Lethargy of laser oscillations and supermodes in free-electron lasers. I

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We propose an analytical solution to the propagation equation of the optical pulse both in free-electron lasers and in optical klystrons. Our results include in a natural way the lethargy and wave-packet spreading.

I. INTRODUCTION

The free-electron laser (FEL) is a coherent source of radiation in which the active medium consists of an ultrarelativistic electron beam moving in an N-period magnetic undulator with on-axis field strength B_0 and spatial period λ_u. The electrons executing transverse oscillations in the undulator emit radiation centered at

$$\lambda=\frac{\lambda_u}{2\gamma^2}(1+K^2) , \tag{1.1}$$

where γ is the relativistic factor and K is the undulator parameter specified in Table I. The emitted radiation is stored in an optical cavity and reinforced by a new copropagating electron beam. Then, if there is enough feedback, the system may work as an oscillator.[1] The electron-beam source is chosen according to the desired laser performance, and in fact a long-wavelength high-power FEL may require an induction LINAC,[2] a moderate-power ir FEL may require a LINAC (Ref. 3) or a microtron,[4] a visible or uv FEL may require a storage ring[5] (see also Ref. 6 where an analysis of the existing accelerators for FEL has been carried out).

If the accelerating system is provided by a radio-frequency (rf) field, the electron beam has a structure characterized by a series of microbunches with a distance fixed by the rf period and with a longitudinal length σ_z fixed by the phase-stable angle.[6]

It is well known that the bunched structure of the electron beam induces an analogous structure in the optical field, giving rise to a type of active phase locking.

The laser and electron bunch move at different velocities so that at the end of the undulator the laser pulse should be ahead of the electron bunch (slippage) by the quantity

$$\Delta=(1-\beta)N\lambda_u\cong N\lambda , \tag{1.2}$$

where β is the reduced electron velocity. Therefore, the laser bunch slips on the electron bunch, the "gain medium" for the optical field. Supposing that the two bunches are initially coincident, the front side of the optical packet will experience a decreasing electron density, while the backward side will undergo a larger amplification. This process may be visualized as a back-shift of the centroid of the laser pulse towards the trailing edge of the electron bunch. The main consequence of this process is a slowing down of the oscillation period of the laser pulse inside the optical cavity. This is the so-called "lethargy."

Let us introduce the dimensionless parameter

$$\mu_c=\Delta/\sigma_z , \tag{1.3}$$

which is a measure of the relative slippage between laser and electron pulse and also determines the number of longitudinal modes coupled by the FEL interaction.

The optical pulse propagation in FEL has been discussed in a number of papers.[7–9] Initially the problem has been treated using a numerical analysis. Only recently have analytical methods been developed,[10,11] thus yielding further insight into the physics of the process.

The proposed method concerns with the so-called long-electron-bunch regime which holds when $\mu_c \ll 1$, i.e., when the slippage is small compared with the electron-bunch length. The interest in this analysis is justified by the fact that for most of the existing rf FEL experiments μ_c is significantly less than the unity.[3,4]

In this paper we use the analytical solution of the pulse evolution found in Refs. 11 and we show that it naturally incorporates the lethargy. We study the problem both for the optical-field spatial configuration and spectrum. We include also the electron-beam emittances and energy spread, thus studying the effect of the inhomogeneous broadening on the FEL pulse dynamics. The possibility of getting a stationary solution of the supermode type[8] is also briefly discussed.

We will consider the FEL and optical klystron (OK) configurations, for which, as stressed in Ref. 11, the "states" characterizing the evolution may be understood as the "non-Hermitian" version of the Yuen and Glauber states, respectively.

The paper is organized into four sections. In Sec. II we discuss the FEL configuration; in Sec. III, the OK configuration; and Sec. IV is devoted to final comments and to the discussion of the inhomogeneous-broadening effects. The calculation details are reported in the Appendix.

TABLE I. List of the symbols used throughout the text.

