Generation of nonclassical photon states using correlated photon pairs and linear feedforward

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It is shown that nonclassical photon states can be produced by generating correlated photon pairs, measuring the idler mode, and manipulating the signal mode using feedforward and linear attenuators, amplifiers, or phase modulators. The minimum achievable Fano factor for such schemes is $(2/\pi\langle n\rangle)^{1/2}$, where $\langle n\rangle$ is the mean output photon number. A nondegenerate parametric oscillator followed by linear manipulators can reach this limit. A nondegenerate parametric amplifier followed by linear phase manipulation can produce phase-squeezed states. The minimum achievable phase noise is a factor $1/2G$ below that of a coherent state, where G is the amplifier gain. The advantages with these schemes are wavelength tunability, potentially very high squeezing bandwidth, and relative simplicity.

I. INTRODUCTION

It has been known for some time now that correlated photon pairs can be used to produce nonclassical photon states. If the photons in two electromagnetic modes are correlated, the measured noise (and this may be either the phase noise or the photon-number noise) in one of the modes can be reduced by measuring the corresponding fluctuations in the other mode and manipulating the photons in the first mode accordingly. In this paper we will try to clarify the limits for such noise reduction schemes which use only *linear* manipulating devices and *feedforward.* One of the reasons for considering such a narrow class of devices is simplicity. Linear devices controlled by feedforward will in general not fully utilize the photon correlation, but on the other hand they have potentially very large bandwidth, and they are simple.

In Sec. IV, some specific examples of linear feedforward noise reduction schemes are elaborated, in which the correlated photons are produced in a parametric down-conversion process. Noise reduction schemes using this specific photon pair generator have already been proposed by various authors,[1–6] but none of them has considered simple linear feedforward, and only in Ref. 6 has a possible phase correlation been considered.

A question which is intimately coupled with the limits of noise reduction schemes which are based on correlated photon pairs, is the question of how perfect the photon correlation may be in the first place. Does quantum mechanics tolerate generation of photon twins, namely, perfectly correlated photons? While we will not try to answer this question definitively, some of the preliminary results in Sec. V indicate that this is impossible if the input states are classical, unless either the phase- or the photon-number fluctuations of the output modes are infinite.

II. PHASE NOISE REDUCTION SCHEME

First we will consider generation of phase-squeezed states using linear feedforward. The idea is outlined in Fig. 1. It is assumed that the generator emits two modes, separated either in frequency, polarization, or in propagation direction. The emitted photons are assumed to be phase correlated. A physical realization of such a generator is a high-gain nondegenerate optical parametric amplifier. In the rest of the paper the two modes will, for simplicity, be referred to as the signal and the idler modes. The phase of the idler is measured using a balanced homodyne receiver.[6,7] The result is fed forward to a phase modulator to manipulate the phase of the signal. Phase is not a well-defined quantity unless the signal photon number is high. There is no phase operator in quantum mechanics, but for high photon numbers it can be approximated by the sine operator

$$\hat{\phi}\approx\hat{S}=\frac{1}{2i}[(\hat{n}+1)^{-1/2}\hat{b}-\hat{b}^{\dagger}(\hat{n}+1)^{-1/2}] . \quad (1)$$

We will approximate this equation further by assuming that the signal and idler photon numbers are large enough to approximate the photon-number operators with their mean values. Recognizing the identity $\hat{b}_2\equiv(1/2i)(\hat{b}-\hat{b}^{\dagger})$, where $\hat{b}_2$ is the quadrature phase amplitude operator, (1) can be rewritten

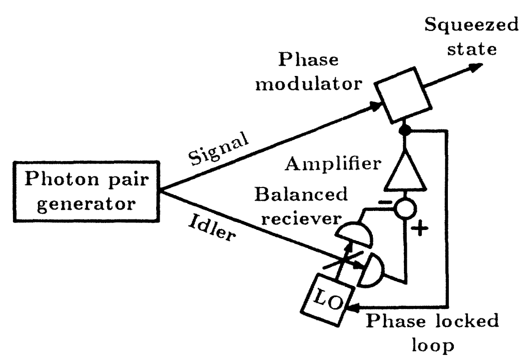

FIG. 1. Phase noise reduction scheme using homodyne detection of the idler phase and a linear phase modulator.

$$\hat{\phi} \approx \frac{\hat{b}_2}{\sqrt{\langle n \rangle + 1}} . \quad (2)$$

Equation (2) may seem like a very rough approximation of (1), but at high mean photon numbers it is actually a very good one.

The operator governing the phase modulator will also be an approximate one. It is reasonable to assume that the classical equation for a phase modulator can be extended to quantum mechanics for any input state which, in some sense, has a defined phase. The relation between the phase of the signal input into the phase modulator and the output signal phase will thus be

$$\hat{\phi}_{\text{out}} - \hat{\phi}_{\text{in}} \approx C\hat{V} , \quad (3)$$

where C is a constant and $\hat{V}$ is the voltage operator over the phase modulator. There are two reasons why we think that (3) describes the phase modulator correctly. First, the phase modulator is dissipation free and does not have to have any fluctuations associated with it. Second, the modulating voltage may be, and in practice is, a macroscopic variable. No further fluctuations, such as thermal noise, need thus be transferred to the signal from the modulation voltage. For these reasons we believe that the fluctuations of the output phase stemming from the phase modulator and the "back action" of the phase modulator on the signal photon number are negligible, even considering quantum-state preparation schemes.

This is an important difference from the photon-number manipulation schemes presented later on. While *linear* manipulation of the photon number always introduces intrinsic fluctuations, the linear phase manipulator is noiseless.

The homodyne detector can be shown[6,7] to noiselessly measure the idler quadrature phase amplitude $\hat{b}_{i2}$, if the output power of the local oscillator is sufficiently high. Due to the local oscillator gain, the output is then a macroscopic variable and need not suffer from any additional noise. With proper adjustment of the feedforward gain, the signal output phase operator will thus be given by

$$\hat{\phi}_{s,\text{out}} = \hat{\phi}_{s,\text{in}} \pm \frac{\hat{b}_{i2}}{\sqrt{\langle n \rangle + 1}} = \frac{\hat{b}_{s2} \pm \hat{b}_{i2}}{\sqrt{\langle n \rangle + 1}} . \quad (4)$$

Equation (4) implies that if the correlation between the signal and idler is perfect (positive or negative), the phase noise of the signal can be completely suppressed. This can only be true if the photon-number fluctuations of the signal are infinite. Heisenberg's uncertainty principle thus imposes a "best" photon phase correlation allowed, for a given signal photon-number fluctuation. It is clear that perfect phase correlation can only be achieved in the limit when the signal photon number goes to infinity. We will probe deeper into this subject in Secs. IV and V.

