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Cosmological implications of the Gibbs ensemble in parametrized relativistic classical mechanics

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Earlier investigators have shown that the free energy of an ideal gas in the ultrarelativistic (UR) regime of parametrized relativistic classical mechanics (PRCM) is different from the free energy calculated with use of the standard theory of relativistic classical mechanics (RCM). These differences do not become significant until very high temperatures are reached. Physical examples where the UR regime may be reached include the interior of stars, relativistic gas dynamics, and the first minute after the Big Bang. Of these, it is shown in the present paper that negligible differences are expected at realistic temperatures for relativistic gas dynamics and stellar models. This demonstration requires an evaluation of the PRCM and RCM free energies over a wide range of temperatures extending from the nonrelativistic limit to the UR limit. The greatest differences between PRCM and RCM are manifested in the density parameter of cosmological models. To study this effect, a general relativistic formalism is derived from variational principles within the context of PRCM. Scale factors and the age of Friedmann universes are calculated.

I. INTRODUCTION

When the temperature of an ideal gas is sufficiently high, the mass-energy density of the stress-energy tensor becomes theory dependent and provides an opportunity for experimentally testing the validity of parametrized relativistic classical mechanics (PRCM). Horwitz *et al.*[1] have shown that the free energy of an ideal gas in the ultrarelativistic (UR) regime of the PRCM is different from the free energy calculated by Juttner[2,3] with use of the standard theory of relativistic classical mechanics (RCM). They did not quantify the calculated differences through a range of temperatures, nor did they consider physical applications where the differences might be significant. The primary purpose of the present paper is to examine the physical consequences of the free-energy differences they observed. An outgrowth of this work which is of interest in its own right is the development of a parametrized general relativistic formalism.

The free-energy difference noted by Horwitz *et al.*[1] is an area of disagreement between PRCM and RCM that may have measurable consequences. To better appreciate the significance of these consequences, a brief review of parametrized relativistic theories is provided in Sec. II. Following this review, the observation that the free-energy difference is insignificant except at high temperatures leads us to consider only physical phenomena where the UR regime may be reached. Examples include the interior of stars, relativistic gas dynamics, and the first minute after the Big Bang. Of these, it is shown in Sec. III that negligible differences are expected at realistic temperatures for relativistic gas dynamics and stellar models. This demonstration requires an evaluation of the PRCM and RCM free energies over a wide range of temperatures extending from the nonrelativistic limit to the UR limit. The greatest differences between PRCM and RCM are manifested in the density parameter of cosmological models. To study this effect, a general relativistic formalism is derived in Sec. IV from variational principles within the context of PRCM. Scale factors of Friedmann universes are then calculated in Sec. V. It is shown that the calculated age of the universe is theory dependent.

II. REVIEW OF PARAMETRIZED THEORIES

Fock[4] and Stückelberg[5] appear to be the first researchers to introduce the idea of a relativistic scalar evolution parameter. The impact of the parameter can be readily seen by considering its role in the symmetry of a physical system. Two types of symmetries are usually recognized:[6] external space-time symmetries, and internal symmetries arising from transformations which do not change the space-time coordinates. External symmetries are usually characterized by the Poincaré group. Internal symmetries, which are most important for understanding particles and their interactions, have been characterized by $SU(2)\otimes U(1)$ in the electroweak theory, SU(3) in quantum chromodynamics (QCD), and SU(5) in grand unified theories and various other groups.[6] Internal symmetries have received more attention than external symmetries in the modern-physics community. Most physicists are content to accept the Poincaré group as the correct group for characterizing the external symmetries of a physical system. Parametrized theories of the type introduced by Fock[4] and Stückelberg[5] replace the Poincaré group, labeled by P, with the direct product group $F=P\otimes T$, where T is the translation group of a relativistic scalar evolution parameter.[7]

Over the years, the conceptualization of the parameter has varied from one researcher to another. I have recently reviewed the history of the parameter interpretation and provided a physically unambiguous procedure for measuring an evolution parameter in terms of familiar observables.[8] This work applies to either the classical or quantum formulations of parametrized relativistic theories.