Symbol	Definition
Physical constants	
e	electron charge
m	electron mass
c	light velocity
Electron-beam parameters	
E	electron-beam energy
$\gamma \equiv E/mc^2$	relativistic factor
σ_ε	rms relative energy spread
σ_z	rms longitudinal bunch length
$\sigma_{x,y}$	rms transverse (x,y) bunch size
$\sigma'_{x,y}$	rms transverse angular size
$\varepsilon_{x,y}=2\pi\sigma_{x,y}\sigma'_{x,y}$	rms (x,y) emittance
T_e	time distance between two adjacent bunches
$\hat{I}$	peak current
Undulator parameters	
λ_u	undulator wavelength
N	number of periods
L	undulator length
B_0	undulator peak magnetic field
$K=\frac{eB_0\lambda_u}{\sqrt{2}2\pi mc^2}$ (linear)	undulator parameter
$K=\frac{eB_0\lambda_u}{2\pi mc^2}$ (helical)	undulator parameter
$h_{x,y}$	undulator sextupolar terms
Optical cavity parameters	
L_c	optical cavity length
$T_c=\frac{2L_c}{c}$	nominal optical cavity round-trip period
Laser-beam parameters	
$\lambda=\frac{\lambda_u}{2\gamma^2}(1+K^2)$,	resonant wavelength
$\left(\frac{\Delta\omega}{\omega}\right)_0=\frac{1}{2N}$	homogeneous width
$\mu_\varepsilon=2\sigma_\varepsilon\left(\frac{\Delta\omega}{\omega}\right)_0^{-1}$	inhomogeneous broadening parameter due to the energy spread
$\mu_{x,y}=\sqrt{2\lvert h_{x,y}\rvert}\,\frac{1}{1+K^2}\frac{\gamma\varepsilon_{x,y}}{\lambda_u}\left(\frac{\Delta\omega}{\omega}\right)_0^{-1}$	inhomogeneous broadening parameter due to the emittance
$\Delta=N\lambda$	slippage distance
$\mu_c=\frac{\Delta}{\sigma_z}$	coupling parameter
δL	cavity detuning $[(L_c-\delta L)\equiv$ effective cavity length]
$\theta=\frac{\delta L}{\Delta g_0}$	delay parameter
$g_h^0=22\times10^{-4}N^2\frac{K^2}{1+K^2}\frac{\hat{I}\,(\mathrm{A})}{\gamma}$	gain coefficient (helical)
$g_1^0=22\times10^{-4}N^2\frac{K^2}{1+K^2}\frac{\hat{I}\,(\mathrm{A})}{\gamma}\left[J_0\left(\frac{K^2}{1+K^2}\right)-J_1\left(\frac{K^2}{1+K^2}\right)\right]$	gain coefficient (linear)
$J_{0,1}$	Bessel function

II. FEL OPTICAL-FIELD EVOLUTION AND LETHARGY

The equation defining the space-time evolution of the optical electric field has been derived in Ref. 8 in the hypothesis of low-gain, small-signal regime and bunched electron beam. Such an equation, written in the form of an integro-differential equation, holds on a time scale large compared with the round-trip period.

Denoting with $E(Z,t)$ the laser electric field, we have

$$2T_c\frac{\partial E(Z,t)}{\partial t}+\left\{\gamma_T+ig_0\theta\left[\nu_0-\pi\left(\frac{\Delta\omega}{\omega}\right)_0^{-1}\right]\right\}E(Z,t)+\Delta\theta g_0\frac{\partial E(Z,t)}{\partial Z}$$
$$=-ig_0\frac{(2\pi)^{3/2}}{\mu_c\Delta^2}\int_0^{\Delta}d\eta\,\eta e^{i\nu_0\eta/\Delta}E(Z+\eta,t)\int_{Z+\eta}^{Z+\Delta}dZ'f(Z') , \quad (2.1)$$

where g_0 is the gain coefficient, ν_0 is the resonance parameter, γ_T denotes the cavity losses, θ is the delay parameter linked to the cavity detuning from the nominal round-trip T_c, and $f(Z)$ is the electron longitudinal distribution (see Table I for further details). The left-hand side (lhs) of Eq. (2.1) accounts for the "free propagation" while the right-hand side (rhs) describes the laser–electron-beam interaction and thus the already discussed dynamics connected with the slippage and the concurrent lethargy.

Expanding the rhs of Eq. (2.1) up to second order in Δ and assuming that the laser pulse is centered about the maximum of the electron-beam distribution, we obtain

$$\frac{\partial}{\partial\tau}\tilde{E}=\left[\left(G_1-\frac{\gamma_T}{g_0}\right)+\sqrt{\mu_c}(G_3-\theta)\frac{\partial}{\partial\bar{Z}}-\mu_c^{3/2}\frac{G_2}{2}\bar{Z}+\mu_c\frac{G_4}{2}\frac{\partial^2}{\partial\bar{Z}^2}-\mu_c\frac{G_1}{2}\bar{Z}^2\right]\tilde{E} , \quad (2.2)$$

with

$$E(\bar{Z},\tau)=\tilde{E}(\bar{Z},\tau)\exp\left\{-i\theta\left[\nu_0-\pi\left(\frac{\Delta\omega}{\omega}\right)_0^{-1}\right](\tau-\tau_0)\right\} ,$$

$$\bar{Z}=Z/\sigma_z\mu_c^{-1/2},\quad \tau=\frac{1}{2}\frac{g_0t}{T_c} ,$$

$$G_1(\nu_0)=-2\pi\frac{\partial}{\partial\nu_0}\left[1+i\frac{\partial}{\partial\nu_0}\right]\left[\frac{\sin\nu_0/2}{\nu_0/2}\exp\left[i\frac{\nu_0}{2}\right]\right] ,$$

$$G_2(\nu_0)=\left[1-i\frac{\partial}{\partial\nu_0}\right]G_1(\nu_0) , \quad (2.3)$$

$$G_3(\nu_0)=-i\frac{\partial}{\partial\nu_0}G_1(\nu_0) ,$$

$$G_4(\nu_0)=-\frac{\partial^2}{\partial\nu_0^2}G_1(\nu_0) .$$

Note that a further term proportional to $\mu_c^2/6(G_1+G_4)$ has been dropped in (2.2). The function $G_1(\nu_0)$ is the complex gain function, whose real and imaginary parts yield the absorptive and dispersive parts of the single-mode gain function, respectively.

Furthermore, the functions $G_{2,3,4}(\nu_0)$ are higher-order corrections to the single-mode gain due to the finite length of the pulse.

