Ion collection by probes in strong magnetic fields with plasma flow

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Fluid-theory calculations are presented of ion collection by electric probes in strongly magnetized plasmas with parallel flow. In the first calculations the problem is treated in a one-dimensional approximation but the cross-field transport of momentum is included in such a way as to model different ratios of viscosity to diffusivity. The results show that the flow deduced from probe measurements is not particularly sensitive to the assumed viscosity, provided it is finite. However, results with zero viscosity are qualitatively different from those with nonzero viscous momentum transport. The second set of calculations is two dimensional but only for fixed (unity) ratio of viscosity to diffusivity. The results are in remarkably good agreement with the corresponding one-dimensional model.

I. INTRODUCTION

The long-recognized difficulty of electric probe theory in the presence of strong magnetic fields[1–5] has received renewed attention recently. In part, this is because of the increasing use of such probes in the edge regions of magnetic confinement fusion experiments[6] to measure such basic parameters as temperature, density, and potential. In part, though, it is also because probe measurements are, in principle, able to determine other quantities such as flow velocity[7] and power flux.[8] The rather crude heuristic approach to probe interpretation which appears sufficient for the more basic parameters is really not satisfactory for obtaining the other parameters quantitatively. Therefore there has been a renewed incentive to obtain a more complete interpretative theory which can indicate whether and how these other parameters can be deduced from probe measurements.

The basic difficulty with probe theory in a magnetic field that is strong enough to give an ion gyroradius ρ_i, substantially smaller than the probe radius a, is that ion collection across the field is diffusive.[9] The quasineutral presheath region, in which acceleration of the ions occurs into the sheath, becomes highly elongated along the field, until the cross-field diffusion is able to balance the parallel collection flow. In such a situation the perpendicular flow cannot be modeled by collisionless probe models of the type pioneered by Langmuir[10] because it is governed by the transport processes. On the other hand, an entirely diffusive theory such as that of Bohm,[1] in which the parallel flow (as well as the perpendicular) is diffusive, is not satisfactory either because for most situations the parallel ion flow is dominated by inertia, not collisions. This is just as well because if the parallel collection were diffusive the ion current would be determined by the diffusivity, which is unknown, rather than the temperature and density, which is what we usually want to measure first.

The approach that has been widely used in the past for deducing the temperature and density from probe characteristics is to assume that the electron current is proportional to a Boltzmann factor in the region of the characteristic close to the floating potential and that the ions are collected by parallel flow at a rate corresponding to the Bohm current density $[\frac{1}{2}n_e(T_e/m_i)^{1/2}]$. These assumptions give plausible values of the electron density (n_e) and electron temperature (T_e) in, for example, the scrape-off layers of magnetic confinement plasmas (although there is rarely any fully independent quantitative verification of the density deduced). However, despite its success, this approach provides no information on another parameter of considerable interest: the parallel flow velocity.

Recent measurements using directional "Mach" or "Janus" probes[11–13] which measure separately the currents collected parallel and antiparallel to the magnetic field have shown that large differences in these currents often exist. As implied by "Mach probe," these differences are usually attributable to plasma flow velocities along the field. However, in the absence of a detailed probe theory, the deduction of the flow velocity can be based only upon *ad hoc* assumptions about the relationship between flow velocity and, for example, the ratio of the upstream to downstream ion saturation currents. Proudfoot *et al.*[11] have advocated a simple expression for the ion current ratio: $\exp(M/0.6)$, where M is the Mach number. This expression is based primarily on fits to their observations within the edge regions of the DITE tokamak.[14] The logical difficulty with this approach is that no independent measurements of the velocities were available and so the coefficient was chosen to match the expected velocities as predicted by edge plasma flow models, which are themselves probably just as uncertain as the unsatisfactory probe theories.

A one-dimensional fluid theory has been developed by Stangeby,[15] which offers a direct solution for the relationship between the ion current ratio and the flow velocity. Harbour and Proudfoot[7] compared Stangeby's results with a naive particle model, which does not take into account ion acceleration in the presheath, and found a very large difference in the predicted ratio (by about a factor of 10 at $M=1$). Their *ad hoc* expression, cited above, lies about halfway between these two extremes.

More recently, Hutchinson[16] has argued that Stangeby's model gives unreliable results because it omits essential cross-field transport terms that correspond to perpendicular viscosity. This work, henceforward referred to as paper I, showed that including a viscosity corresponding to a momentum diffusivity equal to the particle diffusivity leads to a much larger predicted current ratio than Stangeby's model. Although the viscosity value in paper I is arguably the most plausible one to take, there is no complete transport model which could provide a precise prescription of the viscosity because the cross-field transport is inherently anomalous. (That is, it is enhanced relative to the classical collisional theory.) Therefore there remains a degree of uncertainty in the applicability of the paper I results corresponding to the uncertainty in the viscosity-to-diffusivity ratio.

The present work develops an extension of the one-dimensional fluid theory to cases where the viscosity-to-diffusivity ratio can take any prescribed value. Thus the present model encompasses the fluid models of Stangeby and paper I as particular cases of a more general treatment. Numerical solutions of the equations are presented for a range of values of the viscosity. These show that the zero viscosity case of Stangeby is actually singular so that the inclusion of any finite amount of viscosity qualitatively changes the solution. This partly accounts for the large quantitative differences between the Stangeby and paper I results. The present results are much closer to the paper I values, when the viscosity has plausible values, than they are to those of Stangeby. The residual dependence of the ion current ratio on viscosity value is a cause for some concern for velocity diagnosis until there is an independent verification of the best value to adopt. However, one might take a more optimistic view and regard it as an opportunity to use probes to *measure* the viscosity, in plasmas where the flow velocity is known, using the interpertative values presented here.