Classical bases for parametrized relativistic theories have been given by Davidon,[9] Cook,[10] Pearle,[11] and Reuse.[12] Davidon, whose interests were mostly in classical and quantum electrodynamics,[9] did not view the scalar evolution parameter as a measurable quantity. Cook introduced a scalar evolution parameter as a particular calibration of local proper times. Pearle and Reuse used a classical scalar evolution parameter in the sense of Ref. 8. Both researchers worked from a Hamiltonian formalism of relativistic classical mechanics. An interesting construction in Reuse's work is the derivation of Hamilton's equations from a dynamical principle expressed in terms of differential forms.

Most of the work in the development of parametrized theories has focused on demonstrating that parametrized theories are able to describe known physics. A review of the description of hydrogenlike systems by parametrized relativistic quantum mechanics (PRQM) is given by Reuse.[13] Horwitz and Lavie[14] discuss a method of handling two-body scattering. The treatment of electromagnetic interactions is reviewed by Horwitz.[15] An N-body formalism is presented by Fanchi and Wilson,[16] and a model of atomic and molecular systems in the PRQM context is given by Grelland.[17] The foundation of statistical mechanics in the context of PRCM and PRQM has been laid by Horwitz, Schieve, and Piron.[1] A link between the scalar parameter and entropy has been established[18,19] and used to resolve the conceptual conflict between "time" as an arrow and a reversible coordinate.[19] The Klein paradox for spin-0 (Ref. 20) and spin-$\frac{1}{2}$ (Refs. 21,22) particles has been resolved. A field theory has been derived from variation of an action integral, and has been shown to contain Lagrangian quantum field theory as a special case.[7] Model applications of the field theory have been presented.[23]

This growing body of work is a solid theoretical basis upon which to construct tests of PRQM. In all cases considered to date, it appears that PRQM is compatible with experiment while simultaneously removing some conceptual difficulties of the standard paradigm of relativistic quantum mechanics (for a discussion of some of these difficulties, see Ref. 24). Definitive tests of the validity or necessity of expanding the external symmetry group by the introduction of a scalar parameter are needed. These tests must take advantage of differences between the parametrized and standard formulations. As stated earlier, one such difference is the calculation of the free energy of an ensemble of independent, classical relativistic systems. Such an ensemble is usually called a Gibbs ensemble. We now turn to an evaluation of the implications of this difference.

III. FREE ENERGY OF THE RELATIVISTIC GIBBS ENSEMBLE

Horwitz *et al.*[1] have shown that the free energy of a relativistic Gibbs ensemble is theory dependent. Using PRCM, they calculated a free energy of the form

$$E_H=\frac{N}{\beta}\left[2-i\sigma\frac{H_0^{(1)}(i\sigma)}{H_1^{(1)}(i\sigma)}\right] . \tag{3.1}$$

The quantities $H_0^{(1)}$ and $H_1^{(1)}$ are Hankel functions,[25–27] and the variables β and σ are defined by

$$\beta=1/k_BT , \tag{3.2}$$

$$\sigma=M_0c^2\beta . \tag{3.3}$$

Here k_B is Boltzmann's constant, T is the temperature of the system, and c is the speed of light. The number of free particles N is fixed, as is the mass M_0 of each particle. These restrictions are equivalent to specifying a system that is stationary, or independent of the relativistic scalar evolution parameter. They are needed to make possible a comparison of Eq. (3.1) with Juttner's[2,3] result,

$$E_J=\frac{N}{\beta}\left[1-i\sigma\frac{H_2'^{(1)}(i\sigma)}{H_2^{(1)}(i\sigma)}\right] . \tag{3.4}$$

The prime denotes differentiation with respect to the argument of the Hankel function. Differences between Eqs. (3.1) and (3.4) reflect differences in the dynamical laws of PRCM and RCM. A more direct comparison can be made by manipulating the Hankel functions in Eq. (3.4) to arrive at the equivalent form

$$E_J=\frac{N}{\beta}\left[3-i\sigma\frac{H_1^{(1)}(i\sigma)}{H_2^{(1)}(i\sigma)}\right] . \tag{3.5}$$

Equations (3.1) and (3.5) agree in the nonrelativistic limit ($\sigma\rightarrow\infty$),

$$(E_H)_{\rm NR}=(E_J)_{\rm NR}=NM_0c^2+\tfrac{3}{2}Nk_BT . \tag{3.6}$$

At the opposite extreme, the ultrarelativistic limit ($\sigma\rightarrow 0$), the expressions have different values,

$$(E_H)_{\rm UR}=2Nk_BT \tag{3.7}$$

and

$$(E_J)_{\rm UR}=3Nk_BT . \tag{3.8}$$

The question arises: Is this difference observable?

