Charged particle in the presence of a variable magnetic field

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A solution in closed analytic form is obtained for the motion of a charged harmonic oscillator in a variable magnetic field. The time-dependent Schrödinger wave equation is solved in both Cartesian and polar coordinates. The canonical density matrix is calculated via the quasicoherent states.

The problem of a charged particle in a constant magnetic field has been studied extensively in recent years.[1–5] Effort was concentrated on calculating the propagator in order to find the Bloch density matrix. In fact, a knowledge of the density matrix enables us to find the average value of the energy for the system, as well as to throw light on some statistical aspects; for example, in the melting of a Wigner crystal[6,7] in an applied magnetic field it is possible to find the linear restoring force acting on the electrons by means of the Bloch density matrix.

The usual technique for calculating the density matrix is to use a Feynman path integral. However, in the present communication we shall make use of the quasicoherent states to calculate the Green's function and hence the Bloch density matrix. We shall discuss the problem in the general case when the magnetic field is taken to be time dependent, so that the system exhibits a dynamic or ac Zeeman effect.[8] Then the Hamiltonian for a charged particle in a variable magnetic field takes the form[9]

$$H=\tfrac{1}{2}\sum_{i=1}^{2}[p_i^2+\omega^2(t)q_i^2]-\frac{k(t)}{2}(q_1p_2-q_2p_1)\ , \tag{1}$$

where the magnetic field is applied along the z axis and the modulated frequency $\omega(t)$ is

$$\omega(t)=[\omega_0^2+\tfrac{1}{4}k^2(t)]^{1/2}\ . \tag{2}$$

ω_0 and $k(t)$ are the oscillating and Larmor frequencies, respectively. To find the quasicoherent states, we must first calculate the pseudostationary states, found from the Schrödinger wave function

$$H\psi=i\hbar\frac{\partial\psi}{\partial t}\ . \tag{3}$$

From Eqs. (1) and (3) we have

$$\frac{\partial^2\psi}{\partial q_1^2}+\frac{\partial^2\psi}{\partial q_2^2}-\frac{\omega^2(t)}{\hbar^2}(q_1^2+q_2^2)\psi-\frac{ik(t)}{\hbar}\left[q_1\frac{\partial\psi}{\partial q_2}-q_2\frac{\partial\psi}{\partial q_1}\right]=\frac{-2i}{\hbar}\frac{\partial\psi}{\partial t}\ . \tag{4}$$

By using the transformation

$$\begin{bmatrix} q_1\\ q_2\end{bmatrix}=\begin{bmatrix}\cos\gamma(t) & \sin\gamma(t)\\ -\sin\gamma(t) & \cos\gamma(t)\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}\ , \tag{5}$$

where

$$\gamma(t)=\tfrac{1}{2}\int_0^t k(\tau)d\tau\ , \tag{6}$$

$\psi(q_1,q_2,t)\to\bar\psi(x,y,t)$, and hence Eq. (4) becomes

$$\frac{\partial^2\bar\psi}{\partial x^2}+\frac{\partial^2\bar\psi}{\partial y^2}-\frac{\omega^2(t)}{\hbar^2}(x^2+y^2)\bar\psi=\frac{-2i}{\hbar}\frac{\partial\bar\psi}{\partial t}\ . \tag{7}$$

In polar coordinates Eq. (7) is

$$\frac{\partial^2\eta}{\partial r^2}+\frac{1}{r}\frac{\partial\eta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\eta}{\partial\theta^2}-\frac{\omega^2(t)}{\hbar^2}r^2\eta=\frac{-2i}{\hbar}\frac{\partial\eta}{\partial t}\ , \tag{8a}$$

where $\eta(r,\theta,t)\equiv\bar\psi(x,y,t)$,

$$r=(x^2+y^2)^{1/2}$$

and (8b)

$$\theta=\tan^{-1}(y/x)\ .$$

By taking $r=\mu(t)z$, therefore $\eta(r,\theta,t)\to\bar\eta(z,\theta,t)$ and Eq. (8a) takes the form

$$\frac{\partial^2\bar\eta}{\partial z^2}+\frac{1}{z}\frac{\partial\bar\eta}{\partial z}-\frac{2i}{\hbar}\dot\mu(t)\mu(t)z\frac{\partial\bar\eta}{\partial z}+\frac{1}{z^2}\frac{\partial^2\bar\eta}{\partial\theta^2}-\frac{\omega^2\mu^4}{\hbar^2}z^2\bar\eta=\frac{-2i}{\hbar}\mu^2\frac{\partial\bar\eta}{dt}\ , \tag{9}$$

where $\mu(t)$ satisfies the "auxiliary" equation

$$\ddot\mu+\omega^2(t)\mu=1/\mu^3\ . \tag{10}$$

Note that Eq. (10) is a nonlinear differential equation called a Pinney equation,[10] which has a physical interpretation for μ. The one-dimensional motion given by the equation

$$\ddot w+\omega^2(t)w=0 \tag{11}$$

may be considered to be the projection of a two-dimensional motion in which the radius vector has length μ and rotates with angular velocity

$$\dot\rho=\mu^{-2}\ . \tag{12}$$

If w_1 and w_2 are independent solutions of (11), then

$$\mu=(Aw_1^2+Bw_2^2+2Cw_1w_2)^{1/2}\ , \tag{13}$$

where A,B,C are constants given by[11]

$$(AB-C^2)w_1^4=\left[\frac{d}{dt}\left[\frac{w_2}{w_1}\right]\right]^{-2}. \tag{14}$$