Lie algebraic methods to solve equations of the type (2.2) have been thoroughly investigated.[11,12] In order to take advantage of these group-theoretical methods, let us recast (2.2) in the form

$$\frac{\partial}{\partial\tau}\tilde{E}=[\delta\hat{1}+\omega\hat{k}_0+\Omega_1(\hat{k}_++\hat{k}_-)+\bar{\Omega}_1\hat{a}+\bar{\Omega}_2\hat{a}^{\dagger}]\tilde{E} , \quad (2.4)$$

whose rhs is readily recognized as an element of the semidirect sum SU(1,1) $\oplus$h(4), the operators $(\hat{k}_{\pm},\hat{k}_0)$ and $(\hat{a},\hat{a}^{\dagger},\hat{1})$ being the generators of the SU(1,1) and Weyl-Heisenberg [h(4)] groups, respectively.

The explicit expression of the quantities entering Eq. (2.4) is

$$\delta=G_1-\frac{\gamma_T}{g_0},\quad \omega=-\mu_c(G_4+G_1) ,$$

$$\Omega_1=\frac{\mu_c}{2}(G_4-G_1),\quad \bar{\Omega}_1=\sqrt{\mu_c/2}(G_3-\theta)-\tfrac{1}{2}G_2\frac{\mu_c^{3/2}}{\sqrt{2}} , \quad (2.5)$$

$$\bar{\Omega}_2=-\sqrt{\mu_c/2}(G_3-\theta)-\tfrac{1}{2}G_2\frac{\mu_c^{3/2}}{\sqrt{2}} ,$$

$$\hat{a}=1/\sqrt{2}\left[\bar{Z}+\frac{\partial}{\partial\bar{Z}}\right],\quad \hat{a}^{\dagger}=1/\sqrt{2}\left[\bar{Z}-\frac{\partial}{\partial\bar{Z}}\right] ,$$

$$\hat{k}_+=\tfrac{1}{2}\hat{a}^{\dagger 2},\quad \hat{k}_-=\tfrac{1}{2}\hat{a}^2,\quad \hat{k}_0=\tfrac{1}{4}(\hat{a}\hat{a}^{\dagger}+\hat{a}^{\dagger}\hat{a}) .$$

Looking for a solution of the form

$$\tilde{E}(\bar{Z},\tau)=\sum_{n=0}^{\infty}C_n(\tau)u_n(\bar{Z}) , \quad (2.6)$$

where $u_n(\bar{Z})$ are the harmonic-oscillator orthonormal functions, the direct application of the above-quoted algebraic method yields after some algebraic calculation the following expression for the electric field at the first order in μ_c (the details of the calculation are reported in Appendix):

$$\tilde{E}(Z,\tau)\cong(1/\pi^{1/4})\exp\left[\left(G_1-\gamma_T/g_0-\frac{\mu_c}{2}G_4\right)(\tau-\tau_0)\right]\times\exp\left[-\frac{(Z-Z_0)^2}{2\sigma_E^2}\right] , \quad (2.7)$$

with

$$\sigma_E\cong\sqrt{\Delta\sigma_Z}[1+1/2\mu_c(G_4-G_1)(\tau-\tau_0)] ,$$
$$Z_0\cong-\Delta(G_3-\theta)(\tau-\tau_0) . \quad (2.8)$$

The physical meaning of (2.7) is transparent. The electric field has a Gaussian profile with a time-dependent amplitude governed by the gain function and by the cavity losses. The term proportional to μ_c accounts for the coupling between the longitudinal modes (see Ref. 8). The Gaussian is centered at Z_0 which represents the shift of the laser packet with respect to the electron bunch at time τ. Therefore, according to whether

$$\mathrm{Re}G_3(\nu_0)-\theta \lessgtr 0 \ , \tag{2.9}$$

the laser wave packet is ahead or behind the electron bunch. The dependence on ν_0 of the G functions is displayed in Fig. 1. At $\nu_0 \cong 2.6$, which corresponds to maximum gain, $\mathrm{Re}G_3 \simeq 0.45$ and $\mathrm{Im}G_3 \simeq 0$; therefore, after one round trip,

$$Z_0 \simeq -\Delta/2g_0(0.45-\theta) \ , \tag{2.10}$$

thus ensuring the timing between the laser and electron bunches for $\theta \simeq 0.45$. This is just the manifestation of the lethargy effect because increasing θ corresponds to shortening the cavity (see Table I). Indeed, since initially the two bunches are supposed to coincide and the distance between adjacent electron bunches is one round trip (or an integer multiple of it), if the velocity of the interacting laser bunch were exactly the light velocity, the cavity tuned at the nominal round trip would guarantee the timing between electron and laser bunches. On the other side, the fact that a shorter cavity ensures the timing is the consequence of the fact that the laser bunch is slowed down by the interaction.