III. PHOTON-NUMBER NOISE REDUCTION SCHEMES

The starting point of the photon-number noise reduction schemes is pictured in Fig. 2. The emitted photons are assumed to be perfectly correlated and have Poissonian photon counting statistics. An approximate physical realization of such a generator is a nondegenerate optical parametric oscillator, pumped at a few times above threshold. The photon number (or, in a continuous scheme, the photon flux) of the idler mode is measured by a photodetector, which is assumed to have unity quantum efficiency. The measurement is fed forward to a control circuit which regulates the amplification and/or the attenuation of the signal beam.

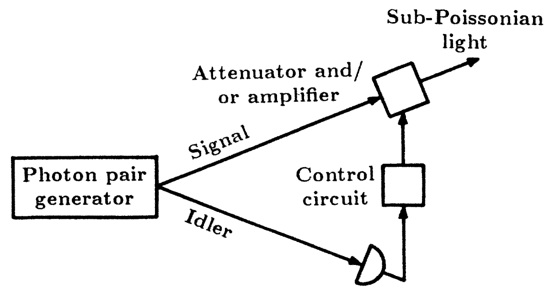

FIG. 2. Photon-number noise reduction scheme using a correlated photon pair generator and linear manipulation of the signal-mode photon number.

Five different schemes will be examined. The difference between them is the device used to manipulate the signal beam. The different schemes are attenuation; amplification; photon adding; attenuation and amplification in tandem; and, finally, attenuation and photon adding in tandem. The schemes can be divided into two groups, the first three schemes, which can only manipulate the photon statistics in one direction, and the last two, which can either subtract or add photons.

A. Attenuation

The first scheme to be considered is the one incorporating an attenuator as the signal photon statistics manipulator. The relation between the field-amplitude input and output operators, $\hat{a}$ and $\hat{b}$, can be written

$$\hat{b} = \sqrt{\epsilon}\hat{a} + \sqrt{1-\epsilon}\hat{c} , \quad (5)$$

where ϵ is the transmittivity of the attenuator and $\hat{c}$ is the field amplitude operator of a vacuum state. The conditional probability that the output mode contains n photons, given that the input mode had k photons and that the attenuator transmittivity is ϵ, can be calculated from (5). It is

$$P(n \mid k) = \begin{cases} \dfrac{k!\epsilon^n(1-\epsilon)^{k-n}}{(k-n)!n!} & \text{if } n \le k \\ 0 \quad \text{if } n > k . \end{cases} \quad (6)$$

The mean value of n, $\langle n \rangle$, given k and ϵ, is $k\epsilon$, while the variance of n, $\langle n^2 \rangle$ is given by $(k\epsilon)^2 + k\epsilon(1-\epsilon)$. The attenuator will be used to selectively attenuate the signal mode every time the measured photon number is greater than some threshold value, N_d. If the measured photon number is less than N_d, the attenuator will be perfectly transparent, $\epsilon=1$. Thus the attenuator will not alter the signal output states with a photon number less than N_d. It is rather obvious that N_d should optimally be chosen smaller than the mean input signal photon number.

What is not obvious is the optimum control law for the attenuator. Given that the idler photon counter registered k photons, how much should the signal be attenuated? We have chosen a very simple strategy, namely, to see that the mean of n, given k, will be N_d every time $k \geq N_d$. While this may not be the optimal strategy, it should be close to it, and at high mean input photon numbers approach the ideal control law. Thus the transmittivity, given the idler count k, will be $\epsilon=N_d/k$ if $k \geq N_d$. The conditional probability of the output becomes

$$P(n\mid k)=\begin{cases} 0 \quad \text{if } n>k \text{ and } k>N_d \\ \dfrac{k!(N_d/k)^n(1-N_d/k)^{k-n}}{(k-n)!n!} \quad \text{if } n\leq k \text{ and } k>N_d \\ \delta_{nk} \quad \text{if } k\leq N_d, \end{cases} \tag{7}$$

where δ_{nk} is Kronecker's δ function. Summing (7) over all possible input states yields the probability density of the output

$$P(n)=\sum_{k=0}^{\infty} P(k)P(n\mid k)\ . \tag{8}$$

To calculate the Fano factor,

$$F=\frac{\langle n^2\rangle-\langle n\rangle^2}{\langle n\rangle}\ , \tag{9}$$

and this is the quantity we want to minimize, the mean and the variance of the output distribution must be calculated. The mean is calculated as

$$\langle n\rangle=\sum_{n=0}^{\infty} n\sum_{k=0}^{\infty} P(k)P(n\mid k)\ . \tag{10}$$

Reversing the order of summation, using the assumption that the input is a Poissonian, and inserting (7), the mean can be written

$$\langle n\rangle=\exp(-\alpha^2)\left[\sum_{k=0}^{N_d}\frac{k\alpha^{2k}}{k!}+\sum_{k=N_d+1}^{\infty}\frac{\alpha^{2k}}{k!}\sum_{n=0}^{k}\frac{nk!(N_d/k)^n(1-N_d/k)^{k-n}}{(k-n)!n!}\right], \tag{11}$$

where α^2 is the mean photon number of the input state. The last sum is simply the mean of a binominal distribution and can be solved:

$$\langle n\rangle=\exp(-\alpha^2)\left[\sum_{k=0}^{N_d}\frac{k\alpha^{2k}}{k!}+\sum_{k=N_d+1}^{\infty}\frac{N_d\alpha^{2k}}{k!}\right]=\alpha^2+\exp(-\alpha^2)\sum_{k=N_d+1}^{\infty}\frac{(N_d-k)\alpha^{2k}}{k!}\ . \tag{12}$$

The variance can be solved in a similar way yielding

$$\begin{aligned}\langle n^2\rangle&=\exp(-\alpha^2)\left[\sum_{k=0}^{N_d}\frac{k^2\alpha^{2k}}{k!}+\sum_{k=N_d+1}^{\infty}\frac{[N_d^2+N_d(1-N_d/k)]\alpha^{2k}}{k!}\right]\\ &=\alpha^4+\alpha^2+\exp(-\alpha^2)\sum_{k=N_d+1}^{\infty}\frac{(N_d^2+N_d-N_d^2/k-k^2)\alpha^{2k}}{k!}\ .\end{aligned} \tag{13}$$

The sums in (12) and (13) and the minimization of the Fano factor cannot be solved analytically. In Fig. 3, the output probability distribution is shown. In Fig. 4, the minimum achievable Fano factor, solved by a computer, is plotted versus the mean output photon number. As can be seen the Fano factor can be reduced substantially below unity, and it decreases with increasing input photon number. In Fig. 5 the normalized optimum decision threshold N_d is plotted versus the mean input photon number.