Free-energy differences can affect the density of the stress-energy tensor used in stellar models (e.g., the Oppenheimer-Volkoff equation[28]), relativistic gas dynamics,[29–32] and cosmology.[28,33,34] It was noted by Horwitz *et al.*[1] that the difference between the calculated free energies in the UR limit was significant for temperatures in excess of 10^{12} K. This estimate assumes M_0 is the electron mass and represents a value of $\sigma=0.001$. For hydrogenlike masses and the same value of σ, we have

$$\sigma_H\approx(1.1\times10^{12}\text{ K})/T . \tag{3.9}$$

At what temperature does the difference in calculated free energies become significant? Is the difference significant only in the UR limit, or does the difference manifest itself at a lower temperature? To answer these questions, we must evaluate the free energies as a function of temperature or, equivalently, σ.

Calculation of the free energies begins by first transforming the Hankel functions to confluent hypergeometric functions,[25]

$$E_H = \frac{N}{\beta}\left[2+\frac{1}{2}\frac{U(\frac{1}{2},1,Z)}{U(\frac{3}{2},3,Z)}\right], \quad Z=2\sigma \tag{3.10}$$

and

$$E_J = \frac{N}{\beta}\left[3+\frac{1}{2}\frac{U(\frac{3}{2},3,Z)}{U(\frac{5}{2},5,Z)}\right], \quad Z=2\sigma \tag{3.11}$$

where $U(a,b,Z)$ is a confluent hypergeometric function. An advantage of expressing the free energies as in Eqs. (3.10) and (3.11) is that no complex numbers are needed. Integral representations of $U(1.5,3,Z)$ and $U(2.5,5,Z)$ were numerically evaluated using both four-point and six-point Gauss-Laguerre quadratures.[35] The difference between each of the four-point and six-point quadrature values was less than 1%, which is sufficient accuracy for our purposes. The Gauss-Laguerre quadrature does not yield asymptotically correct values for the remaining function $U(0.5,1,Z)$. It is evaluated using Lebedev's[36] series expansion for $Z<1$, and by asymptotic approximation[25] for $Z>10$. Intermediate values are interpolated by nonlinear regression.[37] Results of the calculations are shown in Figs. 1–3. The functions are seen to be quite smooth on a log-log plot. A physically more interesting plot is the semilogarithmic graph of E_J/E_H versus Z shown in Fig. 4.

According to Fig. 4, the free energies are equivalent for values of Z greater than 10. The UR regime is reached for values of Z less than 0.1. The relatively narrow region from $Z=0.1$ to $Z=10$ is the region of transition from nonrelativistic to ultrarelativistic behavior. These values are consistent with those of Konigl.[30] Using Eq. (3.9), the transition region occurs for temperatures ranging from $T=1.1\times10^{13}$ K to $T=1.1\times10^{11}$ K, and the UR limit is reached at $T=1.1\times10^{13}$ K. For comparison, the central temperature in the interior of the sun is estimated[38] to be about 1.6×10^{7} K, which is several orders of magnitude less than the temperature associated with the beginning of the transition region. It is unlikely that the different free energies shown in Eqs. (3.10) and (3.11) have observable consequences in either stellar astrophysics or earthbound relativistic gas dynamics. They can, however, affect cosmological models during epochs when temperatures are high enough to be in the UR regime. An illustration of this effect is presented in Sec. V. It is first necessary to construct a general relativistic formalism within the context of PRCM.

