Comparative study of the gyrotron, the free-electron laser, and the wiggler-free free-electron laser

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The gyrotron, the free-electron laser, and the wiggler-free free-electron laser are compared. First, the flows in the three systems are examined in their steady state. Then the interaction of the electron beam with free-space electromagnetic waves is studied by using a fluid picture. The growth rate of the instability is found by calculating the nonreal roots of the three different dispersion relations. It is shown that at off-resonance the interactions in the gyrotron and in the wiggler-free free-electron laser are identical. However, at resonance, the instability in the gyrotron disappears, while the growth rate in the wiggler-free free-electron laser is maximal, as in the usual free-electron laser. The maximum growth rate in the wiggler-free free-electron laser scales as some small parameter to the power of $\frac{2}{5}$, compared to the same small parameter to the power of $\frac{1}{2}$ for the gyrotron, and to the power of $\frac{1}{3}$ for the free-electron laser. Thus the wiggler-free free-electron laser, while having some resemblance to each of the two other systems, the gyrotron and the free-electron laser, is clearly different from both. In terms of gain, it has a higher gain than the gyrotron but lower than the free-electron laser.

I. INTRODUCTION

In the free-electron laser[1] (FEL) an external periodic magnetic field generates a spatially periodic flow of electrons. The periodic electron flow transfers some of its energy to an electromagnetic wave of a frequency related to the periodicity of the flow. In the gyrotron,[2] electrons which execute periodic motion transfer energy to an electromagnetic wave of a frequency related to the periodicity of their motion as well. However, in the gyrotron there is no external periodic field but a uniform magnetic field, and the whole flow is uniform along the axis of propagation and is not periodic. Recently, Fruchtman and Friedland proposed a wiggler-free FEL,[3-5] where a spatially periodic electron flow, which propagates along a uniform magnetic field, transfers energy to an electromagnetic wave.

The wiggler-free FEL resembles the gyrotron in some aspects and the FEL in other aspects. The external magnetic field in the wiggler-free FEL is uniform and forces the electron to move on a helix, the resonant wave frequency is the Doppler-shifted cyclotron frequency. These features characterize the gyrotron as well. However, the electron-beam motion in the wiggler-free FEL is coherent in the sense that the steady-state flow is helical and is spatially periodic, similar to the electron-beam flow along the helical wiggler of a FEL. In contrast, in the gyrotron the individual electrons execute helical motion, but the flow is uniform and is not helical since the gyrophase of the electrons is random. In summary, in the gyrotron both the external field and the steady-state flow are uniform, in the FEL both the external field and the steady-state flow are spatially periodic, and in the wiggler-free FEL the external field is uniform but the steady-state flow is spatially periodic.

The device which we refer to here as a gyrotron is that which is analyzed by Chu and Hirshfeld,[6] and which is commonly referred to as a relativistic gyrotron or a cyclotron autoresonance maser[7] (CARM). We emphasize that even though both the CARM (the gyrotron, as we call it here) and the wiggler-free FEL generate radiation of a considerably Doppler-shifted cyclotron frequency, their aforementioned different steady flows result in different interactions with the electromagnetic wave. Contrary to what is commonly assumed for the CARM, we assume here that the wave vector of the radiation is parallel to the direction of beam propagation, and does not have a perpendicular component. The FEL we study employs a helical magnetostatic wiggler and is referred to as the wiggler FEL.

We compare the interaction of the electrons with electromagnetic waves in these three systems. To that end we perform a linear-stability analysis of the three steady states. We show that in addition to the steady flow the instability in the wiggler-free FEL has similarities to the instability in both these systems. Off-resonance the wiggler-free FEL interaction is identical to the one of the gyrotron. At resonance, on the other hand, there is more resemblance to the FEL.

We analyze the interaction in the three systems through a fluid model for the electron beam and through the Maxwell equations for the wave. The unified analysis enables us to compare the three devices. In the analysis we assume that there are no transverse gradients so that the systems are essentially one dimensional.

The paper is organized as follows. In Sec. II we derive expressions for the electron current in the wiggler FEL and in the wiggler-free FEL. In Sec. III a different version of wiggler-free FEL is discussed, the double-helical-beam wiggler-free FEL. It is shown that in the low-density limit, both versions of the wiggler-free FEL are identical. In Sec. IV the electron currents in the gyrotron are found, by using the results of Secs. II and III. The dispersion relations for the various systems are derived in Sec. V. Approximate expressions for the growth rates in a certain domain of parameters are presented, and are confirmed by numerical solutions of the full dispersion equations. It is shown that the maximal growth rate of the instability in the wiggler-free FEL scales as a small parameter to the power of $\frac{2}{5}$, while in the gyrotron it scales as the same small parameter to the power of $\frac{1}{2}$ and in the wiggler FEL to the power of $\frac{1}{3}$.