A notable limitation of these theories is that they all use a one-dimensional (1D) approximation to what is, in fact, a two dimensional or even three-dimensional situation. The question has thus far been open as to how accurate one can expect such theories to be, given this approximation. Particularly, if one wants to explore the subtleties of the precise viscosity value one might find that these effects are swamped by the errors inherent in making the one-dimensional approximation. For this reason a two-dimensional model has been constructed and solved numerically for comparison with the one-dimensional results. Naturally, the difficulty in a two-dimensional analysis far exceeds that of the one-dimensional approximation. For this reason, the 2D code whose results are presented here treats only the case which corresponds to paper I: unity viscosity-to-diffusivity ratio. However, the results obtained show quite remarkable quantitative agreement with the corresponding 1D results. This agreement lends greatly increased confidence to the whole one-dimensional analysis and its results.

A brief preliminary report of the present work has been given elsewhere.[17] Here, both the methods and results are reported in more complete detail. Section II presents the fluid equations and their reduction to the one-dimensional approximate forms. Section III gives the solution method and the one-dimensional results. In Sec. IV the 2D code is described and its results presented. Section V seeks to explain some of the observed results and outlines remaining issues.

II. FORMULATION

The equations which we take as governing the ion fluid around the probe are

$$\nabla\cdot(n_i \mathbf{v})=0 , \tag{1}$$

$$\nabla\cdot(n_i m_i \mathbf{v} v_\parallel)-\nabla\cdot(\eta\nabla v_\parallel)=-\nabla_\parallel p_i+Zen_iE_\parallel , \tag{2}$$

$$n_i\mathbf{v}_\perp=-D\nabla_\perp n_i . \tag{3}$$

Here, D and η are phenomenological diffusivity and viscosity, $\parallel$ and $\perp$ refer to the magnetic field direction (z), n_i, p_i, and $\mathbf{v}$ are the ion density, pressure, and velocity, respectively, and E is the electric field. These equations are supplemented by an assumption that in the cases of interest the majority of the electrons are repelled by the probe so that their density is governed by a Boltzmann factor. Therefore in the (quasineutral) plasma region the electric field is related to the ion density via

$$E_\parallel=-\nabla_\parallel(T_e/e)\ln(n_i/n_\infty)=-(T_e/en_i)\nabla_\parallel n_i , \tag{4}$$

where the electron temperature T_e is taken as constant and subscript ∞ refers to quantities far from the probe, in the unperturbed plasma. Finally, we need to close the equations with an ion energy equation. For simplicity we take this to be $p_i \propto n_i^\gamma$ so that

$$\nabla_\parallel p_i=\gamma T_i\nabla_\parallel n_i , \tag{5}$$

where T_i is taken as constant.

Some discussion is in order about the anticipated validity of this fluid approach. The treatment of the perpendicular dynamics by a fluid approach will be justified, as is well known, provided that the ion gyro radius is much smaller than the perpendicular length scales of interest: in this case the probe transverse dimension a. The fluid treatment of the parallel dynamics will be less satisfactory unless the ion-ion collisionality is high. This requires the ion-ion mean free path l_{ii} to be much shorter than the length of the collection presheath, L_c, say. This is in fact the case in many of the magnetic fusion applications of interest, but even if it were not, the fluid model turns out to give quite good agreement with kinetic collisionless calculations, as indicated by comparisons of one-dimensional sheaths, for example, Refs. 18–20. A more important issue involves ion-electron collisions, which are ignored in the model. This will be satisfactory if $l_{ie}/L_c \gg 1$, which again is usually well satisfied in fusion plasmas. If ion-electron collisions were not negligible then the ion collection would be diffusive and the present theory would be inapplicable. In order for the quasineutral approximation to be satisfied over the relevant domain requires that the sheath thickness be small. Since the sheath has a thickness typically a few times the debye length λ_D, this requires $\lambda_D/a \ll 1$, again generally easily satisfied.

The viscosity η is generally anisotropic. However, we shall drop the viscous terms arising from parallel gradients $\nabla_\parallel \eta \nabla_\parallel v_\parallel$, so the viscosity appearing in Eq. (2) is to be taken as that for perpendicular transport of parallel momentum. It is this step which strictly requires the ions to be self-collisional.

The ion energy equation (5) is capable of describing a locally adiabatic or isothermal ion fluid, but since the problem is nonlinear we are not fully justified in adopting $T_i=\text{const}$ unless the ions are isothermal. Thus we must regard this as a simplifying assumption and the precise value of γ as open. In the justification of this approach it may be noted that the dominant term on the right-hand side of Eq. (2) is often the second, and, even if we knew the "correct" value for γ, probe measurements do not generally give T_i, so we should still be uncertain as to how to account correctly for the ion pressure term. Within the present theoretical context we avoid having to decide this issue because we write the right-hand side of Eq. (2) as $-m_i c_s^2 \nabla_\parallel n_i$, where

$$c_s \equiv [(ZT_e+\gamma T_i)/m_i]^{1/2} . \tag{6}$$

Then we assume that it is sufficient to express velocities as multiples of the sound speed c_s.

One other major limitation to the applicability of the treatment should be mentioned, namely, that no volumetric particle sources are included. This exclusion of the effects of ionization and recombination is usual in Langmuir probe theory. It will only be justified, in general, if the mean free path for ionization (l_n) of the neutrals formed by recombination at the probe surface is much bigger than the probe dimensions. Otherwise the local buildup of combined neutral and ion density will tend to perturb the results.

As a numerical example of the typical situation in magnetic confinement edge plasmas consider a case where $T_e=T_i=10$ eV, $n_e=10^{19}$ m^{-3}, $B=4$ T, $a=0.002$ m, and $D=T_e/16eB$ (the Bohm value). The presheath length is approximately $L_c=c_s a^2/D$ and then the different characteristic lengths are $\lambda_D=8\times10^{-6}$ m, $\rho_i=(m_i T_i)^{1/2}/eB=8\times10^{-5}$ m, $l_{ii}=8\times10^{-2}$ m, $l_{ie}=100$ m, $l_n\approx5\times10^{-2}$ m, and $L_c=2$ m. These characteristic lengths confirm the remarks made above about the typical validity of the fluid approach.