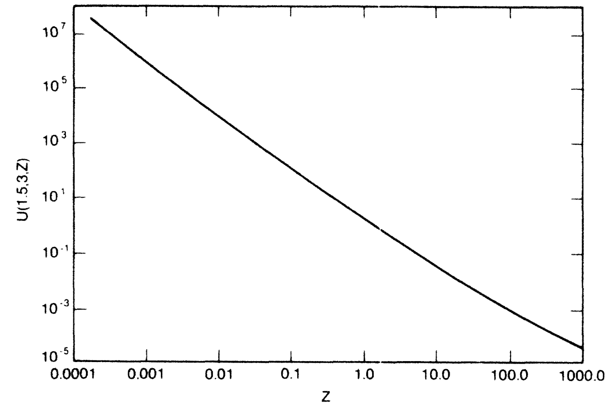

FIG. 2. Function $U(1.5,3,Z)$ vs dimensionless Z.

IV. GENERAL RELATIVITY IN PRCM

One of the most straightforward ways to develop a parametrized general relativistic formalism is to invoke a variational principle that is analogous to the Hilbert-Palatini variational principle.[28] We begin by defining a parameter-dependent action

$$A=\int_1^2\int_R \mathcal{L}\,dx\,d\tau\ , \tag{4.1}$$

where $\mathcal{L}$ is the Lagrangian density. Equation (4.1) is the classical analog of the action used to develop a parametrized quantum field theory.[7] As in that case, the integral over the parameter is for fixed end points. The Lagrangian density $\mathcal{L}$ is a sum of geometric and particle field terms,

$$\mathcal{L}=\mathcal{L}_g+\mathcal{L}_f=\sqrt{-g}\,(L_g+L_f)\ , \tag{4.2}$$

where L_g and L_f are Lagrangians. For simplicity, and because there is no reason to proceed otherwise, we as-

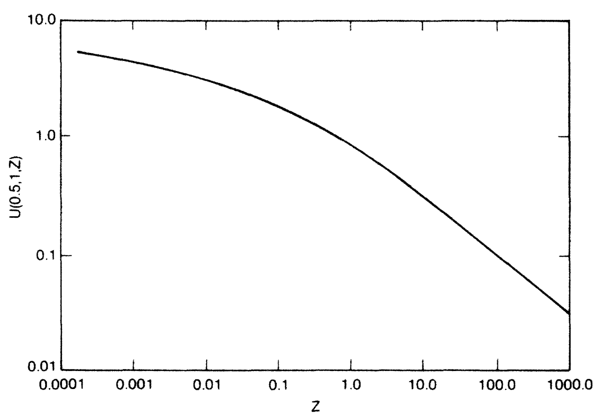

FIG. 1. Function $U(0.5,1,Z)$ vs dimensionless Z.

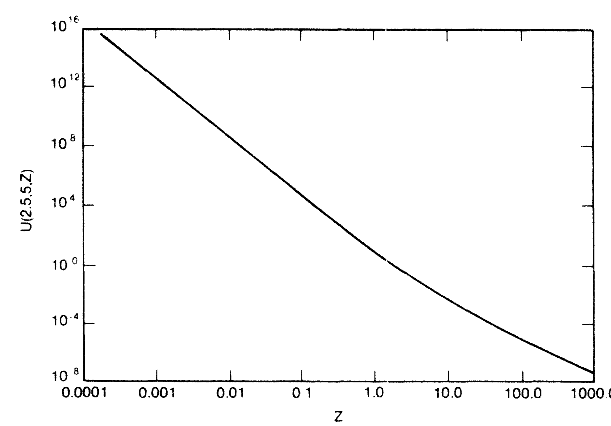

FIG. 3. Function $U(2.5,5,Z)$ vs dimensionless Z.

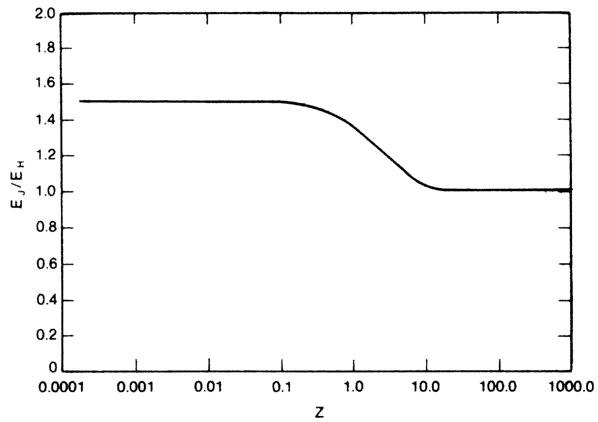

FIG. 4. Ratio E_J/E_H vs dimensionless Z.

sume the geometric factors L_g and g are parameter independent. The particle field term L_f may be parameter dependent.