Finally, the wave-packet width is not constant but varies with time at a rate

$$\frac{d\sigma_E}{d\tau} \propto G_4(\nu_0)-G_1(\nu_0) \ . \tag{2.11}$$

Correspondingly, the derivative of the spectrum width is given by

$$\frac{d\sigma_\nu}{d\tau} \propto -[G_4(\nu_0)-G_1(\nu_0)] \ . \tag{2.12}$$

The results we have presented in this section give a clear idea of the dynamics of the optical pulse in a long-bunch and low-gain FEL experiment. So far we have considered an ideal electron beam, namely, an electron

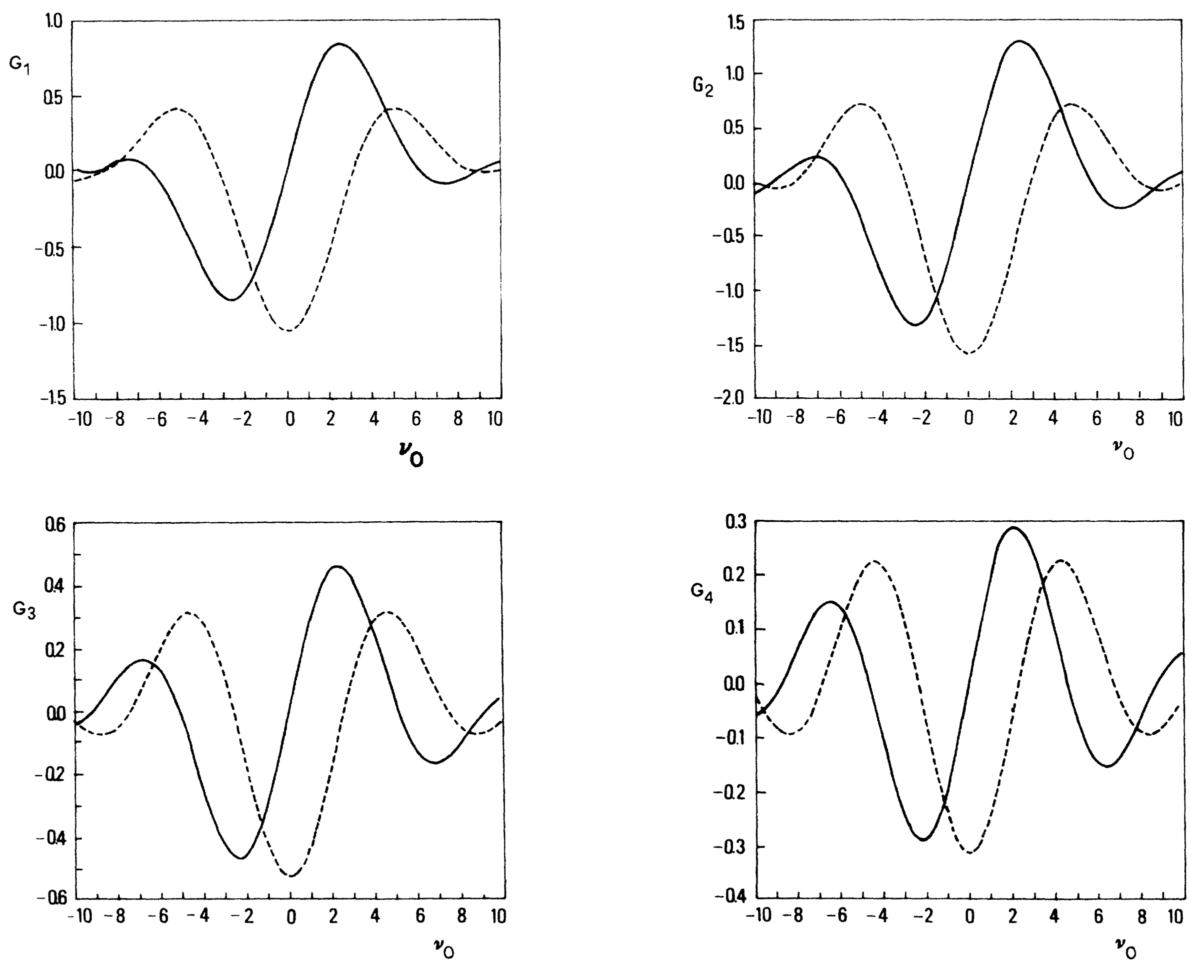

FIG. 1. Real (solid curve) and imaginary (dotted curve) parts of G functions vs ν_0. Homogeneous case ($\mu_\varepsilon = \mu_{x,y} = 0$).

beam with no emittance and energy spread. As we will see in Sec. IV the inclusion of the inhomogeneous broadening effects does not change the main results of this section.

III. OPTICAL-KLYSTRON PULSE DYNAMICS

The equation describing the pulse propagation in the optical-klystron configuration has been derived by Elleaume in Ref. 10. The electric field evolution in our notation reads

$$\frac{\partial}{\partial\tau}E(\bar{Z},\tau)=[\delta'\hat{1}+\omega'\hat{a}^{\dagger}\hat{a}+\Omega'(\hat{a}-\hat{a}^{\dagger})]E(\bar{Z},\tau)\ , \qquad (3.1)$$

where

$$\begin{aligned}\delta'&=(1-\mu_c/2)-\gamma_T/g_0\ ,\\ \omega'&=-\mu_c\ ,\\ \Omega'&=\sqrt{\mu_c/2}(1-\theta/2)\ .\end{aligned} \qquad (3.2)$$

