (22) and it is not important for $a>0$. Figure 6 also shows that quantum noise is still important above threshold in some cases. The distributions numbered 3 and 4 are well reproduced neglecting quantum noise. However, the divergence of $P_{\rm st}$ at $I=0$ for the distributions 1 and 2 obtained when quantum noise is neglected is very strong. As a consequence the normalized distribution is very different when this divergence is suppressed by quantum noise (compare distributions 2 and 5).

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[14]It is possible to consider a complex pump parameter (detuning) in (1) to allow independent fluctuations of the real and imaginary part, but only the real part appears in the equation for the intensity.

[15]Equation (3) supplemented with pump fluctuations is rigorously equivalent to the equation for $\bar{I}$ obtained from (1) when $\bar{a}$ is a fluctuating parameter in (1), only when $p(\bar{t})$ is a white noise. For $\tau \neq 0$ the equation for the intensity has to be considered within a scheme of phenomenological modeling of external noise.

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[20]We stress here that our comparison with the simulations involves no free parameters and it is made for the same values of Q and τ as the simulation (see Ref. 11). Note also that the simulation results for small $\langle I \rangle$ may not be completely reliable (Ref. 3). In addition, the value of τ is somehow large to expect very good quantitative results from a τ expansion.

[21]This equation coincides with the one obtained in the case without quantum noise (Ref. 12), but here it can be used for $a<0$.

[22]Note that the distributions in Fig. 4 correspond to the values of the pump parameter considerably larger than those in Fig. 1: For $a=-270.0$ (P_1) one finds $\langle I \rangle = 0.1026$.

[23]In the general presentation of the τ expansion for a single source of noise in Ref. 12, this approximation consists of replacing $h(q)=g(q)\{1+\tau g(q)[v(q)/g(q)]'\}$ by $g(q)/\{1-\tau g(q)[v(q)/g(q)]'\}$. This reproduces the approximation of R. Fox, Phys. Rev. A **34**, 4525 (1986). The relation between these approximations has been discussed in detail by P. Grigolini, Phys. Lett. **A119**, 157 (1986) and J. Masoliver, B. J. West, and K. Lindenberg, Phys. Rev. A **35**, 3086 (1987). See also the contributions of these authors in Ref. 16.