We now perform the following nondimensionalizing transformations:

$$z'=\int \frac{D}{c_s a^2}dz,\quad x'=\frac{x}{a},\quad y'=\frac{y}{a}\,,\qquad n=n_i/n_\infty,\quad M=v_\parallel/c_s\,. \tag{7}$$

Substituting for the perpendicular velocity from Eq. (3), ignoring perpendicular derivatives of D, and dropping the primes on the new coordinates for brevity, these transformations bring the equations into the form

$$\frac{\partial}{\partial z}(nM)=\nabla_\perp^2 n=0\,, \tag{8}$$

$$\frac{\partial}{\partial z}(nM^2+n)-\nabla_\perp\cdot M\nabla_\perp n-\nabla_\perp\cdot\frac{\eta}{m_i n_\infty D}\nabla_\perp M=0\,. \tag{9}$$

Now, for the purpose of reducing these equations to approximate one-dimensional forms, we subtract M times Eq. (8) from Eq. (9) and ignore the derivative of η (as well as D). Then the momentum equation becomes

$$\frac{\partial n}{\partial z}+nM\frac{\partial M}{\partial z}=(\nabla_\perp M)\cdot(\nabla_\perp n)+\frac{\eta}{m_i n_\infty D}\nabla_\perp^2 M\,. \tag{10}$$

We now substitute for the perpendicular derivatives of any quantity ψ via $|\nabla_\perp\psi|\to(\psi_\infty-\psi)$ and $\nabla_\perp^2\psi\to(\psi_\infty-\psi)$. (Recall that the perpendicular distances have been made dimensionless by scaling to the probe size a.) The one-dimensional equations we then get are

$$M\frac{dn}{dz}+n\frac{dM}{dz}=1-n\,, \tag{11}$$

$$\frac{dn}{dz}+nM\frac{dM}{dz}=(M_\infty-M)\left[1-n+\frac{\eta}{m_i n_\infty D}\right]. \tag{12}$$

III. ONE-DIMENSIONAL SOLUTIONS

We recognize that the viscosity in a medium where transport is via particle exchange has a value $\eta=mnD$. It has been argued in paper I that this value seems the most plausible one in the usual situation of probe measurements, where the cross-field transport is dominated by turbulence. However, our purpose here is to allow different values of viscosity so as to explore its effect. Therefore we put

$$\eta=\alpha m_i n_i D \tag{13}$$

and regard α as a constant. The case $\alpha=0$ is essentially that of Stangeby[15] and $\alpha=1$ is paper I.

With this substitution we follow the approach of paper I, reducing Eqs. (11) and (12) to

$$\frac{dn}{dz}=\frac{(1-n)M-(M_\infty-M)[1-n(1-\alpha)]}{M^2-1}\,, \tag{14}$$

$$\frac{dM}{dz}=\frac{(M_\infty-M)[1-n(1-\alpha)]M-(1-n)}{n(M^2-1)}\,, \tag{15}$$

and hence obtaining

$$\frac{dn}{dM}=n\frac{(1-n)M-(M_\infty-M)[1-n(1-\alpha)]}{(M_\infty-M)[1-n(1-\alpha)]M-(1-n)}\,. \tag{16}$$

The sheath edge is the point at which $d/dz\to\infty$, i.e., $M^2=1$. Choosing the positive sign to denote flow towards the probe, the boundary condition at the probe is thus $M=1$. Naturally we anticipate that the density there will be $n<1$ since an accelerating potential drop will be required to draw in the ions at the sound speed. At $z\to-\infty$, far from the probe, we take $n=1$ and $M=M_\infty$. Then, in order to integrate Eq. (16) from $M=M_\infty$ to $M=1$, we require the slope at $M=M_\infty$. To obtain this condition we expand $n(M)$ as a Taylor series to first order in $M-M_\infty$ about that point. Then we substitute the expansion into Eq. (16), retaining only the lowest-order terms, and obtain a quadratic equation for the slope, whose solution is

$$\left.\frac{dn}{dM}\right|_{M_\infty} = \tfrac{1}{2}(-M_\infty(1-\alpha)\pm\{[M_\infty^2(1-\alpha)^2+4\alpha]\}^{1/2}) . \tag{17}$$

Now consider the nature of the solution in the M-n plane. The plane is divided into different regions by the boundary curves $ndn/dz=0$ and $ndM/dz=0$. In each of these regions the slope of the solution has a specific sign and the sign changes if it crosses a boundary. The point $M=M_\infty$, $n=1$ is at a point of intersection of the boundary lines. It may be shown that this is the vertex of a region which extends uninterrupted as far as the line $M=1$, $0<n<1$. The upper boundary of this region is $ndM/dz=0$, the lower boundary is $ndn/dz=0$, and the slopes of both boundary lines are negative at the vertex for $\alpha>0$. These facts are sufficient to guarantee that any solution of the differential equation which passes through the vertex into the region will remain within the region and extend monotonically to the boundary $M=1$, $0<n<1$. Further analysis indicates also that there is no other continuous solution which links $M=M_\infty$, $n=1$ to $M=1$, $0<n<1$, again provided $\alpha>0$. Therefore the correct choice for the boundary condition is the negative sign in Eq. (17), which gives the required solution at the vertex. When $\alpha=0$ the angle at the vertex becomes zero because the two boundary lines become $n=1$. This causes a singularity in the solution and the only numerically stable solution is obtained with the positive sign in Eq. (17). This problem will be discussed further later.