For large mean input photon numbers the exact Poissonian distribution becomes slightly difficult to handle. The computer the authors used could not handle numbers larger than about 200!. While this can be overcome by more clever programming than the present authors used, an efficient way around this problem is to approximate the statistics of the input state by an exponential distribution. This turned out to be a good approximation for mean photon numbers as low as 10. The approximate output mean and variance becomes

$$\langle n\rangle\approx\frac{1}{\sqrt{2\pi}\alpha}\int_{N_d}^{\infty}(N_d-x)\exp\left[-\frac{(x-\alpha^2)^2}{2\alpha^2}\right]dx \tag{14}$$

and

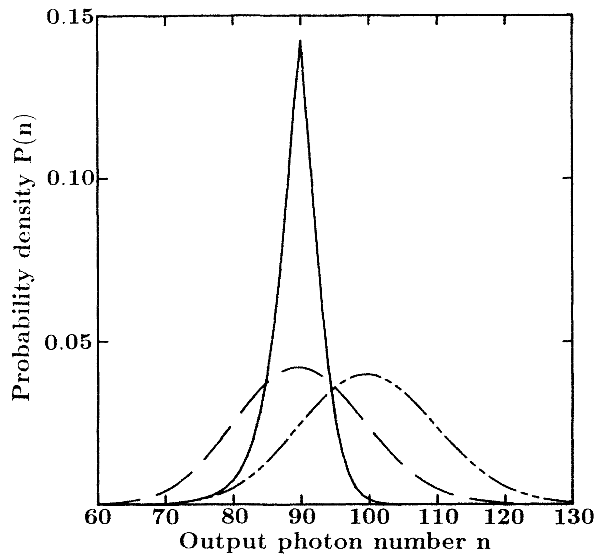

FIG. 3. Probability density vs photon number. The dash-dotted curve is the input distribution (Poissonian with $\langle n \rangle = 100$). The solid line is the output distribution at the optimum decision threshold, $N_d = 91$. The dashed curve is a Poissonian distribution with the same mean photon number as the output distribution, $\langle n \rangle \approx 90$.

$$\langle n^2 \rangle \approx \frac{1}{\sqrt{2\pi}\alpha} \int_{N_d}^{\infty} (N_d^2 + N_d - N_d^2/x - x^2) \times \exp\left[-\frac{(x-\alpha^2)^2}{2\alpha^2} \right] dx \; . \tag{15}$$

Using this approximation the right-hand part of the appropriate curve in Figs. 4 and 5 was computed. At high mean photon numbers the Fano factor for the attenuation scheme is approximately proportional to $\langle n \rangle^{-0.466}$.

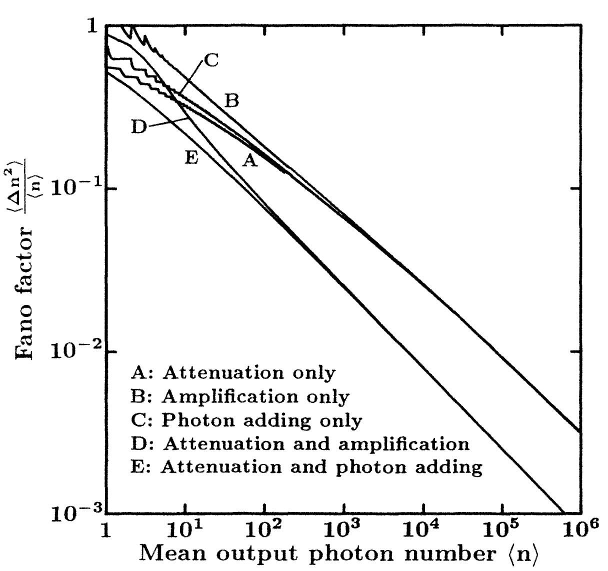

FIG. 4. Optimized Fano factor vs the mean output photon number. The solutions are exact up to a mean photon number of about 200. For higher excitations, the approximate exponential distributions have been used.

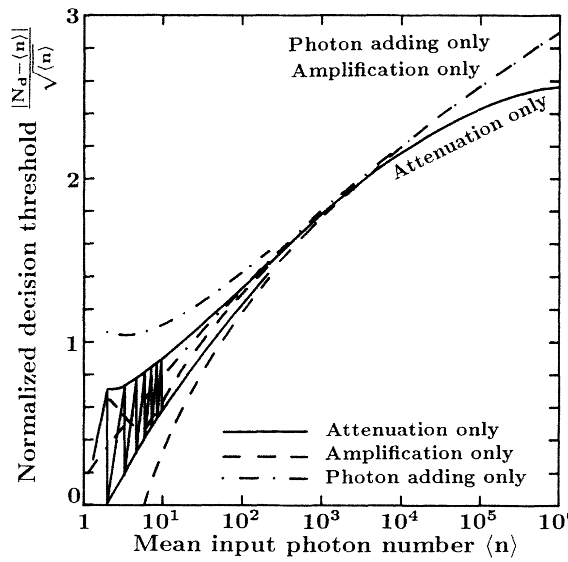

FIG. 5. Normalized optimum decision threshold vs the mean input photon number. The normalized decision threshold oscillates between lower and upper limits due to the fact that N_d must have an integer value, whereas the mean photon number is continuous. The oscillations are shown only for the attenuator scheme for photon numbers up to 10. For the other schemes only the limits are shown.

B. Amplification

An alternative to attenuating all output photon states with a photon number higher than N_d is to amplify all photon states with a photon number smaller than N_d. The equation for the field amplitude annihilation operators for a linear, phase-insensitive amplifier is

$$\hat{b} = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{c} \; , \tag{16}$$

where G is the gain of the amplifier and $\hat{c}$ is the field operator of a vacuum state. In a parametric amplifier, $\hat{c}$ represents the input state of the idler.

Using the proper unitary evolution operator, the conditional probability for finding n photons in the output state, given that k photons were input and that the gain is G, can be expressed as[8,9]

$$P(n \mid k) = \begin{cases} 0 \quad \text{if } n < k \\ \dfrac{1}{k!G(G-1)^k} \left[\dfrac{G-1}{G} \right]^n \dfrac{n!}{(n-k)!} \quad \text{if } n \geq k . \end{cases} \tag{17}$$

the mean value of n, given k and G, is given by

$$\langle n \rangle = G - 1 + Gk \; , \tag{18}$$

while the variance is

$$\langle n^2 \rangle = G^2k^2 + 3G^2k - 3Gk + 2G^2 - 3G + 1 \ . \quad (19)$$

As can be seen from (18) the mean amplification is not directly proportional to the input due to the inevitable amplification of the vacuum state $\hat{c}$. The control law chosen for the amplifier will thus be slightly more difficult than the simple one employed for the attenuator. The basic strategy is the same, though. The gain is to be adjusted so that on the average, all manipulated states will have N_d photons. The gain will thus be given by

$$G = \begin{cases} \dfrac{N_d+1}{k+1} & \text{if } k < N_d \\ 1 & \text{if } k \geq N_d . \end{cases} \quad (20)$$

In Fig. 6 the input distribution and the minimized output distribution can be seen.