The corresponding G functions do not appear explicitly in (3.1) because they have been evaluated at maximum gain. The form of the "Hamiltonian" operator appearing in (3.1) resembles that of the forced harmonic oscillator. The same group-theoretical methods exploited in Sec. II can be utilized to get the solution to Eq. (3.1). Obviously, the algebra involved is much less cumbersome, owing to the simpler structure of the "Hamiltonian." We get, indeed (see the Appendix for the details),

$$E(\bar{Z},\tau)=\frac{1}{\pi^{1/4}}\exp\left[-\frac{\omega'}{2}\int_{\tau_0}^{\tau}\alpha^2(\tau')d\tau'-\tfrac{1}{2}\alpha^2(\tau)+\delta'(\tau-\tau_0)+\bar{Z}_0(\tau)^2\right]\times\exp[-\tfrac{1}{2}(\bar{Z}-2\bar{Z}_0)^2]\ , \qquad (3.3)$$

where

$$\begin{aligned}\alpha(\tau)&=\Omega'\frac{\sinh[\omega'(\tau-\tau_0)/2]}{\omega'/2}\ ,\\ \bar{Z}_0(\tau)&=-1/\sqrt{2}\alpha(\tau)\exp[\omega'/2(\tau-\tau_0)]\ .\end{aligned} \qquad (3.4)$$

Expanding the above functions up to the second order in μ_c, we obtain

$$E(Z,\tau)\cong\frac{1}{\pi^{1/4}}\exp\left[\frac{\mu_c^2}{6}\left[1-\frac{\theta}{2}\right]^2(\tau-\tau_0)^3-\frac{\mu_c}{2}\left[1-\frac{\theta}{2}\right]^2(\tau-\tau_0)^2+\delta'(\tau-\tau_0)+\frac{\mu_c}{4}(\tau-\tau_0)^2\right]\exp\left[-\frac{1}{2\sigma_E^2}(Z-Z_0)^2\right]\ ,$$

$$Z_0\cong-\Delta(1-\theta/2)(\tau-\tau_0),\quad \sigma_E^2\cong\Delta\sigma_Z^2\ . \qquad (3.5)$$

In this case, too, the electric field has a Gaussian shape with a time-dependent amplitude governed by the gain and μ_c. The packet does not spread and the coordinate of its centroid varies with time according to (3.5).

The main result of this section is that the optical pulse has an rms length which is $\sqrt{\mu_c}$ times smaller than the rms length of the electron bunch.

IV. CONCLUSIONS

In this paper we have presented a relatively simple method to study the evolution of the optical pulse in the hypothesis of the long electron bunch and in the low-gain small-signal regime. Our theory does not include (a) the start up effects, (b) the saturation effects, and (c) the inhomogeneous broadening effects.

We have already stressed that Eq. (2.1) holds on a time scale large compared with the round-trip period so that the system no longer has a "memory" of the spontaneous emission or other terms which may affect the optical signal evolution during the first round trips. These terms can be easily incorporated into our model as source terms in Eq. (2.1), which once modified allows an analytical solution within the same simplified hypothesis of the long bunch. This problem is, however, rather complicated and is planned to be more carefully analyzed in a forthcoming paper.

The study of pulse propagation in the saturation regime cannot be accomplished within the framework of a linear theory. The equation of evolution in the strong-signal regime should be presumably a nonlinear Schrödinger equation. The question requires a much deeper insight and is beyond the scope of the purposes of the present paper.

Let us now come to the point (c), namely, the inclusion of the inhomogeneous broadening effects. It is well known that an electron beam with energy spread and finite emittances induces an inhomogeneous broadening of the spontaneous emission line and a consequent reduction of the gain. Assuming that the energy and the angular and spatial (transverse) distributions of the electron beam are Gaussian, Eq. (2.1), modified to unclutter the effects of the energy spread and emittance, reads

$$2T_c\frac{\partial E(Z,t)}{\partial t}+\left\{\gamma_T+ig_0\theta\left[\nu_0-\pi\left[\frac{\Delta\omega}{\omega}\right]_0^{-1}\right]\right\}E(Z,t)+\Delta\theta g_0\frac{\partial E(Z,t)}{\partial Z}$$

$$=-ig_0\frac{(2\pi)^{3/2}}{\mu_c\Delta^2}\int_0^{\Delta}d\eta\frac{\eta\exp[i\nu_0\eta/\Delta-\frac{1}{2}(\pi\mu_\varepsilon\eta/\Delta)^2]}{(1+i\pi\mu_x\eta/\Delta)(1+i\pi\mu_y\eta/\Delta)}E(Z+\eta,t)\int_{Z+\eta}^{Z+\Delta}f(Z')dZ'\ , \qquad (4.1)$$

where $\mu_{\varepsilon,x,y}$ are the inhomogeneous broadening coefficients defined by the ratio of the inhomogeneous width due to the energy spread and (x,y) emittances, respectively, to the homogeneous width (for the explicit definition see Table I, and for further comments, Ref. 10). Equation (4.1) has the same structure as Eq. (2.2), with the only difference being that the G functions are modified according to[11]

$$G_1(\nu_0;\mu_\varepsilon,\mu_x,\mu_y)=-2\pi\frac{\partial}{\partial\nu_0}\left\{1+i\frac{\partial}{\partial\nu_0}\right\}\int_0^1 d\zeta\frac{\exp(i\nu_0\zeta-\frac{1}{2}\pi^2\mu_\varepsilon^2\zeta^2)}{(1+i\pi\mu_x\zeta)(1+i\pi\mu_y\zeta)} \quad . \tag{4.2}$$

The main conclusions of Sec. II hold therefore unchanged. The changes induced by the μ coefficients on the G functions are shown in Fig. 2. It is evident that with increasing μ the G functions both reduce and widen. Consequently, some of the above-described effects might be "smoothed." For instance, the rate of the wave-packet spreading is lower and the lethargy effect grows less.

