Special relativity and interferometers

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The Lorentz group, which is the language of special relativity, is a useful theoretical tool in modern optics. It is shown that the SU(1,1) interferometer of Yurke, McCall, and Klauder can serve as an analog computer for Wigner's little group of the Poincaré group.

One hundred years ago, the Michelson interferometer played a decisive role in the development of the special theory of relativity. These days, it is expected that a new generation of gravitational wave detectors will be based on interferometers.[1] The mathematics of Lorentz transformations[2] is an important theoretical tool both in the design of these new instruments[3] and in the study of coherent-state and squeezed-state light sources.[4,5] At the same time, modern optics may prove useful in studying special relativity.[6] As is illustrated in Fig. 1, this situation is like the analogy between the forced harmonic oscillator and the driven *LCR* circuit through a second-order differential equation.

In their paper,[3] Yurke *et al.* presented a group-theoretical approach to the analysis of interferometers. They noted that the conventional interferometers such as those of the Mach-Zender or Fabry-Perot type can be characterized by the group SU(2). They then introduced a class of interferometers characterized by SU(1,1) which can in principle achieve the phase sensitivity $\Delta\phi$ approaching $1/N$, where N is the total number of photons entering the interferometer.

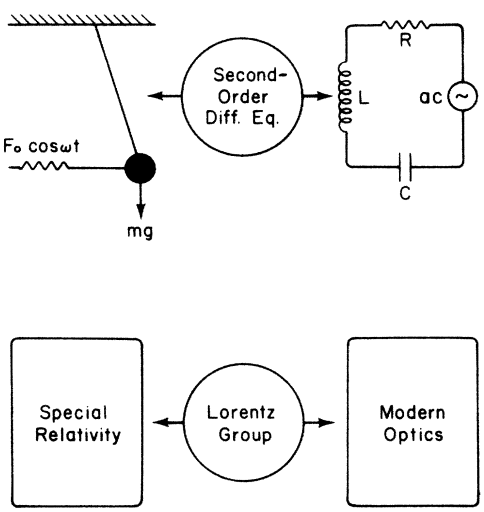

FIG. 1. Analogy of analogies. The analogy between the forced harmonic oscillator and the driven *LCR* circuit is well known. Since the Lorentz group is rapidly becoming one of the standard languages in optical sciences, there will be instances in which one formula in the Lorentz group will describe one physics in optics and another physics in special relativity.

The purpose of this paper is to show that the SU(1,1) interferometer of Yurke *et al.* can serve as an analog computer for Wigner's little group.[2] The role of Wigner's little group is illustrated in Fig. 2. The little group unifies the internal space-time symmetries of massive and massless particles,[7] as Einstein's $E=(m^2c^4+c^2p^2)^{1/2}$ gives the energy-momentum relations for both slow and rapidly moving particles.

Let us start with a massive particle at rest whose four-momentum is

$$(0,0,0,m)\ . \tag{1}$$

We use the four-vector convention: $x^\mu=(x,y,z,t)$. We can boost the above four-momentum along the z direction with a parameter λ,

$$p=m(0,0,\ \sinh\lambda,\ \cosh\lambda)\ . \tag{2}$$

The 4×4 matrix which transforms the four-vector of Eq. (1) to that of Eq. (2) is

$$A(\lambda)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\lambda & \sinh\lambda \\ 0 & 0 & \sinh\lambda & \cosh\lambda \end{bmatrix}\ . \tag{3}$$

	Massive Slow	between	Massless Fast
Energy Momentum	$E=\frac{p^2}{2m}$	Einstein's $E=\sqrt{m^2+p^2}$	$E = p$
Spin, Gauge Helicity	S_3 S_1 S_2	Wigner's Little Group	S_3 Gauge Trans.

FIG. 2. Wigner's little group. As $E=(m^2c^4+c^2p^2)^{1/2}$ is one formula for the energy-momentum relation for massive and massless particles, the little group gives a unified picture of the internal space-time symmetries. This figure is from Ref. 7.

Let us next rotate the four-vector of Eq. (2) using the rotation matrix

$$R(\theta)=\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4}$$

We then boost the four-momentum along the negative x direction using the matrix

$$S(\eta)=\begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} , \tag{5}$$

where $\tanh\eta = 2\alpha(\sin\theta)/[1+\alpha^2(\sin\theta)^2]$, with $\alpha = \tanh\lambda$. We then rotate this vector using the rotation matrix of Eq. (4). The net result is

$$D(\lambda,\theta)=R(\theta)S(\eta)R(\theta) , \tag{6}$$

and this transformation leaves the four-momentum p invariant,

$$D(\lambda,\theta)p=p . \tag{7}$$

The kinematics of these three transformations is described in Fig. 3.

The multiplication of the three matrices is straightforward, and the result is

$$D(\lambda,\theta)=\begin{pmatrix} 1-bu^2(1-\alpha^2)/2 & 0 & -u & \alpha u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1-bu^2/2 & \alpha bu^2/2 \\ \alpha u & 0 & -\alpha bu^2/2 & 1+\alpha^2 bu^2/2 \end{pmatrix} , \tag{8}$$

where

$$u=-[\sin(2\theta)]/[1-\alpha^2(\sin\theta)^2]$$

and

$$b=1+(1-\alpha^2)(\tan\theta)^2 .$$

This complicated expression leaves the four-momentum p of Eq. (2) invariant. Indeed, if the particle is at rest with vanishing velocity parameter α, the above expression becomes a rotation matrix. As the velocity parameter α increases, this D matrix performs a combination of rotation and boost, but leaves the four-momentum invariant.