Using the same procedures as in Sec. III B the mean output photon number can be computed as

$$\langle n \rangle = \exp(-\alpha^2) \left[\sum_{k=0}^{N_d-1} \frac{\alpha^{2k}}{k!} \sum_{n=k}^{\infty} \frac{n!n}{k!(n-k)!G(G-1)^k} \left[\frac{G-1}{G} \right]^n + \sum_{k=N_d}^{\infty} \frac{k\alpha^{2k}}{k!} \right] . \quad (21)$$

The sum over n can be solved because G is independent of n:

$$\langle n \rangle = \exp(-\alpha^2) \left[\sum_{k=0}^{N_d-1} \frac{(G-1+Gk)\alpha^{2k}}{k!} + \sum_{k=N_d}^{\infty} \frac{k\alpha^{2k}}{k!} \right] . \quad (22)$$

Inserting (20) in (22), the mean is simplified to

$$\langle n \rangle = \alpha^2 + \exp(-\alpha^2) \sum_{k=0}^{N_d-1} \frac{(N_d - k)\alpha^{2k}}{k!} . \quad (23)$$

The variance can be expressed as

$$\langle n^2 \rangle = \alpha^4 + \alpha^2 + \exp(-\alpha^2) \sum_{k=0}^{N_d-1} \frac{\left[N_d^2 - N_d - 1 - k^2 - \dfrac{(N_d+1)^2}{(k+1)} \right] \alpha^{2k}}{k!} . \quad (24)$$

Again it is helpful to approximate the Poissonians in (23) and (24) with exponetials,

$$\langle n \rangle \approx \frac{1}{\sqrt{2\pi}\alpha} \int_{-\infty}^{N_d} (N_d - x) \exp \left[-\frac{(x-\alpha^2)^2}{2\alpha^2} \right] dx , \quad (25)$$

and

$$\langle n^2 \rangle \approx \frac{1}{\sqrt{2\pi}\alpha} \int_0^{N_d} \left[N_d^2 - N_d - 1 - x^2 + \frac{(N_d+1)^2}{x+1} \right] \times \exp \left[-\frac{(x-\alpha^2)^2}{2\alpha^2} \right] dx . \quad (26)$$

Note the lower integration limit in (26) to avoid the singularity of the argument at $x=-1$. The minimum Fano factor for this scheme (Fig. 4) is slightly worse at low photon numbers than the attenuator scheme. This may be due to the fact that for input photon numbers in the order of unity, the amplifier will relatively often have to amplify by factors of 2 to 3. At such "high" gain, relatively much noise is added by the inevitable amplification of the vacuum state $\hat{c}$. This beats against the amplified signal and increases the variance of the output state. At

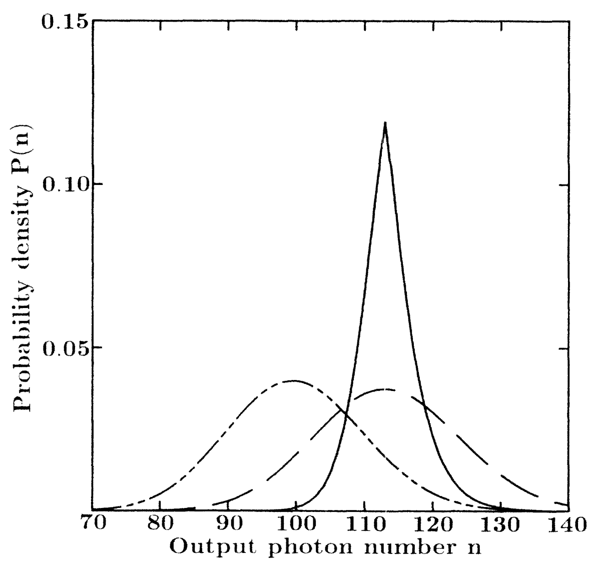

FIG. 6. Probability density vs photon number. The dash-dotted curve is the input distribution (Poissonian with $\langle n \rangle = 100$). The solid line is the output distribution at the optimum decision threshold, $N_d = 112$. The dashed curve is a Poissonian distribution with the same mean photon number as the output distribution, $\langle n \rangle \approx 112$.

high mean inputs, the performance of the feedforward amplifier approaches that of the attenuator.

C. Photon adding

This scheme is a hybrid scheme in the respect that the output is not in a single mode. The principle is outlined in Fig. 7. The output of a laser, whose polarization is orthogonal to the signal mode's, is added via a polarization beamsplitter. The main advantage with this scheme is that is it simple. Low-noise semiconductor lasers are commercially available, and their photon-counting statistics can be made to approach a Poissonian distribution by attenuating the output by a known amount. Furthermore polarization beamsplitters can be made virtually lossless. Commercially available amplifiers and A/O modulators all suffer from insertion losses which degrade the achievable photon-number noise reduction.

The same control strategy will be employed with this scheme as with the previous. Thus, if $k<N_d$, the conditional output from the laser will be

$$P(m\mid k)=\frac{\exp(k-N_d)(N_d-k)^m}{m!}\,, \tag{27}$$

where m is the output photon number of the adding laser. Assuming that the polarization beamsplitter is lossless, the detected photon number n can be written $n=k+m$. The added mode conditional probability will be

$$P(n\mid k)=\begin{cases}0 \text{ if } n<k\\ \dfrac{\exp(k-N_d)(N_d-k)^{n-k}}{(n-k)!} \quad \text{if } n\geq k \text{ and } k<N_d\\ \delta_{nk} \text{ if } k\geq N_d .\end{cases} \tag{28}$$

The output probability distribution will look very similar to that of the previous scheme. At a mean input photon number of 100, the difference between the probability distribution of this scheme and the previous one is imperceptible within the resolution of the figure. The optimum decision threshold will also be very close.

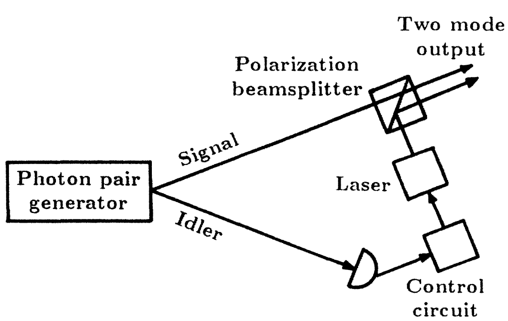

FIG. 7. Photon adding scheme using a polarization orthogonal laser and a polarization beamsplitter.

The mean of the added distribution will be given by (23). The variance can be expressed as

$$\langle n^2\rangle=\alpha^4+\alpha^2+\exp(-\alpha^2)\sum_{k=0}^{N_d-1}\frac{(N_d^2+N_d-k-k^2)\alpha^{2k}}{k!}\,. \tag{29}$$

The exponential approximations for the mean and the variance should be rather obvious. As can be seen in Fig. 4, this scheme is slightly worse than the attenuation scheme, but slightly better than the amplifier scheme. At high photon numbers the difference becomes negligible, and the optimum decision threshold will approach that of the amplification scheme (Fig. 5).

