Comments

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Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag"

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A classical model of a neutron as a current loop is examined. The importance of the mechanical structure (which was overlooked by Boyer) for the dynamical equations is elaborated. It is shown that the current loop has the same behavior in an external static electric field as was obtained for the neutron by Aharonov and Casher from the Dirac equation.

Aharonov and Casher[1] have shown by a number of arguments that a neutron moving past a long uniformly charged wire (with magnetic dipole moment parallel to the wire) will experience no force, and yet neutron wave packets traveling on opposite sides of the wire will experience a relative phase shift. This effect is "dual" to the Aharonov-Bohm[2] effect, where an electron moving past a uniformly magnetized filament experiences no force, and yet there is a phase shift. In the Aharonov-Bohm effect, it is obvious that the electron is not subject to any electromagnetic force, because the magnetic field lies wholly within the filament and so is zero at the electron's location. In the Aharonov-Casher effect, it is not so obvious that the neutron experiences no force because the electric field of the wire certainly extends to the neutron's location.

One of the arguments given by Aharonov and Casher was a straightforward derivation of the equation of motion of the neutron from the Dirac equation with an added Pauli term.[3] This has long been regarded as an appropriate equation to describe the interaction of the neutron with electromagnetic fields.[4] It is a separate, but interesting, question as to whether a classical model of a neutron obeys the same equation of motion that is obtained from the Dirac equation. In particular, will a classical neutron model also experience no force in the physical situation presented by Aharonov and Casher?

Indeed, Boyer[5] argues that a classical neutron does experience a force, causing a velocity change which depends upon the side of the wire that the neutron passes. Then it is the resultant relative positional lag of neutron wave packets passing on opposite sides of the wire that is responsible for the Aharonov-Casher phase shift.

Boyer's argument is based upon the classical model of a neutron as a current loop, a model which has been very successful in explaining the behavior of neutrons encountering magnetized material.[6] He correctly points out that there is an electric force on such a current loop as it passes a charged wire: in the rest frame of the wire the moving current loop appears to have an electric dipole moment $\mathbf{p} \equiv \mathbf{v} \times \boldsymbol{\mu}/c$ ($\mathbf{v}$ is the loop's velocity, $\boldsymbol{\mu}$ is its magnetic dipole moment) which the wire's electric field acts upon (in the rest frame of the loop its magnetic moment experiences a force due to the magnetic field of the moving wire).

Boyer's mistake is to suppose that this electromagnetic force is equal to the (mass)$\times$(acceleration). Actually, this force is equal to the rate of change of momentum. As we shall show, in this case the momentum is not just (mass)$\times$(velocity). There is another contribution to the momentum, arising from the classical neutron's mechanical structure. The electromagnetic force goes into increasing this internal mechanical momentum, while the velocity of the neutron remains constant.

This may be phrased another way if one wishes to define force as (mass)$\times$(acceleration). Then, what Boyer has omitted from his calculation is a mechanical force (the negative of the rate of change of the internal mechanical momentum) exerted by the mechanical structure. This force precisely cancels the electromagnetic force and therefore the neutron moves with constant velocity past the wire.

The importance of mechanical structure is well known in making classical models of particles. For example, in the case of a classical electron model such as a spherical charged shell, it is essential to include the Poincaré stress, or its equivalent, which keeps the charged shell from expanding.[7] Otherwise, the model will not be relativistically invariant.

Similarly, in making a relativistically invariant classical neutron model, such as two uniformly and oppositely charged disks rotating in opposite directions about a central axle, or two uniformly and oppositely charged gas clouds rotating in opposite directions in toroidal tubes (a third, neutral rotating disk or gas cloud may be added to provide spin angular momentum), it is essential to consider the mechanical (disk or gas-tube) contribution to the four-momentum.

We shall demonstrate that such a "neutron" necessarily has mechanical momentum even when its center is *at rest* in the reference frame of the wire. We shall show this by proving the following.

(a) *There is zero total momentum* (electromagnetic plus mechanical) in the rest frame of any finite static configuration.[8]

(b) *There is electromagnetic momentum.*

It follows that *there must be a mechanical momentum,* equal in magnitude and opposite in direction to the electromagnetic momentum. This mechanical momentum was called "hidden momentum" by Shockley and James,[9] and was used to resolve a paradox in a physical configuration which is quite similar to the one considered here.