II. WIGGLER FEL AND WIGGLER-FREE FEL

We describe the electron dynamics in the wiggler FEL and in the wiggler-free FEL by the one-dimensional cold-fluid equations. These are the continuity equation

$$\frac{\partial}{\partial t}(\gamma h)+\frac{\partial}{\partial z}(P_z h)=0 \qquad (1)$$

and the momentum equation

$$\gamma\frac{\partial \mathbf{P}}{\partial t}+P_z\frac{\partial \mathbf{P}}{\partial z}=-\gamma\mathbf{E}-\mathbf{P}\times\mathbf{B} . \qquad (2)$$

Here $\mathbf{P}$ is the normalized momentum (the momentum divided by mc), γ is $(1+\mathbf{P}\cdot\mathbf{P})^{1/2}$, h is the normalized density [the density multiplied by $4\pi e^2/(mc^2\gamma)$], e and m are the electron charge and mass, c is the velocity of light in vacuum, and $\mathbf{E}$ and $\mathbf{B}$ are the normalized electric and magnetic fields (the fields multiplied by e/mc^2). In the helical wiggler FEL the time-independent external field is

$$\mathbf{B}=B_W[\hat{\mathbf{e}}_x\cos(k_0 z)+\hat{\mathbf{e}}_y\sin(k_0 z)] , \qquad (3)$$

while in the wiggler-free FEL it is a uniform axial field

$$\mathbf{B}=B_0\hat{\mathbf{e}}_z . \qquad (4)$$

These two different external fields can support cold helical flows. In the presence of the wiggler the cold flow is

$$\mathbf{P}=-\left[\frac{B_W}{k_0}\right][\hat{\mathbf{e}}_x\cos(k_0 z)+\hat{\mathbf{e}}_y\sin(k_0 z)]+P_z\hat{\mathbf{e}}_z . \qquad (5)$$

In the presence of the uniform axial field the cold flow is

$$\mathbf{P}=-p_\perp\left[\hat{\mathbf{e}}_x\cos\left[\frac{B_0}{p_z}z\right]+\hat{\mathbf{e}}_y\sin\left[\frac{B_0}{p_z}z\right]\right]+p_z\hat{\mathbf{e}}_z . \qquad (6)$$

The density is uniform in both cases, and $p_\perp$, p_z, and P_z in Eqs. (5) and (6) are constant. The electron beam is assumed to be tenuous and the steady-state self-fields are negligible relative to the external fields. We compare the stability of two such flows, when identical, i.e., when

$$k_0=B_0/P_z , \qquad (7)$$

and also

$$p_\perp=B_W/k_0 , \qquad (8)$$

and $p_z=P_z$.

For the linear-stability analysis it is convenient to describe vectors with the aid of the following three orthogonal unit vectors:[8]

$$\hat{\mathbf{e}}_1=-\hat{\mathbf{e}}_x\sin(k_0 z)+\hat{\mathbf{e}}_y\cos(k_0 z) ,$$
$$\hat{\mathbf{e}}_2=-\hat{\mathbf{e}}_x\cos(k_0 z)-\hat{\mathbf{e}}_y\sin(k_0 z) , \qquad (9)$$
$$\hat{\mathbf{e}}_3=\hat{\mathbf{e}}_z .$$

The components of the fields (3) and (4) and of the momentum (5) and (6) with respect to these unit vectors are constant. The wiggler field is

$$\mathbf{B}=-B_W\hat{\mathbf{e}}_2 \qquad (10)$$

and the uniform field is

$$\mathbf{B}=B_0\hat{\mathbf{e}}_3 . \qquad (11)$$

The momentum, with these unit vectors, is

$$\mathbf{P}=p_\perp\hat{\mathbf{e}}_2+p_z\hat{\mathbf{e}}_3 . \qquad (12)$$

Since the vector components of the steady-state quantities are constant, and since

$$\frac{d\hat{\mathbf{e}}_1}{dz}=k_0\hat{\mathbf{e}}_2 \qquad (13a)$$

and

$$\frac{d\hat{\mathbf{e}}_2}{dz}=-k_0\hat{\mathbf{e}}_1 , \qquad (13b)$$

we may seek a solution to the linearized equations, where the unknown quantities are written as

$$\mathbf{X}=\mathbf{x}+\delta\mathbf{x}e^{i(kz-\omega t)} , \qquad (14)$$

and the components of the vector $\delta\mathbf{x}$ are also constant. The cold-fluid and the Maxwell equations turn into a finite set of algebraic equations for the amplitudes $\delta\mathbf{x}$, and k is a root of the dispersion relation.