A final important and delicate point deserving comment is the interpretation of the solution we have presented in terms of harmonic-oscillator functions. In Refs. 10 and 11 the hypothesis has been put forward that they may be understood as the supermodes (SM). Supermodes, firstly introduced in Ref. 8, are a collection of longitudinal modes self-reproducing after each round trip. Mathematically they are defined as the eigenfunctions of Eq. (2.1). It is clear that each "mode" of (2.6) (or better, its Fourier transform) is a collection of longitudinal modes but not self-reproducing; in that case the amplitude function $C_n(\tau)$ should have the simple form

$$C_n(\tau)=\exp[\lambda_n(\tau-\tau_0)] \ , \tag{4.3}$$

where the eigenvalues λ_n specify the gain of each SM.

To clarify this point let us consider first the Hamiltonian operator of the optical klystron, namely,

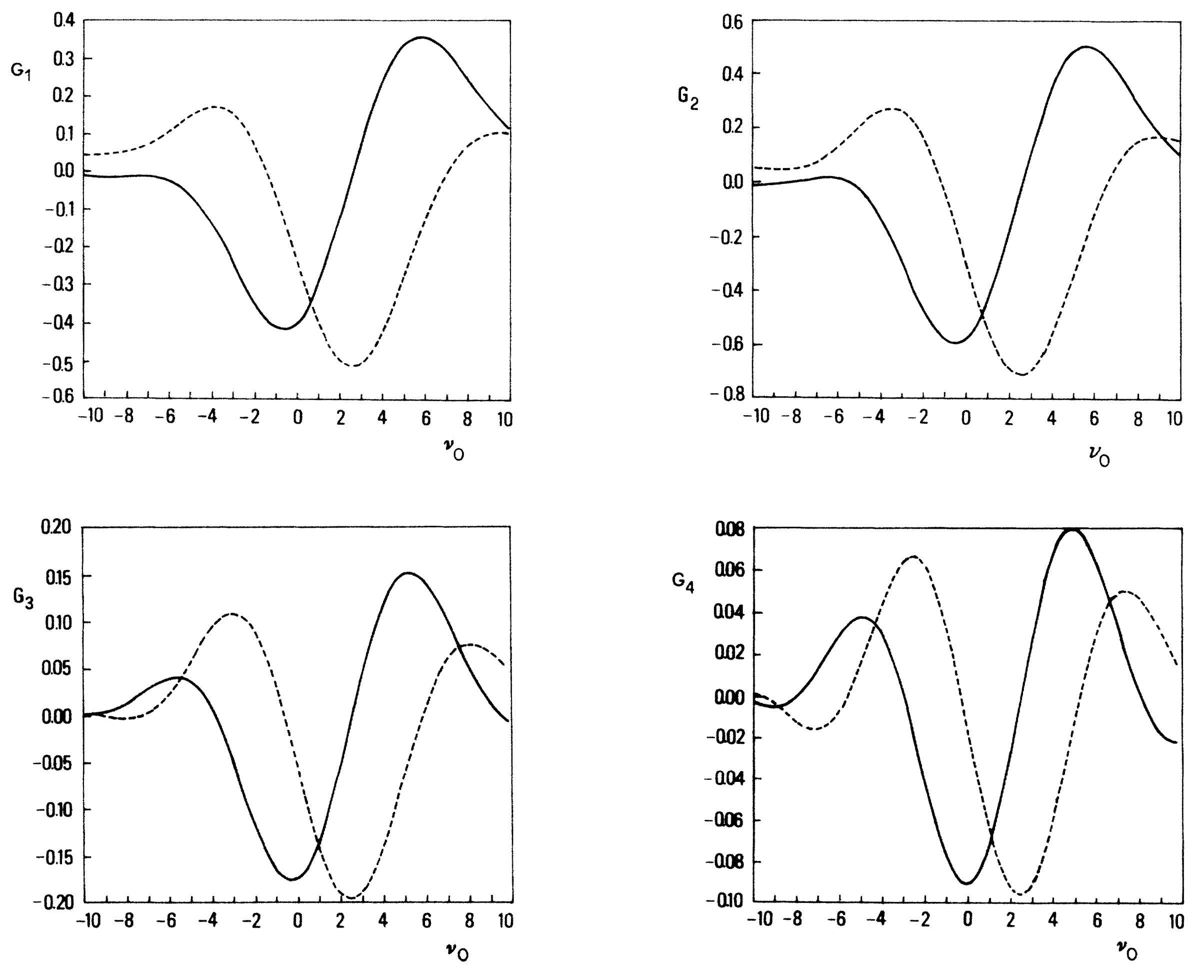

FIG. 2. Real (solid curve) and imaginary (dotted curve) parts of G functions vs ν_0. Inhomogeneous case ($\mu_\varepsilon=\mu_{x,y}=0.5$).