Hamilton's principle requires that the variation of A must vanish,

$$\delta A = 0 = \int_1^2 \int_R [\delta(\sqrt{-g})L_g + \sqrt{-g}\,\delta L_g + \delta\mathcal{L}_f]dx\,d\tau \;. \tag{4.3}$$

The variations of the geometric terms may be treated in the usual manner.[28] In particular, the geometric Lagrangian is expressed in terms of the Ricci tensor as

$$L_g = \frac{1}{16\pi} g^{\alpha\beta} R_{\alpha\beta} \;. \tag{4.4}$$

Given Eq. (4.4), the variation of the geometric Lagrangian is

$$\delta(\sqrt{-g}L_g) = \frac{1}{16\pi}[(R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R)\sqrt{-g}\;\delta g^{\alpha\beta} + \sqrt{-g}\,g^{\alpha\beta}\,\delta R_{\alpha\beta}] \;, \tag{4.5}$$

where we have used the variation of the metric term in the form

$$\delta(\sqrt{-g}) = -\tfrac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\,\delta g^{\alpha\beta} \;. \tag{4.6}$$

Except for straightforward generalizations to allow for a parameter dependence of the particle Lagrangian density, we proceed as in Landau and Lifshitz[39] to evaluate the variation of the particle term in the action integral. The particle Lagrangian density is allowed to have the functional form

$$\mathcal{L}_f = \mathcal{L}_f(q, q_{,\mu}, \dot{q}, g^{\alpha\beta}, \tau) \;, \tag{4.7}$$

where

$$q_{,\mu} = \frac{\partial q}{\partial x_\mu}, \quad \dot{q} = \frac{dq}{d\tau} \;. \tag{4.8}$$

Notice that the evolution parameter τ is not a fifth coordinate as in a Kaluza-Klein theory.[3,40] It is a relativistic scalar as in the derivation of a parametrized quantum field theory.[7] The variation of the particle Lagrangian density is

$$\delta\mathcal{L}_f = \frac{\partial\mathcal{L}_f}{\partial q}\delta q + \frac{\partial\mathcal{L}_f}{\partial q_{,\mu}}\delta q_{,\mu} + \frac{\partial\mathcal{L}_f}{\partial \dot{q}}\delta\dot{q} + \frac{\partial\mathcal{L}_f}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta} + \frac{\partial\mathcal{L}_f}{\partial\tau}\delta\tau \;. \tag{4.9}$$

For fixed parameter end points we have, as the result of an integration by parts, the result

$$\int_1^2 \int_R \frac{\partial\mathcal{L}_f}{\partial\tau}\delta\tau\,dx\,d\tau = 0 \;. \tag{4.10}$$

Another integration by parts applied to the $\delta\dot{q}$ term yields

$$\int_1^2 \int_R \frac{\partial\mathcal{L}_f}{\partial\dot{q}}\delta\left[\frac{dq}{d\tau}\right]dx\,d\tau = \int_1^2 \int_R \frac{\partial\mathcal{L}_f}{\partial\dot{q}}\frac{d}{d\tau}(\delta q)dx\,d\tau = -\int_1^2 \int_R \left[\frac{d}{d\tau}\left[\frac{\partial\mathcal{L}_f}{\partial\dot{q}}\right]\right] \times \delta q\,dx\,d\tau \;, \tag{4.11}$$

where the condition that δq vanishes at the parameter end points has been used. A simplification of the $\delta q_{,\mu}$ term is achieved by invoking the chain rule to find

$$\frac{\partial\mathcal{L}_f}{\partial q_{,\mu}}\delta q_{,\mu} = \frac{\partial\mathcal{L}_f}{\partial q_{,\mu}}\frac{\partial(\delta q)}{\partial x_\mu} = \frac{\partial}{\partial x_\mu}\left[\frac{\partial\mathcal{L}_f}{\partial q_{,\mu}}\delta q\right] - \left[\frac{\partial\pi^\mu}{\partial x_\mu}\right]\delta q \;, \tag{4.12}$$