We start with the cold-fluid equations. The linearized continuity equation has the same form in the two systems,

$$(kp_3-\omega\gamma)\delta h=h(\omega\delta\gamma-k\delta p_3) , \qquad (15)$$

where p_3, h, and γ are the same for both devices. The components of the linearized momentum equation take the following form:

$$i(kp_3-\omega\gamma)\delta p_1+(B_3-k_0p_3)\delta p_2-(B_2+k_0p_2)\delta p_3=-\gamma\delta E_1+p_3\delta B_2 ,$$
$$-(B_3-k_0p_3)\delta p_1+i(kp_3-\omega\gamma)\delta p_2=-\gamma\delta E_2-p_3\delta B_1 , \qquad (16)$$
$$B_2\delta p_1+i(kp_3-\omega\gamma)\delta p_3=-\gamma\delta E_3+p_2\delta B_1 .$$

In these equations p_2, p_3, γ, and k_0 are the same for both the wiggler FEL and the wiggler-free FEL. The external fields, however, are different. In the wiggler FEL B_3 and $B_2+k_0p_2$ are zero, while in the wiggler-free FEL B_2 and $B_3-k_0p_3$ are zero. By using Faraday's law, we express the components of the wave magnetic field as

$$\delta B_1 = \frac{i}{\omega}(ik\delta E_2 + k_0\delta E_1) \, ,$$
$$\delta B_2 = -\frac{i}{\omega}(ik\delta E_1 - k_0\delta E_2) \, . \quad (17)$$

We are interested in the FEL resonance, where $kp_3 - \omega\gamma$ is small. In the wiggler FEL the only resonant momentum component is δp_3, and the momentum components are

$$\delta p_1 = \frac{i}{\omega}\delta E_1 \, ,$$
$$\delta p_2 = -\frac{i}{\omega}\delta E_2 \, , \quad (18)$$
$$\delta p_3 = -i\gamma \frac{[(kp_2/\omega\gamma)\delta E_2 + \delta E_3]}{kp_3 - \omega\gamma} \, .$$

In the wiggler-free FEL, on the other hand, all three momentum components are resonant, and δp_1 is the largest. These components are

$$\delta p_1 = \frac{i(\gamma\delta E_1 - p_3\delta B_2)}{kp_3 - \omega\gamma} - \frac{k_0 p_2^2 \delta B_1}{(kp_3 - \omega\gamma)^2} \, ,$$
$$\delta p_2 = \frac{i(\gamma\delta E_2 + p_3\delta B_1)}{kp_3 - \omega\gamma} \, , \quad (19)$$
$$\delta p_3 = \frac{-ip_2\delta B_1 + i\gamma\delta E_3}{kp_3 - \omega\gamma} \, .$$

By using Eqs. (15), (18), and (19), and the relation

$$\gamma\delta\gamma = p_2\delta p_2 + p_3\delta p_3 \, , \quad (20)$$

which follows from the definition of γ, we express the perturbed density δh for the wiggler FEL as

$$\delta h = \frac{ih(\omega p_3 - k\gamma)[(p_2/p_3)\delta E_2 + \delta E_3]}{(kp_3 - \omega\gamma)^2} \, , \quad (21)$$

and for the wiggler-free FEL as

$$\delta h = ih \frac{[p_2(\omega\delta E_2 + k\delta B_1) + (\omega p_3 - k\gamma)\delta E_3]}{(kp_3 - \omega\gamma)^2} \, . \quad (22)$$

In both the last two expressions we omitted nonresonant terms. The expressions for the perturbed densities in the two systems are similar, the additional term in the wiggler-free system results from the larger δp_2.