D. Attenuation and amplification in tandem

Using both an attenuator and an amplifier, the control law becomes obvious. If the generator signal mode has fewer than the mean number of photons, it is amplified, so that on the average the amplifier output photon number equals the generator mean. If the generator signal mode has too many photons, attenuation is used to do the opposite. The attenuation is thus governed by the control law

$$\epsilon=\begin{cases}1 \text{ if } k\leq\alpha^2\\ \alpha^2/k \text{ if } k>\alpha^2,\end{cases} \tag{30}$$

and the gain is adjusted according to

$$G=\begin{cases}\dfrac{\alpha^2+1}{k+1} \text{ if } k\leq\alpha^2\\ 1 \text{ if } k>\alpha^2.\end{cases} \tag{31}$$

The conditional output probability will be given by (17) if $k\leq\alpha^2$ and by (6) if $k>\alpha^2$. In Fig. 8 the output distribution is shown.

The average output photon number is readily calculated to be the same as the input mean, $\langle n\rangle=\alpha^2$. This is to be expected considering the control law. The variance is found to be

$$\langle n^2\rangle=\alpha^4+\exp(-\alpha^2)\left[\sum_{k=0}^{\alpha^2-1}\left[\frac{(\alpha^2+1)^2}{(k+1)}-\alpha^2-1\right]\frac{\alpha^{2k}}{k!}+\sum_{k=\alpha^2+1}^{\infty}\left[\alpha^2-\frac{\alpha^4}{k}\right]\frac{\alpha^{2k}}{k!}\right]. \tag{32}$$

Note that in (32) it is assumed that α^2 is an integer. This is of course not necessarily true, but the minimum Fano factors are realized when the mean generator output photon number takes on integer values. The curves for this scheme and the next are plotted only for integer values of α^2.

The exponential distribution approximation of the variance is

$$\langle n^2\rangle \approx \alpha^4+\frac{1}{\sqrt{2\pi}\alpha}\int_{-\infty}^{\alpha^2}\left[\frac{(\alpha^2+1)^2}{x+1}-\alpha^2-1\right]\exp\left[-\frac{(x-\alpha^2)^2}{2\alpha^2}\right]dx$$
$$+\frac{1}{\sqrt{2\pi}\alpha}\int_{\alpha^2}^{\infty}(\alpha^2-\alpha^4/x)\exp\left[-\frac{(x-\alpha^2)^2}{2\alpha^2}\right]dx \; . \tag{33}$$

The second term in the first integral and the first term in the second integral cancel. The third term in the first integral is trivial. The integrals can thus be rewritten

$$\langle n^2\rangle \approx \alpha^4-\frac{1}{2}+\frac{(\alpha^2+1)^2}{\sqrt{\pi}}\int_0^{\infty}\frac{\exp(-u^2)du}{\alpha^2+1-\sqrt{2}\alpha u}$$
$$-\frac{\alpha^4}{\sqrt{\pi}}\int_0^{\infty}\frac{\exp(-u^2)du}{\alpha^2+\sqrt{2}\alpha u} \; . \tag{34}$$

Expanding the arguments of the integrals in (34) and keeping only the first-order terms, an approximate solution to the integrals can be found:

$$\langle n^2\rangle \approx \alpha^4-\frac{1}{2}+\frac{(\alpha^2+1)}{\sqrt{\pi}}\int_0^{\infty}\left[1+\frac{\sqrt{2}\alpha u}{\alpha^2+1}\right]\exp(-u^2)du$$
$$-\frac{\alpha^2}{\sqrt{\pi}}\int_0^{\infty}\left[1-\frac{\sqrt{2}u}{\alpha}\right]\exp(-u^2)du$$
$$=\alpha^4+(2\alpha^2/\pi)^{1/2} \; . \tag{35}$$

The minimum achievable Fano factor using correlated photon pairs, idler measurement feedforward, and *linear* manipulation of the signal photon number is thus approximately

$$F_{\min}=\left[\frac{2}{\pi\alpha^2}\right]^{1/2}=\left[\frac{2}{\pi}\right]^{1/2}\langle n\rangle^{-1/2} \; . \tag{36}$$

The curve for the attenuation and amplification scheme in Fig. 4 approaches this limit asymptotically. The reason this limit cannot be surpassed is that the linear manipulation of photon number always introduces noise, manifested by the vacuum field operator $\hat{c}$ in (5) and (16). If the photon correlation is less than perfect, the minimum Fano factor will be even greater than indicated by (36).

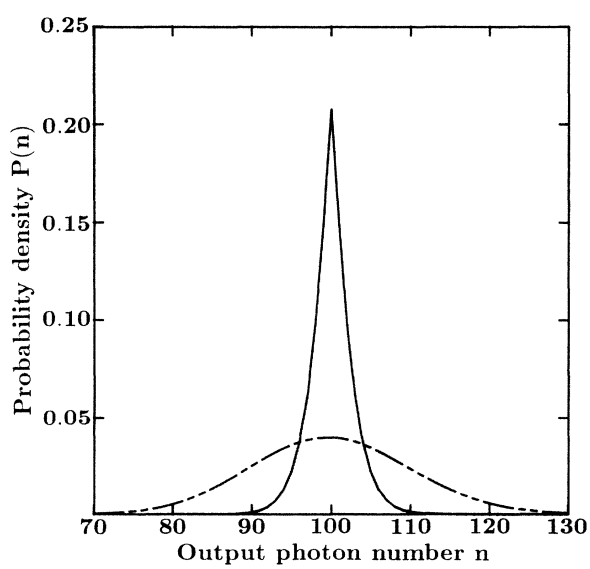

FIG. 8. Probability density vs photon number. The dash-dotted curve is the input distribution (Poissonian with $\langle n\rangle=100$). The solid line is the output distribution.

E. Attenuation and photon adding in tandem

The last scheme to be treated is again a hybrid scheme. The signal is first passed through an attenuator with variable attenuation, and then it is combined with the output of a laser which has an orthogonal polarization. The control law for the attenuator is given by (30). The control law for the laser is that its average output photon number m, given the measured idler photon count k, is

$$\langle m\rangle=\begin{cases}\alpha^2-k & \text{if } k<\alpha^2\\ 0 & \text{if } k>\alpha^2.\end{cases} \tag{37}$$

The mean is again the same as that of the initial signal distribution, $\langle n\rangle=\alpha^2$. The variance is

$$\langle n^2\rangle=\alpha^4+\alpha^2-\exp(-\alpha^2)\left[\sum_{k=0}^{\alpha^2-1}\frac{k\alpha^{2k}}{k!}+\sum_{k=\alpha^2}^{\infty}\frac{\alpha^{2(k+2)}}{k!k}\right] . \tag{38}$$

Using exponential approximations of the Poissonian in (38), and expanding the arguments in a similar way as in Sec. III D, it is easy to show that for high photon numbers, (35) and (36) are retained. The curve for the attenuation and photon adding scheme in Fig. 4 also approaches the limit (36) asymptotically. Photon adding, employing a coherent state, can thus be considered to be a linear process. The uncertainty associated with photon adding is due to the uncertainty in the added photon number.