Equation (16) is solved for n as a function of M, given M_∞ and the boundary conditions discussed, by simple finite differences supplemented by conditions which prevent the solution from crossing the boundaries of the region indicated above. These conditions assist in stabilizing the solution in the vicinity of the vertex but otherwise have no effect.

Figure 1 shows a sample of the solutions for the density n as a function of Mach number M. Each of the subfigures is for a specific value of α. The family of curves shown corresponds to solutions with different values of M_∞. Each curve starts at the point $n=1$, $M=M_\infty$ and ends at $M=1$. Thus the starting points give the external flow field, with negative M_∞ corresponding to the downstream side of the probe and positive M_∞ to the upstream side, while the end points give the value of the density at the sheath edge. Since the velocity at the sheath edge is equal to the sound speed, the density also gives the ion flux into the sheath, and hence to the probe.

Solutions like Fig. 1 are sufficient to give the measurable quantities required for probe interpretation. If we want instead to obtain the variation of density (or potential or velocity) with *position* then we must integrate Eq. (14) or (15), regarded as an equation for z in terms of n,M. An example of this process was shown in paper I; it gives the presheath structure in space and, as expected, gives a presheath in which the perturbation falls off in a characteristic distance $\sim c_s a^2/D$ and tends to zero ($n=1$, $M=M_\infty$) as z tends to infinity.

The exception to this behavior is the $\alpha=0$ case. As shown by Fig. 1(d) the solutions in this case are qualitatively different from the finite-α cases in that they are not monotonic. Moreover, for $M_\infty<0$ the solutions all pass through the point $n=1$, $M=0$. [The curves of Fig. 1(d) were actually generated by the same numerical code used to solve the other cases but the results are the same as the analytical solution given by Stangeby.] The result of an integration of Eq. (15) to express the results in terms of spatial variation shows that the branch of the $M_\infty<0$ curves from $M=1$ to $M=0$ transforms to the interval $z=0$ to $z=-\infty$. In other words, the point $n=1$, $M=1$ corresponds to the point at infinity. Mathematically this is because $dM/dz\to 0$ there. What this result indicates is that M does not tend to the external value M_∞ as $z\to-\infty$. Rather, the presheath length, from the viewpoint of velocity perturbation, is infinite, even though the density-perturbation length is finite. The physical explanation for this interesting result is that in the complete absence of viscosity, the only momentum transport is by convection. Thus if the particle transport tends to zero, because there is no density difference between the inside of the presheath and the surrounding plasma outside, then momentum transport also tends to zero even if there is a velocity difference. The inner velocity can thus tend to a value different from M_∞, and it does: to zero. Clearly, the slightest amount of viscosity prevents this behavior; so $\alpha=0$ is a singular case. Stangeby showed solutions in which the $M<0$ part of the curve was at a finite distance from the probe. This was because he assumed that the source of particles in the sheath, modeling the cross-field flux, was a positive constant, independent of space. This is clearly an unphysical assumption because it would imply particle diffusion *up* the density gradient. Our present formulation avoids this assumption and, as a consequence, the $M<0$ part of the $\alpha=0$ solution no longer has any physical significance.

The important results of a series of solutions of the type shown in Fig. 1 can be summarized by plotting the density at the sheath edge ($M=1$) versus flow Mach number (M_∞) for various values of α. This is shown in Fig. 2. Note how the $\alpha=0$ results deviate substantially even from the $\alpha=0.01$ results when $M_\infty<0$. Note also that the $\alpha=0.1$ case is closer to the case $\alpha=1$ at $M_\infty\sim-1$ than it is to $\alpha=0$. These facts just emphasize that for any finite viscosity the $\alpha=0$ solution is a bad choice.

The important quantities for diagnosis are the mean value and the ratio of the ion collection flux densities upstream and downstream. In Fig. 3 is shown the mean ion flux, normalized to the product $n_\infty c_s$, as a function of plasma flow velocity, for several values of the viscosity ratio α. The mean flux proves to be a relatively weak function of both M_∞ and α. This is fortunate because it means that density measurements using the Bohm formula for ion saturation current,

$$I_{si}=0.5en_e A(ZT_e/m_i)^{1/2} , \tag{18}$$

will give reasonably accurate results, even in the presence of parallel flow, using for A the area of the projection of the probe in the parallel direction. One anticipates that the values of flux rather less than $0.5n_\infty c_s$ for $\alpha\sim1$ are partly compensated by the extra term γT_i within the definition of c_s, as discussed in paper I.

The ratio of the fluxes to the upstream and downstream sides of the probe are shown in Fig. 4. Such curves enable one to deduce the velocity from measurements, for any specific choice of α, using the flux ratio, which will be the ratio of ion saturation currents. Notice that the uncertainty in deduced flow Mach number arising from uncertainty in α is about $\pm 15\%$ for ratios less than 5 if α lies between 0.1 and 1. We exclude the $\alpha=0$ case because of its singularity. It is interesting to note that Harbour and Proudfoot's *ad hoc* formula $\exp(M/0.6)$ lies extremely close to the $\alpha=0.1$ curve, although this in itself is no real indication of its appropriateness.

Another way of showing the α dependence of the result is given in Fig. 5, where we show the slope at $M_\infty=0$ of the flux ratio versus Mach number curve $dR/dM_\infty|_0$ plotted as a function of α. This parameter determines the velocity "calibration" of a Janus probe at low velocities. We include α values up to 2 since there is no reason in principle why values greater than 1 should be excluded.