Now if we write

$$\bar{\eta}(z,\theta,t)=\phi(z,\theta,t)\exp\left[\frac{i}{2\hbar}\dot{\mu}(t)\mu(t)z^2\right], \tag{15}$$

then Eq. (9) reads

$$\frac{\partial^2\phi}{\partial z^2}+\frac{1}{z}\frac{\partial\phi}{\partial z}+\frac{1}{z^2}\frac{\partial^2\phi}{\partial\theta^2}-\frac{z^2}{\hbar^2}\phi=\frac{-2i}{\hbar}\mu^2\frac{\partial\phi}{\partial t}-\frac{2i}{\hbar}\dot{\mu}(t)\mu(t)\phi . \tag{16}$$

The above equation is easy to solve, and leads to the general solution of Eq. (8a) in the form

$$\eta_{ls}(r,\theta,t)=\left[\frac{\mu^{-2}(t)}{\hbar\pi}\right]^{1/2}\left[\frac{l!}{(l+s)!}\right]^{1/2}\exp(\pm is\theta)L_l^{(s)}(r^2/\mu^2\hbar)(r^2/\mu^2\hbar)^{s/2}$$
$$\times\exp\left[-\frac{1}{2\hbar}[\mu^{-2}(t)-i(\dot{\mu}(t)/\mu(t))]r^2\right]\exp\left[-i(2l+s+1)\int_0^t\mu^{-2}(\tau)d\tau\right], \tag{17}$$

where l and s are the radial and azimuthal quantum numbers, respectively, and $L_l^{(s)}$ are associated Laguerre polynomials. The corresponding solution in Cartesian coordinates is given by

$$\psi_{mn}(q_1,q_2,t)=\left[\frac{\mu^{-2}(t)}{\pi\hbar}\right]^{1/2}(2^{n+m}n!m!)^{-1/2}$$
$$\times H_n\left[\frac{\hbar^{-1/2}}{\mu}[q_1\cos\gamma(t)-q_2\sin\gamma(t)]\right]H_m\left[\frac{\hbar^{-1/2}}{\mu}[q_2\cos\gamma(t)+q_1\sin\gamma(t)]\right]$$
$$\times\exp\left[-\frac{1}{2\hbar}[\mu^{-2}(t)-i(\dot{\mu}(t)/\mu(t))](q_1^2+q_2^2)\right]\exp\left[-i(n+m+1)\int_0^t\mu^{-2}(\tau)d\tau\right]. \tag{18}$$

The connection between the quasicoherent states and pseudostationary states is given by the equation[12]

$$\psi_{\alpha\beta}(q_1,q_2,t)=\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^n(t)\beta^m(t)}{\sqrt{n!}\sqrt{m!}}\psi_{mn}(q_1,q_2,t), \tag{19}$$

where α and β are the eigenvalues for boson operators a and b which satisfy the commutation relation $[a_i,a_j^\dagger]=\delta_{ij}$. Then from Eqs. (18) and (19) the quasicoherent states are

$$\psi_{\alpha\beta}(q_1,q_2,t)=\left[\frac{\mu^{-2}(t)}{\pi\hbar}\right]^{1/2}\exp[-\tfrac{1}{2}(\alpha^2(t)+\beta^2(t)+|\alpha|^2+|\beta|^2)]$$
$$\times\exp\left[-\frac{1}{2\hbar}\left[\mu^{-2}(t)-i\frac{\dot{\mu}(t)}{\mu(t)}\right](q_1^2+q_2^2)\right]$$
$$\times\exp\left[\left[\frac{2}{\hbar\mu^2(t)}\right]^{1/2}[q_1(\alpha(t)\cos\gamma(t)+\beta(t)\sin\gamma(t))+q_2(\beta(t)\cos\gamma(t)-\alpha(t)\sin\gamma(t))]\right]. \tag{20}$$

In order to calculate the Green's function, we shall use the following relation:

$$G(q_1,q_2,\bar{q}_1,\bar{q}_2,t)=\frac{1}{\pi^2}\int\psi_{\alpha\beta}(q_1,q_2,t)\psi^*_{\alpha\beta}(\bar{q}_1,\bar{q}_2,0)d^2\alpha\, d^2\beta . \tag{21}$$