The result (36) should be compared with the minimum Fano factors achievable with other schemes. They are

$$F_{\min}=\begin{cases}0 & \text{for the number state}\\ (\tfrac{3}{4})^{7/6}\langle n\rangle^{-2/3} & \text{for the self-phase modulation}\\ \tfrac{3}{4}\langle n\rangle^{-1/3} & \text{for the in-phase squeezed state}\\ 1 & \text{for the classical states.}\end{cases}$$

The value for the self-phase modulation is from Ref. 10. It can be seen that in spite of the fact that linear feedforward manipulation of the photon number cannot fully utilize the photon correlation, it is a competitive scheme for generation of sub-Poissonian photon states.

IV. PARAMETRICALLY GENERATED PHOTON PAIRS

The parametric process is one of the most promising ways of producing correlated photon pairs. Expanding the signal and idler annihilation operators in quadrature components, $\hat{a}_s \equiv \hat{a}_{s1} + i\hat{a}_{s2}$, etc., (16) can be rewritten

$$\hat{b}_{s1} = \hat{b}_{i1} + \exp(-r)(\hat{a}_{s1} - \hat{a}_{i1}) , \quad (39)$$

$$\hat{b}_{s2} = -\hat{b}_{i2} + \exp(-r)\hat{a}_{s2} + \hat{a}_{i2}) , \quad (40)$$

where

$$r = \operatorname{arccosh}(\sqrt{G}) \approx \tfrac{1}{2}\ln(G) \quad \text{when } G \gg 1 . \quad (41)$$

It is clear from these equations that the signal and idler in-phase correlation (photon-number correlation) is positive, whereas the quadrature-phase correlation (phase correlation) is negative. As the gain approaches infinity, the correlation becomes perfect.

Let us first consider the parametric amplifying process. The operator Manley-Rowe relation,[11] which is quite different from (39) and (40), although related to them, states that an equal number of signal and idler photons are *created* by the parametric process,

$$\hat{n}_s(t) - \hat{n}_s(0) = \hat{n}_i(t) - \hat{n}_i(0) . \quad (42)$$

Here, $\hat{n}(t)$ is the signal photon number at time t, and in a traveling wave amplifier, t an be interpreted as a spatial variable. The first experimental confirmation of (42) was made more than 15 years ago.[12] Since then, the parametric fluorescence experiment has been repeated,[13,14] showing that the photon space and time correlation is very high indeed.

From (42) it is obvious that the only way to get perfect photon-number correlation in a nondegenerate optical parametric amplifier, is to let both the signal and idler input states be vacuum states. The output phases are then completely random (although correlated). However, the output signal photon-number statistics are not Poissonian, but exponential:[15]

$$P(n) = \frac{1}{G}\left[\frac{G-1}{G}\right]^n . \quad (43)$$

The signal output statistics for the case where the signal input state is in a coherent state while the idler is in a vacuum state can be calculated from (6) and (8). Although the photon-number correlation approaches unity as $1/G$, the increasing generator output Fano factor,

$$F_{\text{signal}} = \frac{2G^2\alpha^2 - G\alpha^2 + G^2 - G}{G\alpha^2 + G - 1} \approx 2G \quad \text{when } G, \alpha^2 \gg 1 , \quad (44)$$

will prevent us from reaching (36). It can be concluded that the parametric amplifier is not a suitable device for photon-number fluctuation suppression using linear feedforward. It has been shown,[5] however, that using linear *feedback*, to control the pumping intensity, the photon number fluctuations can be suppressed to a factor $1/G$ below the Poisson limit using this photon pair generator.

A better idea is to use the parametric amplifier with linear feedforward to generate phase-squeezed states. Using (4), the modulator output phase operator can be written

$$\hat{\phi}_{\text{out}} \approx \hat{S} \approx \frac{\hat{b}_{s2} + \hat{b}_{i2}}{\sqrt{\langle n \rangle + 1}} . \quad (45)$$

Note that this equation is not valid if both the input states $\hat{a}_s$ and $\hat{a}_i$ are in vacuum states because, in that case, the idler output phase cannot be measured. Inserting (40) in (45), and assuming that the signal input is in a coherent state while the idler input is in a vacuum state, the modulator output phase fluctuation variance can be found to be

$$\langle \Delta\hat{\phi}^2 \rangle \approx \langle \Delta\hat{S}^2 \rangle \approx \frac{[2G - 1 - 2(G^2+G)^{1/2}](\langle \Delta\hat{a}_{s2}^2 \rangle + \langle \Delta\hat{a}_{i2}^2 \rangle)}{\langle n \rangle + 1} \approx \frac{1 + 1/2G}{8G(\langle n \rangle + 1)} \quad \text{for large gain} . \quad (46)$$

This is clearly much smaller than the limit $\langle \Delta\hat{S}^2 \rangle = 1/4\langle n \rangle$ for a coherent state. If the amplifier has 10 dB gain, the phase noise can be reduced to about 13 dB below that of a coherent state. For each additional order of magnitude the gain increases; the achievable squeezing also increases by one order of magnitude. The limit (46) is due to the imperfect correlation between the generated photons, in contrast to the limit (36) which was due to the finite noise imposed by the linear-photon-number manipulation. It is important to point out that (16), and thus (46), is in the strict sense valid only if the pumping mode is in a coherent state. If a laser is used as the pump source, the phase noise can only be reduced above the pump laser cavity bandwidth.[6]

To check that (46) does not violate Heisenberg's uncertainty relation, the photon-number uncertainty and the cosine operator have to be calculated. The first can be calculated using (18) and (19). The number-phase uncertainty product is thus

$$\langle \Delta\hat{n}^2 \rangle \langle \Delta\hat{S}^2 \rangle = (2G^2\alpha^2 - G\alpha^2 + G^2 - G) \times \left[\frac{1 + 1/2G}{8G(G\alpha^2 + G)}\right] \approx \left[\frac{1}{4} + \frac{1}{8\alpha^2}\right]\frac{\alpha^2}{\alpha^2 + 1} . \quad (47)$$

The uncertainty relation for the number and sine operator is[16]

$$\langle \Delta\hat{n}^2 \rangle \langle \Delta\hat{S}^2 \rangle \geq \tfrac{1}{4} |\langle \hat{C} \rangle|^2 . \quad (48)$$