We are now ready to calculate the current, which is the source in the Maxwell equations. The components of the perturbed transverse current are

$$\delta j_1 = -h\delta p_1 \, ,$$
$$\delta j_2 = -h\delta p_2 - p_2\delta h \, . \quad (23)$$

In the wiggler FEL only the density modulation contributes a resonant term to the current and thus

$$\delta j_1 = 0 \, ,$$
$$\delta j_2 = -\frac{ih(\omega p_3 - k\gamma)[(p_2^2/p_3)\delta E_2 + p_2\delta E_3]}{(kp_3 - \omega\gamma)^2} \, . \quad (24)$$

In the wiggler-free FEL the perturbed transverse momentum contributes resonant terms to the current as well, and thus

$$\delta j_1 = -\frac{ih(\gamma\delta E_1 - p_3\delta B_2)}{kp_3 - \omega\gamma} + \frac{hk_0 p_2^2 \delta B_1}{(kp_3 - \omega\gamma)^2} \, ,$$
$$\delta j_2 = -\frac{ih(\gamma\delta E_2 + p_3\delta B_1)}{kp_3 - \omega\gamma} \quad (25)$$
$$- \frac{ih[p_2^2(\omega\delta E_2 + k\delta B_1) + p_2(\omega p_3 - k\gamma)\delta E_3]}{(kp_3 - \omega\gamma)^2} \, .$$

In the next section we analyze a different version of the wiggler-free FEL, the double-helical-beam wiggler-free FEL.

III. DOUBLE-HELICAL-BEAM WIGGLER-FREE FEL

Let us assume that two helical beams propogate along a uniform magnetic field. One of the beams has the same steady-state momentum as the beam in the wiggler-free FEL of Sec. II,

$$\mathbf{p}^{(1)} = p_2\hat{\mathbf{e}}_2 + p_3\hat{\mathbf{e}}_3 \, , \quad (26)$$

while the other beam is also a helical beam but with a different gyrophase

$$\mathbf{p}^{(2)} = -p_2\hat{\mathbf{e}}_2 + p_3\hat{\mathbf{e}}_3 \, . \quad (27)$$

We also assume that each of the two beams has a density of $h/2$. Solving the linearized cold-fluid equations for the two beams, as in Sec. II, we obtain

$$\delta p_{1,2}^{(2)} = \delta p_{1,2}^{(1)} = \delta p_{1,2} \quad (28a)$$

and

$$\delta p_3^{(1)} = \delta p_3 \, , \quad (28b)$$

where $\delta p_{1,2,3}$ are given in Eq. (19). The expression for $\delta p_3^{(2)}$ is different and is

$$\delta p_3^{(2)} = \frac{ip_2\delta B_1 + i\gamma\delta E_3}{kp_3 - \omega\gamma} \, . \quad (29)$$

We find also that

$$\delta h^{(1)} = \delta h/2 \, , \quad (30)$$

where δh is given by Eq. (22), but that

$$\delta h^{(2)} = \frac{ih[-p_2(\omega\delta E_2 + k\delta B_1) + (\omega p_3 - k\gamma)\delta E_3]}{2(kp_3 - \omega\gamma)^2} \, . \quad (31)$$

The perturbed currents are

$$\delta j_1 = -\frac{h}{2}\delta p_1^{(1)} - \frac{h}{2}\delta p_1^{(2)} = -h\delta p_1 \, ,$$
$$\delta j_2 = -\frac{h}{2}\delta p_2^{(1)} - \frac{h}{2}\delta p_2^{(2)} - p_2\delta h^{(1)} + p_2\delta h^{(2)} \quad (32)$$
$$= -h\delta p_2 - \frac{ihp_2^2(\omega\delta E_2 + k\delta B_1)}{(kp_3 - \omega\gamma)^2} \, .$$

Examining the last expressions, we see that the term that arose from the longitudinal electric field δE_3 has been canceled out. This field component contributed to the perturbed current in the one-beam wiggler-free FEL, but not in the double-helical-beam wiggler-free FEL. This same result was also obtained by Fruchtman and Friedland in a analysis of the wiggler-free FEL (Ref. 4) and later by An *et al.*[9] In the low density beam limit, when space-charge effects are small, δE_3 can be neglected. The perturbed currents in both systems, the helical-beam wiggler-free FEL and the double-helical-beam wiggler-free FEL, are then identical. If we limit our analysis to the low-density case and neglect δE_3, the results hold for both wiggler-free systems. We note that for the double-helical-beam wiggler-free FEL the regime of validity is wider and is not limited to the low-density case alone. Having completed the derivation of the currents in the wiggler FEL and in the wiggler-free FEL, we turn to the gyrotron.