where

$$\pi^\mu = \frac{\partial\mathcal{L}_f}{\partial q_{,\mu}} \;. \tag{4.13}$$

Collecting the results in Eqs. (4.10) through (4.13) gives

$$\int_1^2 \int_R \delta\mathcal{L}_f\,dx\,d\tau = \int_1^2 \int_R \left\{\left[\frac{\partial\mathcal{L}_f}{\partial q} - \frac{\partial\pi^\mu}{\partial x_\mu} - \frac{d}{d\tau}\left[\frac{\partial\mathcal{L}_f}{\partial\dot{q}}\right]\right]\delta q + \frac{\partial}{\partial x_\mu}(\pi^\mu\,\delta q) + \frac{\partial\mathcal{L}_f}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}\right\}dx\,d\tau \;. \tag{4.14}$$

The divergence term, $\int_1^2 \int_R (\partial/\partial x_\mu)(\pi^\mu\,\delta q)dx\,d\tau$, vanishes upon transformation by Gauss's theorem and integration over all space.[39] Furthermore, the metric variation term can be written as

$$\frac{\partial\mathcal{L}_f}{\partial g^{\alpha\beta}} = \frac{\partial\sqrt{-g}L_f}{\partial g^{\alpha\beta}} = \frac{\partial\sqrt{-g}}{\partial g^{\alpha\beta}}L_f + \sqrt{-g}\,\frac{\partial L_f}{\partial g^{\alpha\beta}} = \left[-\tfrac{1}{2}g_{\alpha\beta}L_f + \frac{\partial L_f}{\partial g^{\alpha\beta}}\right]\sqrt{-g} \;. \tag{4.15}$$

Substituting the results of Eqs. (4.4)–(4.15) into Eq. (4.3) lets us write

$$\delta A=0=\int_1^2\int_{\mathbf{R}}\left\{\frac{1}{16\pi}(R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R)\sqrt{-g}\;\delta g^{\alpha\beta}+\sqrt{-g}\,g^{\alpha\beta}\,\delta R_{\alpha\beta}+\left[\frac{\partial L_f}{\partial g^{\alpha\beta}}-\tfrac{1}{2}g_{\alpha\beta}L_f\right]\sqrt{-g}\;\delta g^{\alpha\beta}\right.$$
$$\left.+\left[\frac{\partial \mathcal{L}_f}{\partial q}-\frac{\partial \pi^\mu}{\partial x_\mu}-\frac{d}{d\tau}\left[\frac{\partial \mathcal{L}_f}{\partial \dot{q}}\right]\right]\delta q\right\}dx\,d\tau\;. \tag{4.16}$$

Hamilton's principle, as embodied in Eq. (4.16), can hold for arbitrary variations only if the coefficients of the variations vanish. The resulting equations are the general relativistic formalism that we seek.

The vanishing of the coefficient of the variation of the Ricci tensor $R_{\alpha\beta}$ leads to the standard equation for the connection coefficients.[28] Parametrized Euler-Lagrange field equations result from the vanishing of the coefficient of δq,

$$\frac{\partial \mathcal{L}_f}{\partial q}-\frac{\partial \pi^\mu}{\partial x_\mu}-\frac{d}{d\tau}\left[\frac{\partial \mathcal{L}_f}{\partial \dot{q}}\right]=0\;. \tag{4.17}$$

Finally, the requirement that the coefficient of the variation of the metric tensor vanish yields

$$\frac{1}{16\pi}(R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R)=\left[\frac{\partial L_f}{\partial g^{\alpha\beta}}-\tfrac{1}{2}g_{\alpha\beta}L_f\right]. \tag{4.18}$$

Identifying, in the usual way, the Einstein curvature tensor,

$$G_{\alpha\beta}\equiv R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R\;, \tag{4.19}$$

and the stress-energy tensor,

$$T_{\alpha\beta}\equiv g_{\alpha\beta}L_f-2\frac{\partial L_f}{\partial g^{\alpha\beta}}\;, \tag{4.20}$$

lets us rewrite Eq. (4.18) as

$$G_{\alpha\beta}=8\pi T_{\alpha\beta}\;. \tag{4.21}$$

Equation (4.21) is Einstein's second-order equation for space-time geometry.