The signal cosine operator can be approximated in the same way as the sine operator,

$$\hat{C} \approx \frac{\hat{b}_{s1}}{\sqrt{\langle n \rangle + 1}} . \tag{49}$$

If the signal input state into the parametric amplifier is a coherent state and the idler input is in a vacuum state, as assumed, the mean of the signal cosine operator can be calculated to be

$$\langle \hat{C} \rangle \approx \frac{\sqrt{G}\,\alpha}{(G\alpha^2 + G)^{1/2}} = \frac{\alpha}{(\alpha^2+1)^{1/2}} . \tag{50}$$

Looking at the last four equations, two things become clear. First, the phase-manipulated signal state is close to a number-phase minimum uncertainty state (NUS) if the amplifier input is reasonably large. Secondly, it can be seen that the signal and idler phase correlation was bought at the cost of an increasing photon-number uncertainty. In Fig. 9, a variation of the above scheme is shown. Here, the signal input is replaced by idler input. Part of the idler-input mode is used to homodyne detect the output-idler phase fluctuations. This scheme can also attain the squeezing predicted by (46). The output will also be close to a NUS.

While the paramp is not a good photon pair generator for linear feedforward photon-number noise reduction, the nondegenerate parametric oscillator is. To show this, it is convenient to use the photon flux operator $\hat{r} = R + \Delta\hat{r}_1 + i\Delta\hat{r}_2$, instead of the photon field amplitude annihilation operator. The input-output relation of the amplifier and attenuator, (5) and (16), will remain the same with $\hat{r}$ substituted for $\hat{a}$. Attenuation will be used if the idler photon flux exceeds the mean flux,

$$\epsilon = \frac{R_i^2}{R_i^2 + 2R_i\Delta\hat{r}_{i1}} \quad \text{if } \Delta\hat{r}_{i1} > 0 , \tag{51}$$

and amplification will be used if the photon flux is too weak,

$$G = \frac{R_i^2}{R_i^2 + 2R_i\Delta\hat{r}_{i1}} \quad \text{if } \Delta\hat{r}_{i1} < 0 . \tag{52}$$

Linearizing the equations, and dropping all terms of the order Δr^2, yields the following output:

$$\hat{r}_s = R_s + \Delta\hat{r}_{s1} + i\Delta\hat{r}_{s2} - \frac{R_s}{R_i}\Delta\hat{r}_{i1} + \left[\frac{2\,|\Delta\hat{r}_{i1}|}{R_i}\right]^{1/2} (\Delta\hat{r}_{v1} + i\Delta\hat{r}_{v2}) , \tag{53}$$

where $\hat{r}_v$ is the fluctuation operator of a vacuum state. In a parametric oscillator, the mean signal photon flux R_s and the mean idler photon flux R_i are equal, so the ratio between them is unity.

Working in continuous time, it is practical to express the attenuator and amplifier output statistics in the spectral domain. The output's double-sided in-phase spectral density per cycle per second can be expressed as

$$S_{\Delta\hat{r}_{\text{out }1}} = S_{\Delta\hat{r}_{s1} - \Delta\hat{r}_{i1}}(\Omega) + \frac{2}{R_s} S_{|\Delta\hat{r}_{i1}|^{1/2}\Delta\hat{r}_{v1}}(\Omega) , \tag{54}$$

where the last term is defined by

$$S_{|\Delta\hat{r}_{i1}|^{1/2}\Delta\hat{r}_{v1}}(\Omega) \equiv \int_{-\infty}^{\infty} \langle \sqrt{|\Delta\hat{r}_{i1}(t+\tau)|\,|\Delta\hat{r}_{i1}(t)|}\,\Delta\hat{r}_{v1}(t+\tau)\Delta\hat{r}_{v1}(t)\rangle \exp(-i\Omega\tau)d\tau . \tag{55}$$

Since the idler fluctuations and the vacuum fluctuations are independent, the ensemble averages in (55) can be taken separately. Using the Markoff approximation, the ensemble average over the vacuum fluctuations will yield $\delta(\tau)/4$, where δ is Dirac's δ function.[5] The integration can then be performed:

$$\begin{aligned} S_{|\Delta\hat{r}_{i1}|^{1/2}\Delta\hat{r}_{v1}}(\Omega) &= \tfrac{1}{4}\langle [\,|\Delta\hat{r}_{i1}(t)|^2]^{1/2}\rangle \\ &= \tfrac{1}{4}\langle\,|\Delta\hat{r}_{i1}(t)|\,\rangle . \end{aligned} \tag{56}$$

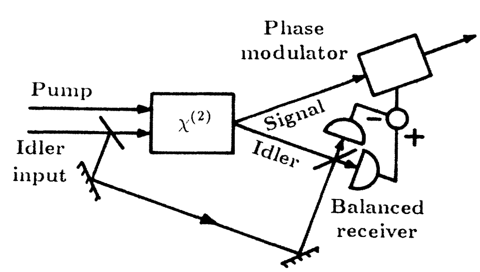

FIG. 9. Phase noise reduction scheme using a nondegenerate OPO. The homodyne detection of the idler phase is accomplished splitting off most of the idler input.

Assuming that the idler fluctuation operator has a Gaussian distribution with zero mean and the variance $\frac{1}{4}$, the ensemble average is easily solved and (54) can be written

$$S_{\Delta\hat{r}_{\text{out }1}} = S_{\Delta\hat{r}_{s1} - \Delta\hat{r}_{i1}}(\Omega) + \frac{1}{4R_s}\left[\frac{2}{\pi}\right]^{1/2} . \tag{57}$$

From the analysis in Ref. 5 it can be deduced that the first term will be proportional to Ω^2 at low frequencies. The noise "floor" for a parametric oscillator with linear feedforward manipulation of the photon number will thus be set by the last term. It will be lower than the standard quantum limit by a factor $\sqrt{2/\pi}R_s^{-1}$, corresponding exactly to the result obtained in Sec. III. Figure 10 shows the normalized spectral density (what is shown is four times the actual spectral density) of the linear feedforward coupled parametric oscillator. In the figure it has been assumed that the bandwidth of the feedforward loop is much higher than the oscillator cavity bandwidth.