$$\hat{H}=\delta'\hat{1}+\omega'\hat{a}^{\dagger}\hat{a}+\Omega'(\hat{a}-\hat{a}^{\dagger}) , \quad (4.4)$$

we have already stressed that it recalls the Hamiltonian of a forced harmonic oscillator. Using a standard procedure let us introduce the operators

$$\hat{b}^{\dagger}=\hat{a}^{\dagger}+x\hat{1}, \quad \hat{b}=\hat{a}-x\hat{1}, \quad x=\Omega'/\omega' , \quad (4.5)$$

and rewrite (4.4) as

$$\hat{H}=\omega'\hat{b}^{\dagger}\hat{b}+\delta'\hat{1}+\Omega'^2/\omega'\hat{1} . \quad (4.6)$$

In the $\hat{b}$ basis the "Hamiltonian" is that of a harmonic oscillator and it can be immediately diagonalized, thus yielding for the λ_n's

$$\lambda_n=1-(2n+1)\frac{\mu_c}{2}-\frac{1}{2}\left[1-\frac{\theta}{2}\right]^2-\frac{\gamma_T}{g_0} , \quad (4.7)$$

that coincides with that given by Elleaume[10] and that in Ref. 11 using different arguments.

A similar procedure but a more involved algebra provides the following result for the FEL case:

$$\lambda_n=\delta-(2n+1)\mu_c(1+G_1G_4) - \frac{1}{2G_4}(G_3-\theta)^2 . \quad (4.8)$$

In both cases the eigenvalues are recovered in an auxiliary basis and in the case of the FEL are also the result of some simplifications linked to the perturbative nature of μ_c. The harmonic-oscillator functions can be considered the SM of our problem only approximately. However, rigorous techniques have been developed by the mathematicians to deal with the eigenfunctions of operators of the type[12]

$$\hat{A}=aZ^2+bZ\frac{\partial}{\partial Z}+c\frac{\partial^2}{\partial Z^2}+\alpha Z+\beta\frac{\partial}{\partial Z}+\gamma\hat{1} . \quad (4.9)$$

The identification of these eigenfunctions and of the relevant eigenvalues will be the topic of a forthcoming paper.

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APPENDIX

We will treat first the optical klystron case because it is, by far, simpler than the FEL case. Since Eq. (3.1) is an equation of evolution we can exploit well-developed methods of quantum mechanics.[11] In fact, we look for a nonunitary evolution operator defined by the equation

$$\frac{\partial}{\partial\tau}\hat{U}(\tau,\tau_0)=[\delta'\hat{1}+\omega'\hat{a}^{\dagger}\hat{a}+\Omega'(\hat{a}-\hat{a}^{\dagger})]\hat{U}(\tau,\tau_0) , \quad \hat{U}(\tau_0,\tau_0)=\hat{1} , \quad (A1)$$

whose solution may be formally written as

$$\hat{U}(\tau,\tau_0)=\hat{U}_0(\tau,\tau_0)\hat{U}_I(\tau,\tau_0) , \quad (A2)$$

with

$$\hat{U}_0(\tau,\tau_0)=\exp[\delta'(\tau-\tau_0)+\omega'(\tau-\tau_0)\hat{a}^{\dagger}\hat{a}]\hat{1} , \quad (A3)$$

and $\hat{U}_I(\tau,\tau_0)$ obeying the equation

$$\frac{\partial}{\partial\tau}\hat{U}_I(\tau,\tau_0)=(\Omega'e^{\omega'(\tau-\tau_0)}\hat{a}-\Omega'e^{-\omega'(\tau-\tau_0)}\hat{a}^{\dagger})\hat{U}_I(\tau,\tau_0) , \quad \hat{U}_I(\tau_0,\tau_0)=\hat{1} . \quad (A4)$$

The explicit expression of $\hat{U}_I$ can be inferred using the Magnus ordering theorem,[13] thus getting

$$\hat{U}_I=\exp\left[-\frac{\omega'}{2}\int_{\tau_0}^{\tau}\alpha^2(\tau')d\tau'\right]\exp[-\tfrac{1}{2}\alpha^2(\tau)] \times\exp\left[\frac{\Omega'}{\omega'}(e^{-\omega'(\tau-\tau_0)}-1)\hat{a}^{\dagger}\right] \times\exp\left[\frac{\Omega'}{\omega'}(e^{\omega'(\tau-\tau_0)}-1)\hat{a}\right]\hat{1} . \quad (A5)$$

Let us expand the field $E(\bar{Z},\tau)$ as

$$E(\bar{Z},\tau)=\sum C_n(\tau)u_n(\bar{Z}) , \quad (A6)$$

u_n being the harmonic-oscillator eigenfunctions.