IV. TWO-DIMENSIONAL CALCULATIONS

Since the one-dimensional results involve an approximation whose accuracy is uncertain, it is of considerable interest to obtain some results based on fully two-dimensional solutions to the fluid equations (8) and (9), which can be compared with our 1D results. For this purpose a code has been developed to solve the equations

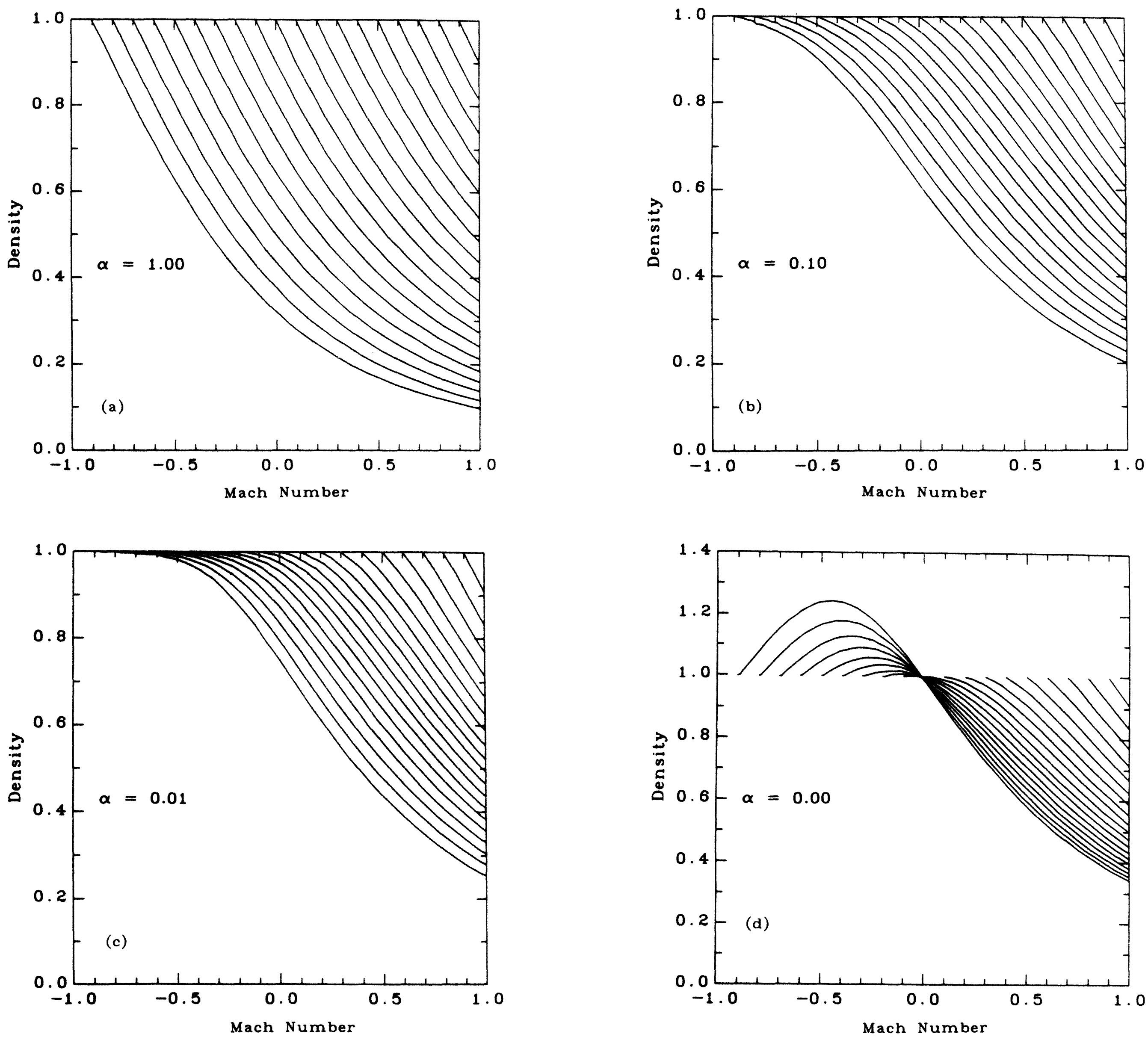

FIG. 1. Solutions for density as a function of the Mach number in the presheath. Different values of the viscosity-to-diffusivity ratio are shown: (a) $\alpha=1$, (b) $\alpha=0.1$, (c) $\alpha=0.01$, (d) $\alpha=0$.

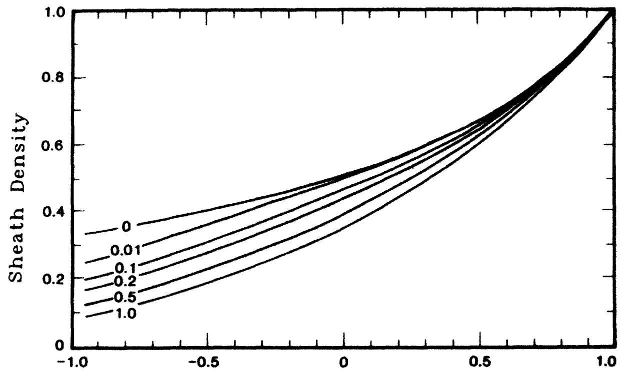

FIG. 2. Variation of the density at the sheath edge with the external flow velocity, for various viscosity-to-diffusivity ratios. The flow velocity at the sheath edge is equal to the sound speed. Therefore the collection flux density is equal to this density (times the sound speed).

corresponding to the $\alpha=1$ case. Although much more general codes exist (such as that developed by Braams[21]) which include general electron and ion momentum and energy equations, their generality is more of a handicap than an asset when handling a simplified model such as this. The equations that are integrated by the present code are

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial z}Mn-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial n}{\partial r}=0\ ,$$
$$\frac{\partial}{\partial t}Mn+\frac{\partial}{\partial z}(M^2n+n)-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}Mn=0\ . \tag{19}$$