Then by inserting Eq. (20) and the corresponding complex conjugate at $t=0$ into Eq. (21), after considerable calculations one finds

$$G(q_1,q_2,\bar{q}_1,\bar{q}_2,t)=(2\hbar\pi|R(t)|)^{-1}\exp\left[\frac{i}{2\hbar R(t)}\{U(t)(\bar{q}_1^2+\bar{q}_2^2)+V(t)(q_1^2+q_2^2)-2[(\bar{q}_1q_1+\bar{q}_2q_2)\cos\gamma(t)+(q_1\bar{q}_2-\bar{q}_1q_2)\sin\gamma(t)]\}\right], \tag{22}$$

where

$$R(t)=\mu(0)\mu(t)\sin\delta(t)\ , \tag{23a}$$

$$U(t)=\frac{\mu(t)}{\mu(0)}\cos\delta(t)-\dot{\mu}(0)\mu(t)\sin\delta(t)\ , \tag{23b}$$

$$V(t)=\frac{\mu(0)}{\mu(t)}\cos\delta(t)+\dot{\mu}(t)\mu(0)\sin\delta(t)\ , \tag{23c}$$

and

$$\delta(t)=\int_0^t \mu^{-2}(\tau)d\tau\ . \tag{23d}$$

In the case of a constant magnetic field so that k and ω are constants Eq. (22) assumes the form

$$G(q_1,q_2,\bar{q}_1,\bar{q}_2,t)=[(\omega/2\hbar\pi)\csc\omega t]\exp\left[\frac{i\omega}{2\hbar\sin\omega t}[(\bar{q}_1^{\,2}+\bar{q}_2^{\,2}+q_1^2+q_2^2)\cos\omega t-2(\bar{q}_1q_1+\bar{q}_2q_2)\cos(k/2)t -2(q_1\bar{q}_2-\bar{q}_1q_2)\sin(k/2)t]\right]\ . \tag{24}$$

This result can be compared with Eq. (12) of Ref. (3).

The Bloch density matrix $\bar{\rho}$ defined by

$$\bar{\rho}(r,\bar{r},\bar{\beta})=\sum_i \psi_i^*(\bar{r})\psi_i(r)\exp(-\bar{\beta}\epsilon_i)\ , \tag{25}$$

where the ψ_i's and the corresponding energies ϵ_i are solutions of the Schrödinger equation

$$\bar{H}\psi_i=\epsilon_i\psi_i\ , \tag{26}$$

$\bar{\beta}=(\kappa_\beta T)^{-1}$, κ_β is Boltzmann's constant, and T is the temperature. Instead of solving the Bloch equation to get the density matrix in explicit form, we can find this matrix immediately by substituting t for $(-i\hbar\bar{\beta})$ in Eq. (24). Thus

$$\bar{\rho}(q_1,q_2,\bar{q}_1,\bar{q}_2,\bar{\beta})=[\omega/2\hbar\pi\sinh(\bar{\beta}\hbar\omega)]\exp\left\{-[\omega/2\hbar\sinh(\bar{\beta}\hbar\omega)]\left[(\bar{q}_1^{\,2}+\bar{q}_2^{\,2}+q_1^2+q_2^2)\cosh(\bar{\beta}\hbar\omega) -2(\bar{q}_1q_1+\bar{q}_2q_2)\cosh\left[\bar{\beta}\frac{\hbar}{2}k\right] +2i(q_1\bar{q}_2-\bar{q}_1q_2)\sinh\left[\bar{\beta}\frac{\hbar}{2}k\right]\right]\right\}. \tag{27}$$

To obtain the average ensemble energy at temperature T, we need to calculate the partition function which is given by

$$Q=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(q_1,q_2,\bar{\beta})dq_1dq_2\ . \tag{28}$$

From Eqs. (27) and (28) we have

$$Q=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\exp\left[-\bar{\beta}\hbar\left[m+\frac{1}{2}\right]\left[\omega+\frac{k}{2}\right]\right]\times\exp\left[-\bar{\beta}\hbar\left[n+\frac{1}{2}\right]\left[\omega-\frac{k}{2}\right]\right] \tag{29}$$

so that the energy eigenvalues are

$$\epsilon(n,m)=\hbar\left[\omega(n+m+1)+\frac{k}{2}(m-n)\right] \tag{30}$$

and the average energy can be calculated from the equation[13]

$$\langle E\rangle=-\frac{\partial}{\partial\bar{\beta}}\ln Q\ . \tag{31}$$

Thus

$$\left\langle\frac{E}{h}\right\rangle=\left[\left[\omega+\frac{k}{2}\right]\coth\left[\omega+\frac{k}{2}\right]\hbar\bar{\beta} +\left[\omega-\frac{k}{2}\right]\coth\left[\omega-\frac{k}{2}\right]\bar{\beta}\hbar\right]\ . \tag{32}$$

Since the density matrix is a result of having the Green's function which may be obtained from the solution in the Heisenberg picture, we would like to point out that the derivation of the Green's function via the quasicoherent states shows how useful these states are in moving from one picture to the other and how much complication can be avoided by employing them in calculations.

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