Coming back to the original situation, where the neutron moves relative to the wire, we shall see that the force calculated by Boyer solely accounts for the change of this intrinsic mechanical momentum, leaving no room for the "acceleration" of the neutron.

We will begin by proving part (a) of our assertion. Consider a neutron at rest, together with a source of static external electric field (e.g., a charged wire[8]). The energy-momentum stress density tensor $T^{\mu\nu}$ satisfies

$$\partial_\mu T^{\mu\nu}=\partial_\mu T_e^{\mu\nu}+\partial_\mu T_m^{\mu\nu}=0 \ , \tag{1}$$

where $T_m^{\mu\nu}$ is the mechanical part of the tensor, and $T_e^{\mu\nu}$ is the electromagnetic part satisfying

$$-\partial_\mu T_e^{\mu\nu}=\frac{1}{c}F^{\nu\lambda}j_\lambda \tag{2}$$

($F^{\nu\lambda}$ is the electromagnetic tensor, j_λ is the current density). The total momentum is $P^i=(1/c)\int T^{i0}dV$.

In this situation, the total tensor $T^{\mu\nu}$ is independent of time (our neutron's internal charge and mass are rotating, but the charge, mass, internal velocity, and stress distributions are static). Therefore it follows from Eq. (1) that $\sum_i \partial_i T^{i0}=0$. But if $\nabla\cdot\mathbf{W}=0(W^i\equiv T^{i0}/c)$ and the vector $\mathbf{W}$ vanishes at infinity faster than $1/r^3$ (certainly the case here, where $T^{\mu\nu}\rightarrow T_e^{\mu\nu}\sim 1/r^4$ or faster at infinity), then

$$\begin{aligned} P^i&=\int W^i dV=\int \nabla\cdot(\mathbf{W}x^i)dV-\int x^i\nabla\cdot\mathbf{W}\,dV \\ &=\oint(\mathbf{W}x^i)\cdot d\mathbf{S}=0 \ . \end{aligned}\tag{3}$$

Next we prove part (b) of our assertion that the electromagnetic momentum for this static configuration does not vanish. The electromagnetic momentum density is $\mathbf{E}\times\mathbf{B}/4\pi c$. The magnetic field $\mathbf{B}$ of the neutron may be written as $\mathbf{B}=4\pi\mathbf{M}-\nabla\phi$, where the magnetization density $\mathbf{M}$ is localized (nonvanishing only within the neutron volume), and the potential ϕ of the magnetic intensity $\mathbf{H}=-\nabla\phi$ falls off at infinity as $1/r^2$. The static charge are the sole source of electric field $\mathbf{E}$ (i.e., the "neutron" is not electrically polarizable), and its value $\mathbf{E}_0$ over the volume of the neutron is assumed to be essentially constant. Then

$$\begin{aligned} \mathbf{P}_e&=\frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,dV=\frac{1}{4\pi c}\int \mathbf{E}x(4\pi\mathbf{M}-\nabla\phi)dV \\ &=\frac{1}{c}\mathbf{E}_0\times\int \mathbf{M}\,dV+\frac{1}{4\pi c}\int \phi\,\nabla\times E\,dV \\ &=\frac{1}{c}\mathbf{E}_0\times\boldsymbol{\mu} \ . \end{aligned}\tag{4}$$

It follows from (3) and (4) that *there is mechanical momentum* in this configuration, whose value is

$$\mathbf{P}_m=\mathbf{P}-\mathbf{P}_e=\frac{1}{c}\boldsymbol{\mu}\times\mathbf{E}_0 \ . \tag{5}$$

This mechanical momentum cannot lie in the mechanical structure of the charge source. Nothing is moving there, so the momentum density T_m^{i0} vanishes over the charge source. Therefore it is the nonvanishing T_m^{i0} for the neutron that is responsible for the mechanical momentum (5). This completes our proof that a neutron at rest has mechanical momentum. We will later explain in detail how this dynamical momentum arises.

We now have all the essential elements with which to write down the equation of motion for the neutron, say, in the rest frame of the wire (the calculation in the rest frame of the neutron will appear later).