Let us approach this problem in the traditional framework.[4] The above transformation clearly leaves the four-

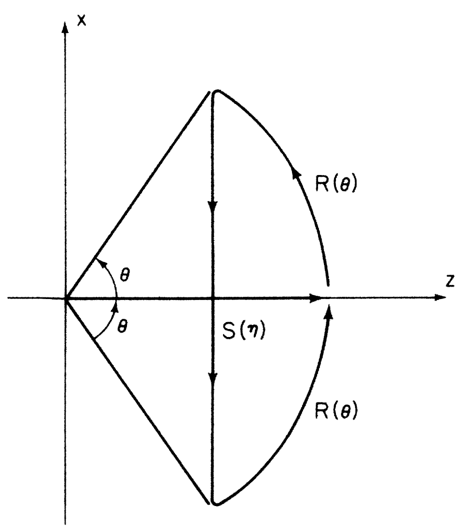

FIG. 3. The Lorentz kinematics based on the mathematics of the SU(1,1) interferometer of Yurke *et al.* The starting point is a massive particle moving along the z direction with its four-momentum given by Eq. (2). This momentum is rotated around the y axis, boosted along the x axis, and then rotated around the y axis, as shown in this figure. The net effect is a transformation which does not change the initial momentum. This is not an identity transformation, but a Lorentz-boosted rotation. This Lorentz-boosted rotation becomes a gauge transformation in the infinite-momentum, zero-mass limit.

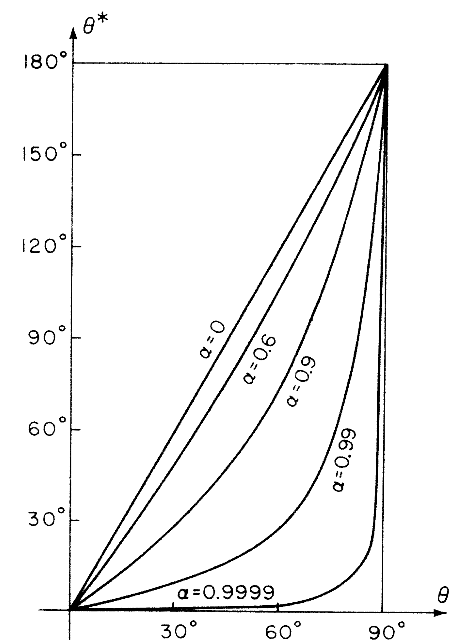

FIG. 4. The Wigner rotation angle vs laboratory-frame rotation angle. We have plotted θ^* as a function of θ for various values of α using Eq. (11). $\theta^*=2\theta$ when $\alpha=0$. θ^* is nearly equal to 2θ for moderate values of α, but it approaches 0 as α becomes 1.

momentum p invariant. Then we can boost the particle with its four-momentum p by A^{-1} until the four-momentum p by A^{-1} until the four-momentum becomes that of Eq. (1), rotate it around the y axis, and then boost it by A until the four-momentum becomes p of Eq. (2). This rotation in the rest frame is called the *Wigner rotation.*[8] The transformation of the O(3)-like little group constructed in this manner should take the form

$$D(\lambda,\theta)=A(\lambda)W(\theta^*)A^{-1}(\lambda)\ , \tag{9}$$

where W is the Wigner rotation matrix

$$W(\theta^*)=\begin{bmatrix} \cos\theta^* & 0 & \sin\theta^* & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta^* & 0 & \cos\theta^* & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{10}$$

The D matrix of Eq. (9) serves the same purpose as that of Eq. (6). Thus

$$A(\lambda)W(\theta^*)A(-\lambda)=R(\theta)S(\eta)R(\theta)\ . \tag{11}$$

By taking the traces of both sides of the above equation, we can calculate the Wigner rotation angle θ^*. The result is

$$\cos\theta^*=[1-(1-\alpha^2)(\tan\theta)^2]/[1+(1-\alpha^2)(\tan\theta)^2]\ . \tag{12}$$

In Fig. 4, the Wigner rotation angle θ^* is plotted as a function of θ. θ^* becomes 2θ when $\alpha=0$ and remains approximately equal to 2θ when α is smaller than 0.6. θ^* vanishes when $\alpha\rightarrow 1$. Indeed, for a given value of θ, it is possible to determine the value of θ^* which is the rotation angle in the Lorentz frame in which the particle is at rest. In the limit of $\alpha\rightarrow 1$, the D matrix takes the form

$$D(u)=\begin{bmatrix} 1 & 0 & -u & u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1-u^2/2 & u^2/2 \\ u & 0 & -u^2/2 & 1+u^2/2 \end{bmatrix} . \tag{13}$$

This matrix performs a gauge transformation on the four-vector of a photon traveling in the z direction.[9] This means that Lorentz-transformed rotations become gauge transformations in the infinite-momentum, zero-mass limit, as is illustrated in Fig. 2.

Let us go back to Eq. (11). This equation allows us to design a new interferometer.[3] This equation allows us to make a quantitative analysis of Wigner's little group. Indeed, the interferometer of Yurke *et al.* serves two important purposes. It also suggests that there may be many more cases in which modern optics will produce analog computers for special relativity.

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