These are the cylindrical forms of Eqs. (8) and (9) with the substitution $\eta=n_i m_i D$, including the time derivative terms omitted previously. [$r=(x^2+y^2)^{1/2}/a$ is the radius normalized to the probe radius a.] The solution is

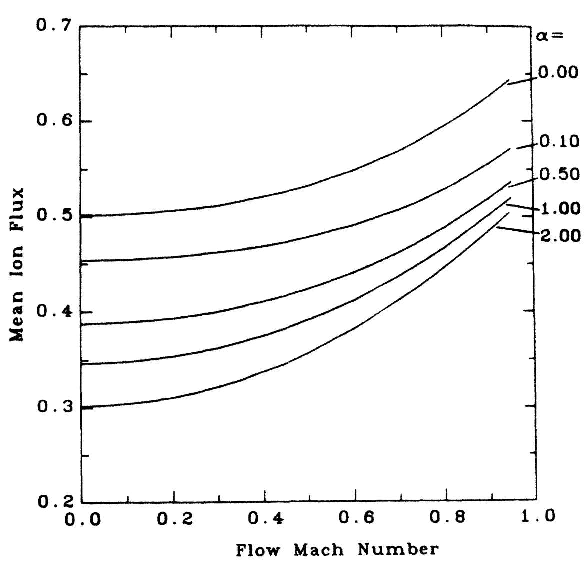

FIG. 3. Average of the upstream and downstream flux densities to the probe as a function of the external flow velocity.

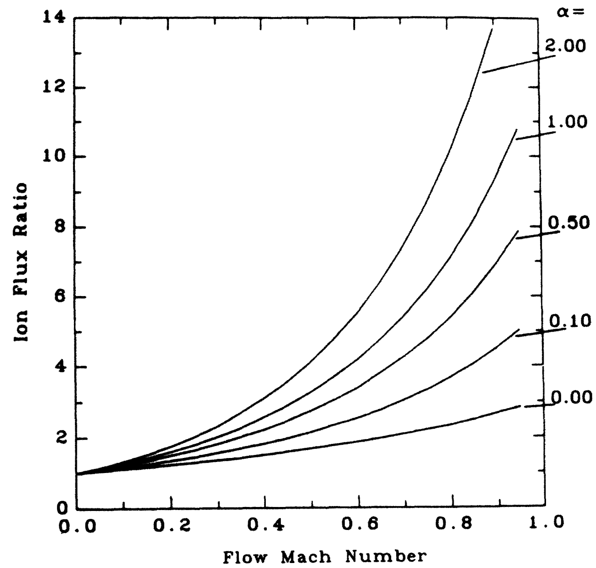

FIG. 4. Ratio of the upstream to downstream ion collection flux vs external flow velocity.

obtained by stepping forward in time until convergence, which then gives the steady state.

The method used to advance the equations is to regard them as conservation equations for the two dependent variables n and $\mu=nM$. An alternating direction scheme is used, in which the perpendicular direction step is implicit and in the parallel direction a two-step Lax-Wendroff scheme, of the type described by Richtmyer and Morton,[22] is used. This has the merit of treating the shock transition at the probe quite accurately but the disadvantage of requiring small time steps for stability. The equations are solved on a spatial mesh which is uniform in the r direction but nonuniformly spaced, proportional to $|z|^{1/2}$, in the parallel direction, with size 16×40 over the region $0<r/a<2$, $-4<z<+4$. Tests with different mesh spacings and solution-domain extents show that the results are adequately converged with these choices.

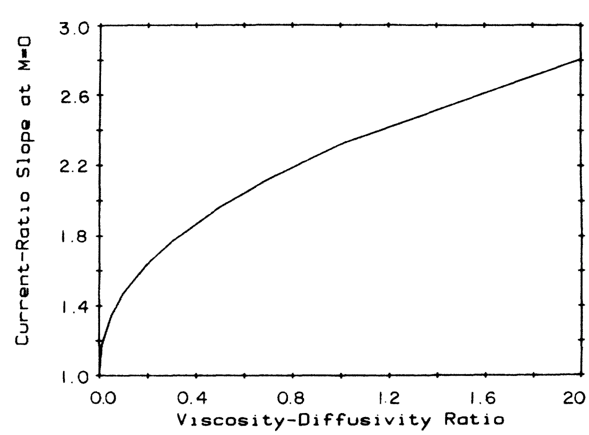

FIG. 5. Slope of the ion current ratio (curves of Fig. 4) at the point $M=0$, plotted against the viscosity-to-diffusivity ratio α.

The boundary conditions used are $\partial n/\partial r=\partial(nM)/\partial r=0$ at $r=0$; $n=n_\infty$, $\mu=\mu_\infty$ at $r=2$; $M=\mu/n=\pm 1$ at $z=0\mp\delta$ as $\delta\to 0$ from above, $r<1$; n,μ continuous at $z=0$, $r>1$; $n=n_\infty$ at $z=\pm 4$. These conditions are sufficient for the order of the equations. It should be noticed that no explicit boundary condition on μ is required at $z=\pm 4$. An implicit condition, necessary for the numerical scheme, is derived from the first of Eqs. (18): $\partial\mu/\partial z=0$.

The steady-state solutions for the density and flux are shown in Fig. 6 for the case of zero flow, $M_\infty=0$. This case, of course, gives rise to symmetric density and antisymmetric flux solutions about the line $z=0$. Because of the scaling of the parallel mesh proportional to $|z|^{1/2}$, the singularities there are removed. The density then has finite slope and the flux has zero slope at $|z|^{1/2}=0$. When there is nonzero flow in the plasma, the solution is no longer symmetric, as Fig. 7 illustrates. For flow Mach numbers greater than about 0.5, the upstream flux is perturbed very little by the probe. Thus in Fig. 7(b) the flux is almost uniform for $z<0$. On the downstream side, however, an increasing flux variation requires, as expected, an increasing potential and hence density depression. The presheath also lengthens in the downstream direction to the point where the boundary condition begins to introduce artificial oscillations in the parameters. These should not be considered physically significant. They do not appear to change noticeably the flux at the probe, which is the parameter of experimental significance.