The force on the neutron's electric dipole moment $\mathbf{p}=(\mathbf{v}\times\boldsymbol{\mu})/c$ due to the electric field $\mathbf{E}$ of the charged wire, which Boyer correctly calculated, is $(\mathbf{p}\cdot\nabla)\mathbf{E}$. The neutron's momentum is $m\mathbf{v}+\boldsymbol{\mu}\times\mathbf{E}/c$ [m is the neutron's mass; the hidden momentum is given by Eq. (5) even when the neutron moves, omitting terms of order $(v/c)^2$]. Thus the equation of motion is

$$\frac{1}{c}(\mathbf{v}\times\boldsymbol{\mu}\cdot\nabla)\mathbf{E}=m\mathbf{a}+\frac{1}{c}\frac{d}{dt}\boldsymbol{\mu}\times\mathbf{E} \tag{6}$$

or

$$\frac{1}{c}[(\mathbf{v}\times\boldsymbol{\mu}\cdot\nabla)\mathbf{E}+(\mathbf{v}\cdot\nabla)(\mathbf{E}\times\boldsymbol{\mu})]=m\mathbf{a} \ , \tag{7}$$

where Eq. (7) follows from Eq. (6) using $d/dt=\partial/\partial t+\mathbf{v}\cdot\nabla$ and the fact that $\mathbf{E}$ depends upon time only through its dependence upon the particle's position. Using vector identities Eq. (7) may be written as

$$\frac{1}{c}[-(\mathbf{v}\times\boldsymbol{\mu})\times(\nabla\times\mathbf{E})+(\mathbf{v}\times\boldsymbol{\mu})\nabla\cdot\mathbf{E}-(\boldsymbol{\mu}\cdot\nabla)(\mathbf{v}\times\mathbf{E})] =m\mathbf{a} \ . \tag{8}$$

Since $\nabla\times\mathbf{E}=\mathbf{0}$ and $\nabla\cdot\mathbf{E}=0$ for the wire's electric field at the location of the neutron, we obtain

$$-\frac{1}{c}(\boldsymbol{\mu}\cdot\nabla)(\mathbf{v}\times\mathbf{E})=m\mathbf{a} \ . \tag{9}$$

Now, in the physical situation envisaged by Aharonov and Casher, **E** does not depend upon the coordinate z parallel to the wire. However, $\boldsymbol{\mu}\cdot\nabla=\mu\partial/\partial z$, so the force on the left-hand side of Eq. (9) vanishes. Thus even this classical neutron moves with constant-velocity motion, as Aharonov and Casher claimed. In fact, Eq. (9) is identical to the equation of motion of the neutron obtained from the Dirac equation, for this situation.[1]

We have made our point, but we wish to comment further on two of its aspects, the dynamical origin of the hidden momentum and the equation of motion in the neutron's rest frame.

Before giving examples of hidden momentum, whose manifestation is model dependent, we shall prove its existence in another model-independent way. This time, consider the local equation obtained by combining Eqs. (1) and (2),

$$\frac{1}{c}F^{\nu\lambda}j_\lambda=\partial_\mu T_m^{\mu\nu} , \tag{10}$$

whose $\nu=0$ component is

$$\frac{1}{c}\mathbf{j}\cdot\mathbf{E}=\sum_{i=1}^{3}\partial_i T_m^{i0} . \tag{11}$$

Equation (11) describes the energy exchange between the external electric field and the neutron: The existence of the hidden momentum can be obtained directly from it and from current conservation $\nabla\cdot\mathbf{j}=0$. We integrate the identity $-\mathbf{r}(\mathbf{E}\cdot\mathbf{j})=(\mathbf{r}\times\mathbf{j})\times\mathbf{E}-\mathbf{j}(\mathbf{E}\cdot\mathbf{r})$ over the volume of the neutron, obtaining

$$-\int \mathbf{r}\sum_i \partial_i T_m^{i0}dV=2\boldsymbol{\mu}\times\mathbf{E}-\frac{1}{c}\mathbf{E}\cdot\int \mathbf{r}\mathbf{j}\,dV , \tag{12}$$

where we have used the definition $\boldsymbol{\mu}\equiv(1/2c)\int \mathbf{r}\times\mathbf{j}dV$ and the constancy of **E** over the volume of the neutron. Integration by parts of the left-hand side of Eq. (12) yields $c\mathbf{P}_m$. As for the last term on the right-hand side of Eq. (12), we may use the identity $\int(x_m j_n+x_n j_m)dV=-\int x_m x_n\nabla\cdot\mathbf{j}\,dV=0$ to write it as $-(1/2c)\mathbf{E}\cdot\int(\mathbf{r}\mathbf{j}-\mathbf{j}\mathbf{r})dV=\mathbf{E}\times\boldsymbol{\mu}$. Thus we obtain Eq. (5) once more.