IV. GYROTRON

In the gyrotron, a randomly gyrophased beam whose equilibrium distribution function is

$$f_0(P_\perp,P_z)=\frac{h\gamma}{2\pi p_\perp}\delta(P_\perp-p_\perp)\delta(P_z-p_3)\ , \tag{33}$$

propogates along a uniform magnetic field. The perturbed current due to the interaction with the wave is easily found by solving the Vlasov equation.[6] Here we derive the expression for the perturbed current through the fluid model. We describe the beam in its steady state as composed of cold helical beamlets each of a momentum

$$\mathbf{p}=\hat{\mathbf{e}}_2'(\theta)p_\perp+p_z\hat{\mathbf{e}}_3 \tag{34a}$$

and density

$$dh=\tilde{h}(\theta)\frac{d\theta}{2\pi}\ , \tag{34b}$$

where we define for each helical beamlet the unit vectors

$$\begin{aligned}\hat{\mathbf{e}}_1'(\theta)&=-\hat{\mathbf{e}}_1\cos\theta-\hat{\mathbf{e}}_2\sin\theta\ ,\\ \hat{\mathbf{e}}_2'(\theta)&=-\hat{\mathbf{e}}_1\sin\theta+\hat{\mathbf{e}}_2\cos\theta\ .\end{aligned} \tag{35}$$

The beamlets are characterized by θ, the gyrophase angle. The components of the perturbed current of each beamlet $d\tilde{\delta}j_1$ and $d\tilde{\delta}j_2$ relative to the unit vectors defined in Eq. (35), have the same form as in Eq. (25), where δE_i and δB_i $(i=1,2)$ are replaced by $\delta\tilde{E}_i$ and $\delta\tilde{B}_i$. We now write $d\delta j_1$ and $d\delta j_2$, the components relative to $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, as a function of δE_i and δB_i, employing the relations

$$\begin{aligned}d\delta j_1&=-d\tilde{\delta}j_1\cos\theta-d\tilde{\delta}j_2\sin\theta\ ,\\ d\delta j_2&=-d\tilde{\delta}j_1\sin\theta+d\tilde{\delta}j_2\cos\theta\ ,\\ \delta\tilde{E}_1&=-\delta E_1\cos\theta-\delta E_2\sin\theta\ ,\\ \delta\tilde{E}_2&=-\delta E_1\sin\theta+\delta E_2\cos\theta\ .\end{aligned} \tag{36}$$

We thus obtain the contribution of each beamlet to the current. Integrating over θ we find the total current as a sum of these contributions. If the beam is randomly gyrophased,

$$\tilde{h}(\theta)=h\ , \tag{37}$$

and the total perturbed current is

$$\begin{aligned}\delta j_1&=\frac{ih(\gamma\delta E_1-p_3\delta B_2)}{kp_3-\omega\gamma}\\ &\quad+\frac{hp_2^2(k_0\delta B_1+ik\delta B_2-i\omega\delta E_1)}{2(kp_3-\omega\gamma)^2}\ ,\\ \delta j_2&=-\frac{ih(\gamma\delta E_2+p_3\delta B_1)}{kp_3-\omega\gamma}\\ &\quad+\frac{hp_2^2(-ik\delta B_1+k_0\delta B_2-i\omega\delta E_2)}{2(kp_3-\omega\gamma)^2}\ .\end{aligned} \tag{38}$$

Thus we completed the derivation of expressions for the perturbed currents in the wiggler FEL (24), in the wiggler-free FEL (25), and in the gyrotron (38). We are now ready to substitute these expressions into the Maxwell equations.

V. DISPERSION RELATIONS AND NUMERICAL EXAMPLES

The linearized Maxwell equations have the form

$$\begin{aligned}(k^2+k_0^2-\omega^2)\delta E_1+2ikk_0\delta E_2&=i\omega\delta j_1\ ,\\ -2ikk_0\delta E_1+(k^2+k_0^2-\omega^2)\delta E_2&=i\omega\delta j_2\ .\end{aligned} \tag{39}$$