Although the singly or doubly resonant nondegenerate optical parametric oscillator (OPO) is a suitable device for sub-Poissonian light generation, it is not an appropriate device for phase-squeezed-state generation. This is obvious from the fact that the output photon-number

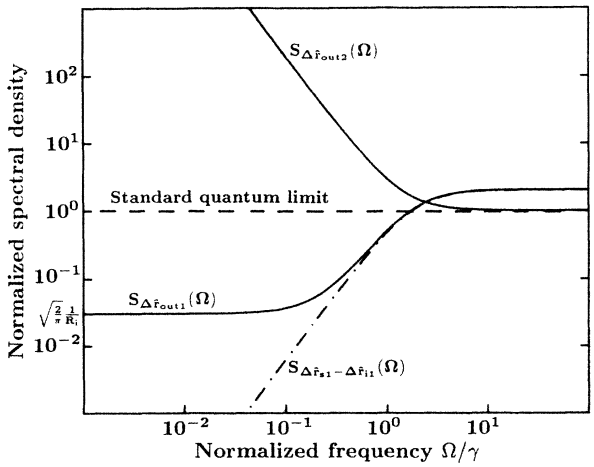

FIG. 10. Spectral density of a singly resonant nondegenerate parametric oscillator with idler flux measurement followed by linear manipulation of the signal photon number. The cavity decay rate is denoted by γ. The pumping rate is well above threshold pumping.

statistics approach the Poisson limit at pumping levels a few times above oscillation threshold pumping.[5] The phase correlation is rapidly degraded as the pumping increases above the threshold.[6]

To sum up the conclusions in this section, it has been shown that the nondegenerate OPO is a suitable generator of correlated photon pairs for producing sub-Poissonian light using linear feedforward manipulation. The minimum achievable Fano factor is approximately $\langle n \rangle^{-1/2}$. The nondegenerate optical parametric amplifier (OPA) is a good photon pair generator for producing squeezed states using linear feedforward manipulation of the phase. The phase noise may be reduced by a factor of $1/2G$ where G is the amplifier gain.

V. LIMITS TO PHOTON PAIR CORRELATION

As pointed out before, the ultimate squeezing limit using correlated photon pairs is the correlation itself. The seemingly perfect device for generating correlated photon pairs is the "photon duplicator," a device which generates the state $|\psi\rangle_1|\psi\rangle_2$ from the state $|\psi\rangle$. However, in an elegant paper,[17] Yuen proved that such a device can only exist is all the possible input states are mutually orthogonal. Since our aim is to produce nonclassical light from classical light, and classical light always can be expanded in coherent-state base vectors, which are not orthogonal, a photon duplicator for our purposes does not exist.

However, to produce correlated photon pairs, a photon duplicator is an unnecessarily sophisticated device. What we want is essentially a device which produces the output state $|\psi\rangle_1|\psi\rangle_2$ from a classical input state $|\varphi\rangle$. In addition, to produce photon twins, it is desirable that the output photon statistics and phase noise be as small as possible. As shown in Sec. IV, the nondegenerate process produces correlated photon pairs; however, it approximately produces the output state $|\psi\rangle_1(|\psi\rangle)_2^\dagger$, so that the output states are not identical, but rather conjugate. Furthermore, as seen in Sec. IV, the photon-number correlation is only perfect when both the input states are vacuum states. The phase is then completely random. On the other hand, phase correlation is only perfect when the gain is infinite. The output photon number and its fluctuation variance are then infinite, too. Both these relations are necessary to preserve Heisenberg's uncertainty principle in conjunction with feedforward signal manipulation schemes (which do not necessarily have to use linear manipulation). We believe that they apply for any photon-number correlation scheme, not only the parametrically produced photons.

In trying to find a device which produces identical output states, it is tempting to try devices with the Hamiltonian

$$\hat{H} = \hbar(\kappa \hat{a}_s^\dagger \hat{a}_i + \kappa^* \hat{a}_s \hat{a}_i^\dagger) , \quad (58)$$

instead of the Hamiltonian for the parametric device,

$$\hat{H} = \hbar(\kappa \hat{a}_s^\dagger \hat{a}_i^\dagger + \kappa^* \hat{a}_s \hat{a}_i) , \quad (59)$$

which produces conjugate output states. The directional coupler, the beamsplitter, and the frequency converter all have the Hamiltonian (58). The input-output relations calculated from (58) are, with the proper phase references,

$$\hat{b}_s = \hat{a}_s \cos(\kappa L) + \hat{a}_i \sin(\kappa L) \quad (60)$$

and

$$\hat{b}_i = \hat{a}_i \cos(\kappa L) - \hat{a}_s \sin(\kappa L) , \quad (61)$$

where L is the coupling length. The correlation between the output modes is "best" when $\kappa L = \pi/4, 3\pi/4, \ldots$. The relation between the output states can then be written

$$\hat{b}_s = \hat{b}_i + \sqrt{2}\hat{a}_i . \quad (62)$$

It is clear that no nonclassical photon states can be produced using this class of devices with classical input states. This has been confirmed by experiments.[2,18] The reason the photon correlation is so poor is that little or no energy is added to the input modes to produce the output modes. In order to preserve Heisenberg's uncertainty limit, the noise term on the right-hand side in (62) is needed.

VI. CONCLUSIONS

The feasibility of generating nonclassical light using correlated photon pairs, idler measurement, and feedforward linear manipulation of the signal photon number has been studied. The advantage of feedforward manipulation is that it is, in principle, free from any bandwidth restriction. Feedback manipulation always has a finite squeezing bandwidth, due to its finite delay time, while the squeezing bandwidth of cavity devices are always limited by the cavity bandwidth. The advantage of linear manipulating devices is that they are simple.

It has been shown that linear manipulation (attenuation and/or amplification) of the photon number always introduces fluctuations. The photon-number correlation cannot thus be fully utilized. Photon adding can, in principle, be done noiselessly, but only if the added mode is in a number state. If classical light is added, the uncertainty in added photon number introduces an uncertainty comparable to that introduced by amplification.

If the photon-number correlation is perfect and the initial photon-number distribution is Poissonian, photon states with substantially reduced photon-number fluctuations can be produced. The minimum achievable Fano factor is approximately $\langle n \rangle^{-1/2}$. A physical realization of such a generator is a nondegenerate optical parametric oscillator pumped at a few times above the oscillation threshold.

In contrast to linear-photon-number manipulation, it is suggested that linear phase manipulation, a nondissipative process, does not introduce any noise intrinsically. The full phase correlation can thus be utilized. It has been shown that a high gain nondegenerate OPA with coherent-state input will generate highly phase-correlated photon pairs. Using linear feedforward, the phase noise can be reduced to a factor $1/2G$ below that of a coherent state. The output state is very close to a number-phase minimum uncertainty state. An advantage of this squeezed-state generator over the degenerate OPA is that, in the former, the wavelength is tunable over a wide range. In the latter, the wavelength is fixed at two times the wavelength of the pump laser.

It has also been shown that at high gain, the output states from the parametric frequency down-conversion process are approximately conjugate. This relation is different from, although related to, the operator Manley-Rowe relation. Perfect phase correlation is only possible when the parametric gain is infinite. The photon-number noise is then infinite, too. Perfect photon-number correlation is only possible when both the input states are in vacuum states. The phase is then completely random. The above relations preserve Heisenberg's uncertainty relation. It is suggested that they are true for any photon pair generator, not only the parametric process. It is finally hoped that this paper will spark a renewed interest in correlated photon pairs to answer the question we have left unsettled.

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