Now let's take a look at some models, to understand where the hidden momentum comes from. Consider first the neutron model we have cited involving counter-rotating charged gas clouds. Suppose the neutron is at rest, the axis of rotation lying perpendicular to this piece of paper, with the positive (negative) charges rotating in the clockwise (counterclockwise) direction, and suppose the external electric field **E** points to the top of this page.

Let us examine the energy exchange between the external electric field and the moving charges in the neutron. $\mathbf{j}\cdot\mathbf{E}$ in Eq. (11) is power density put into the left half of the neutron, and taken out of the right half: the speed and kinetic energy of the positively (negatively) charged particles are increased by the electric field as they travel from the bottom (top) of the page to the top (bottom) of the page and are decreased on the return trip. As a result, the positively (negatively) charged particles have greater momenta in the upper (lower) part of their trajectory. The density of the charged particles is decreased in the region where the speed is increased, so a nonrelativistic calculation gives no net momentum. However, taking into account the relativistic mass increase together with the speed increase, one obtains a net momentum toward the right. Penfield and Haus,[10] in considering charged particles moving in a rectangular current loop in an external electric field, show how just this mechanism produces the mechanical momentum (5) (see also Ref. 9).

In the model of the neutron involving counter-rotating charged disks, the mechanism behind the hidden momentum is quite different. The rigidity of the disk does not allow acceleration and deceleration of the charges, and the external electric field causes only stresses. Both normal and shearing stresses can occur (their proportions depend on the elastic properties of the disks). A Lorentz transformation of the stress-energy tensor converts stress to momentum density. This contribution leads to the net mechanical momentum (5) in the rest frame of the center of the neutron.[11]

Finally let us consider the equation of motion for the neutron in its instantaneous rest frame. This equation of motion is the integral of Eq. (10) over the volume of the neutron. On the left-hand side of Eq. (10) we write $F^{\nu\lambda}=F_{\text{ext}}^{\nu\lambda}+F_{\text{self}}^{\nu\lambda}$, where $F_{\text{ext}}^{\nu\lambda}$ contains only the external electromagnetic fields, and $F_{\text{self}}^{\nu\lambda}$ contains the fields whose sources lie within the neutron. The integral of $(1/c)\,F_{\text{ext}}^{i\lambda}j_\lambda$ is the external force on the current loop $\nabla(\boldsymbol{\mu}\cdot\mathbf{B})$ that Boyer correctly calculated. The integral of $(1/c)F_{\text{self}}^{i\lambda}j_\lambda$ is the electromagnetic self-force which, as in the case of the classical electron, is equal to the (negative of the) electromagnetic mass $[(1/8\pi)\int \mathbf{B}_{\text{self}}^2 dV$ in this case] multiplied by the acceleration. The integral of the right-hand side of Eq. (10) consists of the rate of change with time of two parts, the field-independent mechanical momentum, and the field-dependent "hidden momentum" (5). Thus we obtain the equation of motion in the rest frame:

$$\nabla(\boldsymbol{\mu}\cdot\mathbf{B})+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}\times\boldsymbol{\mu}=m\mathbf{a} , \tag{13}$$

where m is the total rest mass (mechanical plus electromagnetic).

Since $\partial E/\partial t=c\nabla\times\mathbf{B}$ (assuming the neutron is not passing through a current-carrying region of space) and $(\boldsymbol{\mu}\cdot\nabla)\mathbf{B}=\nabla(\boldsymbol{\mu}\cdot\mathbf{B})-\boldsymbol{\mu}\times(\nabla\times\mathbf{B})$, we may also write (13) as

$$(\boldsymbol{\mu}\cdot\nabla)\mathbf{B}=m\mathbf{a} . \tag{14}$$

This shows that the current-loop neutron experiences the force [defined as (mass)$\times$(acceleration)] that a dipole composed of a pair of magnetic charges of opposite signs would experience in this situation.

Specializing Eq. (14) to the Aharonov-Casher situation, where the moving wire creates a magnetic field $\mathbf{B}=-(1/c)\mathbf{v}\times\mathbf{E}$, Eq. (9) is once again obtained for which we showed that the force vanished.

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[11]This remarkable mechanism for obtaining hidden momentum [which is not (mass)$\times$(velocity)] has not previously been discussed, and we hope to give a more detailed presentation elsewhere.