The formal expression of the coefficients $C_n(\tau)$ can be easily inferred as

$$C_n(\tau)=\sum_{m=0}^{\infty}U_{m,n}(\tau,\tau_0)C_m(\tau_0) , \quad (A7)$$

where $C_m(\tau_0)$ are the initial conditions of the problem and the matrix elements $U_{m,n}(\tau,\tau_0)$ are given by

$$U_{m,n}(\tau,\tau_0)=\langle m\mid\hat{U}(\tau,\tau_0)\mid n\rangle , \quad (A8)$$

where $|m\rangle$ synthetically denotes u_m. Assuming $C_m(\tau_0)=\delta_{m,0}$, we get

$$C_n(\tau)=\frac{(-1)^n}{\sqrt{n!}}\exp\left[-\frac{\omega'}{2}\int_{\tau_0}^{\tau}d\tau'\alpha^2(\tau')-\tfrac{1}{2}\alpha^2(\tau) +\delta'(\tau-\tau_0)+n\omega'(\tau-\tau_0)\right] \times\left[\alpha(\tau)\exp\left[-\frac{\omega'}{2}(\tau-\tau_0)\right]\right]^n . \quad (A9)$$

Finally, using the generating function of Hermite polynomials we end up with

$$E(\bar{Z},\tau)=\frac{1}{\pi^{1/4}}\exp\left[-\frac{\omega'}{2}\int_{\tau_0}^{\tau}\alpha^2(\tau')d\tau'-\tfrac{1}{2}\alpha^2(\tau)+\delta'(\tau-\tau_0)-\bar{Z}^2/2\right]$$
$$\times\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\frac{\alpha(\tau)}{\sqrt{2}}\exp\left[\frac{\omega'}{2}(\tau-\tau_0)\right]\right]^n H_n(\bar{Z})$$
$$=\frac{1}{\pi^{1/4}}\exp\left[-\frac{\omega'}{2}\int_{\tau_0}^{\tau}\alpha^2(\tau')d\tau'-\tfrac{1}{2}\alpha^2(\tau)+\delta'(\tau-\tau_0)+\bar{Z}_0^2(\tau)\right]\exp\{-\tfrac{1}{2}[\bar{Z}-2\bar{Z}_0(\tau)]^2\}\ , \qquad (A10)$$

according to Eq. (3.3).

Let us come now to the more cumbersome problem of finding a solution to Eq. (2.4). We will sketch the main steps of the calculus, the details of which can be found in Ref. 11.

We have already realized that the operator

$$\hat{H}=\delta\hat{1}+\omega\hat{k}_0+\Omega_1(\hat{k}_++\hat{k}_-)+\bar{\Omega}_1\hat{a}+\bar{\Omega}_2\hat{a}^\dagger \qquad (A11)$$

belongs to the semidirect sum SU(1,1)⊕h(4). To get an ordered form for the evolution operator, in this case, one can use a method based on the Wei-Norman algebraic procedure,[14] which yields for $C_n(\tau)$ the following expression:

$$C_n(\tau)=e^{\delta(\tau-\tau_0)}e^{(\omega/4)(\tau-\tau_0)}\mathcal{H}^{-1/2}\mathcal{T}$$
$$\times[Z(\tau)]^n\frac{1}{2^{n/2}\sqrt{n!}}H_n(X(\tau))\ , \qquad (A12)$$

where at the lowest order in μ_c we have

$$\mathcal{H}(\tau)\cong 1-\frac{\omega^2}{4}(\tau-\tau_0)^2\ ,$$
$$Z(\tau)\simeq i\left[\frac{\mu_c}{2}(G_4-G_1)(\tau-\tau_0)\right]^{1/2}\left[1+\frac{\omega}{4}(\tau-\tau_0)\right]\ ,$$
$$X(\tau)\simeq i\left[\frac{(\tau-\tau_0)}{2(G_4-G_1)}\right]^{1/2}(G_3-\theta)\left[1+\frac{\omega}{4}(\tau-\tau_0)\right]\ , \qquad (A13)$$

$$\mathcal{T}(\tau)\simeq\exp\left[-\frac{\mu_c}{4}(G_3-\theta)^2(\tau-\tau_0)\right]\ .$$

Exploiting again the generating function of Hermite polynomials we finally get[11]

$$\tilde{E}(\bar{Z},\tau)=e^{\delta(\tau-\tau_0)}\mathcal{T}(\tau)[\phi(\tau)]^{1/2}\exp\left[\frac{X^2Z^2}{1+Z^2}\right]$$
$$\times\exp\left[-\frac{1+Z^2}{2(1-Z^2)}\left[\bar{Z}-\frac{2XZ}{1+Z^2}\right]^2\right]\ ,$$
$$\phi(\tau)=\frac{\mathcal{H}^{-1}}{\sqrt{\pi}}\frac{e^{\omega/2(\tau-\tau_0)}}{(1-Z^2)}\ , \qquad (A14)$$

with

$$\frac{2XZ}{1+Z^2}\simeq-\sqrt{\mu_c}(G_3-\theta)(\tau-\tau_0)\ ,$$
$$\frac{1-Z^2}{1+Z^2}\simeq 1+\frac{\mu_c}{2}(G_4-G_1)(\tau-\tau_0)\ , \qquad (A15)$$
$$\frac{X^2Z^2}{1+Z^2}\simeq\mu_c/4(G_3-\theta)^2(\tau-\tau_0)^2$$

at the lowest order in μ_c. Neglecting the last term proportional to g_0^2 we easily recover Eq. (2.7).

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