Three values of the ion collection flux, as a function of flow Mach number, are shown in Fig. 8: the mean value across the probe of the probe flux from the two-dimensional calculations, the value of the flux at $r=0$, and for comparison the one-dimensional result from Fig. 2 for $\alpha=1$. The remarkable fact about this comparison is that the three values are so similar. For the two two-dimensional results this reflects the fact, evident in Figs. 6 and 7, that the flux is very uniform across the surface of the probe. The experimental significance of these two flux values is that a simple probe can measure the mean flux, while some types of Mach probes are designed with collector elements located at the probe axis which are much smaller than the entire probe shield; thus they measure the flux at $r=0$. The excellent agreement with the one-dimensional calculations provides greatly increased confidence that the one-dimensional results for a variety of viscosity values, presented in Sec. III, are good approximations to what would result from a two-dimensional calculation.

Focusing on the differences, which are most important at large flow velocities on the downstream side, the progressive falling off of the mean value below that at $r=0$

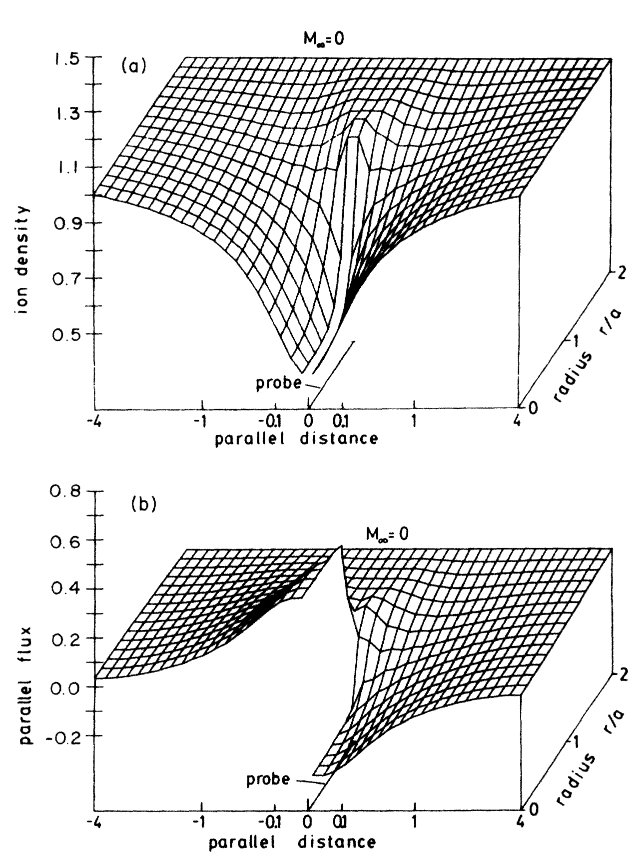

FIG. 6. Axonometric plot of the two-dimensional solution in the case when the external flow is $M_\infty=0$. (a) Density, (b) parallel flux.

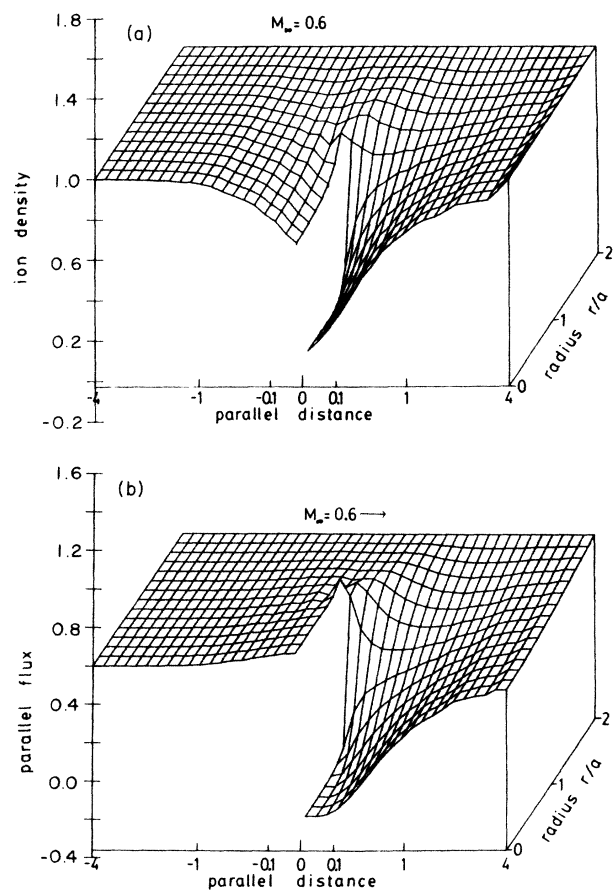

FIG. 7. Plots corresponding to Fig. 6 except that the external flow velocity is nonzero, $M_\infty=0.6$.

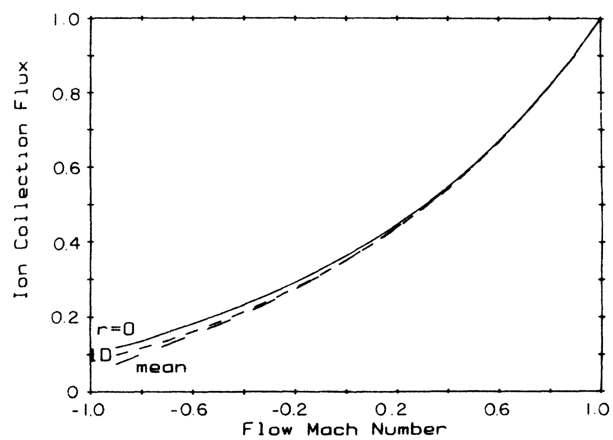

FIG. 8. Ion collection flux as a function of external flow velocity for three cases of interest: the two-dimensional solution value at "$r=0$," its "mean" value across the probe surface, and the "1D" one-dimensional solution for corresponding α.

reflects the increasing importance of reduced flux at the probe edge. The flux just beyond the edge is close to the external value, which is directed away from the probe. The large radial gradient of the flux which is necessary at the probe edge leads to an important boundary effect on the mean collection flux.