We limit ourselves to the case of low density, where space-charge effects are small, and neglect δE_3. In this case, the equations for the wiggler FEL are

$$\begin{aligned}&(k^2+k_0^2-\omega^2)\delta E_1+2ikk_0\delta E_2=0\ ,\\ &-2ikk_0\delta E_1+(k^2+k_0^2-\omega^2)\delta E_2\\ &\qquad=\frac{\omega h(\omega-k\gamma/p_3)}{(k-\omega\gamma/p_3)^2}(p_2/p_3)^2\delta E_2\ ,\end{aligned} \tag{40}$$

those for the wiggler-free FEL are

$$\begin{aligned}&(k^2+k_0^2-\omega^2)\delta E_1+2ikk_0\delta E_2\\ &\quad=-h\delta E_1-\frac{ihk_0\delta E_2}{k-\omega\gamma/p_3}\\ &\qquad-\frac{hk_0}{(k-\omega\gamma/p_3)^2}(p_2/p_3)^2(ik\delta E_2+k_0\delta E_1)\ ,\\ &-2ikk_0\delta E_1+(k^2+k_0^2-\omega^2)\delta E_2\\ &\quad=-h\delta E_2+\frac{ihk_0\delta E_1}{k-\omega\gamma/p_3}\\ &\qquad-\frac{h(p_2/p_3)^2[(\omega^2-k^2)\delta E_2+ikk_0\delta E_1]}{(k-\omega\gamma/p_3)^2}\ ,\end{aligned} \tag{41}$$

and those for the gyrotron are

$$(k^2+k_0^2-\omega^2)\delta E_1+2ikk_0\delta E_2 = -h\delta E_1 - \frac{ihk_0\delta E_2}{k-\omega\gamma/p_3} - \frac{h(p_2/p_3)^2[(k^2+k_0^2-\omega^2)\delta E_1+2ikk_0\delta E_2]}{2(k-\omega\gamma/p_3)^2} ,$$

$$-2ikk_0\delta E_1+(k^2+k_0^2-\omega^2)\delta E_2 = -h\delta E_2+\frac{ihk_0\delta E_1}{(k-\omega\gamma/p_3)^2} - \frac{h(p_2/p_3)^2[-2ikk_0\delta E_1+(k^2+k_0^2-\omega^2)\delta E_2]}{2(k-\omega\gamma/p_3)^2} . \tag{42}$$

In writing Eqs. (41) and (42) we used Eq. (17). We define a normalized eigenvalue Δ,

$$\Delta \equiv \frac{k}{k_0} - \frac{\omega}{k_0}\frac{\gamma}{p_3} , \tag{43}$$

and assume

$$\Delta \ll 1 . \tag{44}$$

We also define a mismatch parameter

$$\xi \equiv \frac{1}{k_0}[k-(k_0+\omega)] - \Delta = \frac{\omega}{k_0}\left[\frac{\gamma}{p_3}-1\right]-1 , \tag{45}$$

and since we look for modes which are close to vacuum electromagnetic modes, ξ is also assumed to be small. The frequency is then

$$\omega \cong 2k_0\gamma^2 , \tag{46}$$

the usual resonant frequency in free electron lasers. We note that

$$\frac{\omega}{k_0} - \frac{k}{k_0}\frac{\gamma}{p_3} \cong -2 , \tag{47}$$

since Δ and ξ are much smaller than 1, and since γ/p_3 is approximately 1 for a relativistic beam. We define a coupling coefficient

$$s \equiv \frac{h}{2k_0^2}\left[\frac{p_2}{p_3}\right]^2 , \tag{48}$$

which is also much smaller than 1, a normalized density

$$\bar{h} \equiv h/k_0^2 , \tag{49}$$

and a normalized frequency

$$f \equiv \omega/k_0 . \tag{50}$$

We are interested in the relativistic regime of a considerable Doppler upshift,

$$f \gg 1 . \tag{51}$$

This last inequality and the smallness of both Δ and ξ enable us to make the following approximations:

$$(k-k_0)^2-\omega^2 \cong 2k_0^2 f(\Delta+\xi) , \quad (k+k_0)^2-\omega^2 \cong 4k_0^2 f . \tag{52}$$

Using the variables

$$\delta E_\pm \equiv \frac{1}{\sqrt{2}}(\delta E_1 \mp i\delta E_2) , \tag{53}$$

and the definitions and assumptions (43)–(53), we obtain a more compact form of Eqs. (40)–(42). The equations for the wiggler FEL become

$$(\Delta+\xi)\delta E_+ + 2\delta E_- = 0 , \quad (\Delta+\xi)\delta E_+ - 2\delta E_- = -\frac{2s}{\Delta^2}(\delta E_+ - \delta E_-) , \tag{54}$$

those for the wiggler-free FEL become

$$2f(\Delta+\xi)\delta E_+ = \left[\frac{\bar{h}}{\Delta} - \frac{2s}{\Delta^2}f(\Delta+\xi)\right]\delta E_+ + \frac{2s}{\Delta^2}f\delta E_- , \quad 4f\delta E_- = \left[-\frac{\bar{h}}{\Delta} - \frac{4s}{\Delta^2}f\right]\delta E_- + \frac{2s}{\Delta^2}f\delta E_+ , \tag{55}$$

and those for the gyrotron

$$2f(\Delta+\xi)\delta E_+ = \left[\frac{\bar{h}}{\Delta} - \frac{2s}{\Delta^2}f(\Delta+\xi)\right]\delta E_+ , \quad 4f\delta E_- = \left[-\frac{\bar{h}}{\Delta} - \frac{4s}{\Delta^2}f\right]\delta E_- . \tag{56}$$

We can now examine the roots of the dispersion relations. Equation (54) yields for the wiggler FEL

$$\Delta^2(\Delta+\xi) = -s , \tag{57}$$

which is the well-known cubic polynomial dispersion relation for the wiggler FEL in the strong-pump regime.[10] The growth rate is given by the imaginary part of the nonreal root, and its maximal value is

$$\mathrm{Im}\Delta = \frac{\sqrt{3}}{2}s^{1/3} . \tag{58}$$

In the gyrotron the two polarizations δE_+ and δE_- decouple. The first of Eqs. (56) may be written

$$2f(\Delta+\xi)\left[1+\frac{s}{\Delta^2}\right] - \frac{\bar{h}}{\Delta} = 0 . \tag{59}$$

When the mismatch parameter ξ is not too small, off-resonance, the last term on the left-hand side can be neglected, and

$$1+\frac{s}{\Delta^2} = 0 , \tag{60}$$

which yields the result for the growth rate

$$\mathrm{Im}\Delta = s^{1/2} . \tag{61}$$

Since s is much smaller than 1, the growth rate in the gyrotron (61) is smaller than in the wiggler FEL (58). At resonance, when ξ is small,if

$$f\frac{s}{\bar{h}} = \frac{f}{2}\left[\frac{p_2}{p_3}\right]^2 \ll 1 , \tag{62}$$

Eq. (59) can be approximated as

$$2f\Delta - \frac{\bar{h}}{\Delta} = 0 , \tag{63}$$

Δ is real, and the gyrotron instability disappears.

We now turn to the wiggler-free FEL. From Eq. (55) it is clear that the diagonal terms of the dielectric tensor in the wiggler-free FEL and in the gyrotron are identical. The difference is in the off-diagonal terms which are zero in the gyrotron and proportional to s^2 in the wiggler-free FEL. Since s is small one would expect this difference to be small. However, at resonance these off-diagonal terms determine the growth rate of the instability, and make the wiggler-free FEL very different from the gyrotron. To show this we write the dispersion relation for the wiggler-free FEL as follows:

$$\left[2f(\Delta+\xi)\left[1+\frac{s}{\Delta^2}\right]-\frac{\bar{h}}{\Delta}\right]\left[4f\left[1+\frac{s}{\Delta^2}\right]+\frac{\bar{h}}{\Delta}\right] - \frac{4s^2}{\Delta^4}f^2 = 0 . \tag{64}$$

The dispersion relation for the gyrotron is the same except for the last small coupling term. When ξ is not small, this coupling term is negligible and the roots of the dispersion equation (64) are identical to those in the gyrotron. At resonance, when ξ is small, and if

$$\frac{\bar{h}}{2f} \ll (2s^2)^{2/5} , \tag{65}$$

the dispersion relation is simplified,

$$\Delta^4(\Delta+\xi) = \frac{s^2}{2} . \tag{66}$$

When ξ is zero, this fifth-order polynomial has two pairs of nonreal roots, and the maximal growth rate is

$$\mathrm{Im}\Delta = 0.83 s^{2/5} . \tag{67}$$

Comparing the growth rates, we see that the growth rate of the wiggler-free FEL (67) is larger than that of the gyrotron (61) and smaller than that of the wiggler FEL (58). The behavior of the growth rate in the wiggler-free FEL is similar to that in the wiggler FEL in that both are larger on resonance and decrease off resonance. The gyrotron differs in this respect as the growth rate is uniform near resonance and usually vanishes at resonance.