The ion flux ratios corresponding to these three cases are shown in Fig. 9. Again, these are the "calibration curves" for the use of Janus-type probes for velocity measurements. In the interest of having a convenient approximate analytic form for use in probe interpretation, one can fit these curves with equations of the type used by Harbour *et al.* The one-dimensional approximation is well fitted by $\exp(M_\infty/0.41)$ and the two-dimensional ($r=0$) by $\exp(M_\infty/0.45)$. Thus the deduced velocities using these two curves would differ by $\sim 10\%$. The curve of mean flux is less well fitted by this functional form.

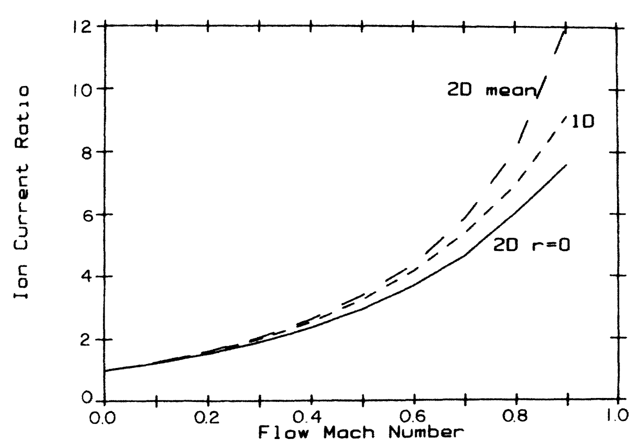

FIG. 9. Current ratios for the three solutions of Fig. 8.

V. DISCUSSION

The results we have given show that it is indeed possible to obtain the parallel flow Mach number from probe measurements provided we can decide an appropriate value to take for the ratio of the viscosity to the diffusivity. It has been argued in paper I that $\alpha \sim 1$ is the most plausible value to take when the diffusion is anomalous. In support of this contention one may cite also the brief discussion of the turbulent case by Braginskii[23] in his classic paper on collisional diffusion in plasmas. Nevertheless some uncertainty remains, and the present results show how the viscosity uncertainty translates into velocity uncertainty.

It is worth emphasizing again that our treatment has shown that neither the absolute value nor the spatial variation of the diffusivity has any direct influence on the ion collection flux, provided the conditions discussed in Sec. II are satisfied. This fact, which is demonstrated by the ability to transform the diffusivity away in our choice of nondimensionalized parallel distance (Eq. 7), can be used to help understand why the one-dimensional approximation seems to give such good agreement with the two-dimensional calculations. The argument is as follows.

Consider Eqs. (19). They are in the form of conservation equations. Therefore if we consider a tube of radius $r=a$ and integrate these equations over the perpendicular direction, we obtain equations for the average parameters within the tube. The cross-field terms are then in the form of derivatives $\partial n/\partial r$ and $(\partial/\partial r)(Mn)$ evaluated at the tube boundary. The one-dimensional approximation replaces these derivatives by differences $(n_\infty - n)/a$, etc. Clearly, this is in itself a relatively poor approximation, because the perpendicular scale length (at $r=a$) far from the probe becomes considerably longer than that near $z=0$, as Figs. 6 and 7 show. However, this variation of the perpendicular scale length with z is similar to parallel variation in D. It can be transformed away by a new scaling of the parallel coordinate. Thus the resulting probe flux values are unaffected by it. Actually, it cannot be completely scaled away because the scale length for n and for Mn need not be identical. Nevertheless, the scale-length variation is qualitatively similar to diffusivity variation in causing a variation primarily in the parallel extent of the presheath.

There may well be occasions when the presence of the presheath itself affects the value of the diffusivity, by exciting additional instabilities, for example. In these cases too it seems likely that the probe flux value should be little affected by this process because there is no direct dependence of the results on diffusivity.

It should be noted that the present results are limited to subsonic flow velocities. The upstream side collection can reasonably be taken as equal to the unperturbed flux in supersonic cases, such a value representing a straightforward recognition of the fact that no presheath need necessarily form. However, neither the one-dimensional nor the two-dimensional numerical schemes can deal with the downstream side of the probe when the external plasma flow velocity exceeds c_s. It seems that this difficulty is associated with the formation of shocks in the presheath. Mathematically it manifests itself as instability in the two-dimensional code and as the absence of continuous solutions in the one-dimensional analysis.

VI. CONCLUSION

A theoretical study has been presented of ion collection by probes in strong magnetic fields, using a fluid description of the plasma. The results obtained allow quantitative interpretation of Janus-type probe measurements to give the parallel flow velocity. Limited two-dimensional calculations agree very well with the one-dimensional approximate treatment, giving greater confidence in the wider one-dimensional study. Some uncertainty remains in the precise value of the ratio of viscosity to diffusivity that should be used. The sensitivity of the results to this ratio is not excessive, provided that the singular case of zero viscosity is avoided. However, some independent experimental measurements of flow velocity would be helpful to verify whether the ratio advocated here is indeed appropriate. The results indicate that the dependence on the Mach number of the upstream-to-downstream ion current ratio, required for the velocity measurement, may be described approximately by the expression $\exp(M/M_c)$, where the calibration Mach number M_c lies in the range of about 0.4–0.45.

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