We turn now to a numerical example. The parameters in Fig. 1 are $s = 10^{-5}$, $\bar{h} = 10^{-3}$, and $f = 100$. The inequality (65) is satisfied. We calculate the growth rates for the three systems by solving the dispersion equations (57), (59), and (64). For the wiggler-free FEL and for the gyrotron the solutions of the approximate dispersion relations (66) and (60) are also shown. There is good agreement between the results of the exact and approximate calculations of the gains. Note that the wiggler-free FEL has two growing modes and that far from resonance both eigenvalues converge to those of the gyrotron.

We demonstrate how one finds the actual physical parameters for these devices which lead to the dimensionless parameters of the numerical example. Following the definitions (48) and (49), the ratio of the perpendicular to

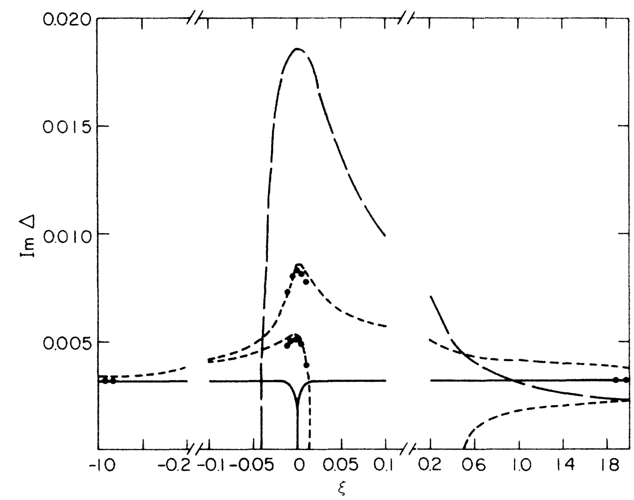

FIG. 1. Normalized growth rate vs the mismatch parameter for the three devices: the gyrotron (solid line), the wiggler-free FEL (short dashes) and the wiggler FEL (long dashes). The dots represent values found by the approximate expressions.

parallel momentum (p_2/p_3) is 0.141 in the numerical example. Also, because of Eq. (46), γ is 7.1, which corresponds to beam energy of about 3 MeV. By specifying a value of k_0 we determine the intensity of the uniform field in the gyrotron and in the wiggler-free FEL through Eq. (7) and the intensity of the wiggler-field in the wiggler FEL through Eq. (8). Moreover, from Eq. (49) we find the beam current density and from Eq. (50) we find the wavelength of the radiation. Finally, by multiplying ImΔ by k_0 we obtain the value of the growth rate in cm^{-1}. As a first example let us assume that k_0 is 0.86 cm^{-1}. The uniform field B_0 is found ot be 10 kG and the wiggler field B_W is 1.46 kG. The current density is 7 $\mathrm{A/cm}^2$ and the wavelength of the radiation is 0.73 mm. The maximum growth rate is found to be 0.0159 cm^{-1} for the wiggler FEL, 0.0027 cm^{-1} for the gyrotron, and 0.0071 cm^{-1} for the wiggler-free FEL. We now assume k_0 to be 4.3 cm^{-1}. In this case B_0 is 50 kG and B_W is 7.3 kG. The current density becomes 175 $\mathrm{A/cm}^2$ and the wavelength is 0.15 mm. The maximum growth rate is now 0.0795 cm^{-1} for the wiggler FEL, 0.0135 cm^{-1} for the gyrotron, and 0.0355 cm^{-1} for the wiggler-free FEL.

In summary, we derived the dispersion relations for the gyrotron, the wiggler FEL, and the wiggler-free FEL in a unified fluid model. Approximate expressions for the growth rates were found. The same small parameter characterizes the strength of the instability in the three systems. The maximal growth rate in the wiggler FEL scales as the small parameter to the power of $\frac{1}{3}$ and in the gyrotron to the power of $\frac{1}{2}$. The maximum gain in the gyrotron is off-resonance and at resonance the gain usually vanishes. In the wiggler-free FEL the growth rate off-resonance is similar to that of the gyrotron. However, at resonance the growth rate increases as in the wiggler FEL. It scales as the small parameter to the power of $\frac{2}{5}$, larger than in the gyrotron and smaller than in the wiggler FEL.

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