Each of the above equations is expressed in terms of Lagrangian functions. Most applications of the parametrized formulation have been in terms of Hamiltonian functions. All parametrized Hamiltonians used to date are based on a flat space-time metric. The classical Lagrangian and Hamiltonian formulations can be related in the usual way[41,42] for flat space-time. The presence of a curved background metric complicates the connection.

V. COSMOLOGICAL IMPLICATIONS

Einstein's curvature tensor $G_{\alpha\beta}$ is parameter dependent if the stress-energy tensor $T_{\alpha\beta}$, through the Lagrangian L_f, is parametrized. If L_f is stationary with respect to the parameter τ, i.e., L_f is not parametrized, then Eq. (4.21) is the usual RCM expression. It was pointed out in Sec. III that this is the case for the comparison we are considering. We can therefore use the standard model of cosmology[33,34] as our test model for evaluating the impact of Eqs. (3.10) and (3.11) on cosmology. In particular, we are interested in studying the effect of Eqs. (3.10) and (3.11) on the Friedmann scale factor. We recognize that the classical treatment, especially at the high temperatures needed to reach the ultrarelativistic regime, is only approximate.[33] Nevertheless, the comparison given below indicates that a more rigorous treatment could lead to significant differences between the cosmologies of parametrized and nonparametrized formulations of relativistic physics.

We begin our comparison by summarizing the derivation of the standard cosmological model. Invoking the cosmological principle, i.e., three-space is homogeneous and isotropic, and assuming that the metric is Riemannian, leads to the Robertson-Walker metric,

$$ds^2=-dt^2+R^2(t)\left[\frac{dr^2}{1-K_0r^2}+r^2(d\theta^2+\sin^2\theta\,d\varphi^2)\right]. \tag{5.1}$$

$R(t)$ is a time-dependent scale factor with the initial condition

$$R_0=R(0)=1\;. \tag{5.2}$$

K_0 is an arbitrary constant. The model universe is either closed, open, or spatially flat if K_0 is $+1$, -1, or 0, respectively. The Gaussian curvature of three-space is given in terms of $R(t)$ and K_0 as

$$K(t)=K_0/R^2,\quad K(0)=K_0\;, \tag{5.3}$$

where the initial condition of $K(t)$ is derived from Eq. (5.2).

Substitution of Eq. (5.1) into the Einstein field equations leads to the Lemaître equations,

$$\left[\frac{1}{R}\frac{dR}{dt}\right]^2=\tfrac{8}{3}\pi G\rho(t)-c^2\frac{K_0}{R^2}+\frac{\Lambda}{3}\;, \tag{5.4}$$

$$\frac{d\rho(t)}{dt}=-3\left[\rho(t)+\frac{p(t)}{c^2}\right]\left[\frac{1}{R}\frac{dR}{dt}\right], \tag{5.5}$$

where $\rho(t)$ is the mass density including the mass equivalent of all energy present, $p(t)$ is pressure, G is the gravitational constant, and Λ is the cosmological constant. The Lemaître equations, together with an equation of state relating p and ρ, determine the behavior of $R(t)$.

The Friedmann scale factor is the solution of the Friedmann equation, which is derived from Eqs. (5.4) and (5.5) by making the assumption $p \ll \rho c^2$. In this case, Eq. (5.5) can be solved to give

$$\rho(t)=\rho_0/R^3,\quad \rho(0)=\rho_0\;, \tag{5.6}$$

and Eq. (5.4) takes on the simplified form

$$\left[\frac{1}{R}\frac{dR}{dt}\right]^2=\frac{8}{3}\pi G\frac{\rho_0}{R^3}-c^2\frac{K_0}{R^2}+\frac{\Lambda}{3}\ . \tag{5.7}$$

Following Felten and Isaacman,[34] we introduce the Hubble parameter and dimensionless density (or closure) parameter with the definitions

$$H(t)=\frac{1}{R}\frac{dR}{dt},\quad H_0=H(0) \tag{5.8}$$

and

$$\Omega(t)=\frac{8}{3}\pi G\frac{\rho(t)}{H^2(t)},\quad \Omega_0=\Omega(0)=\frac{8}{3}\pi G\frac{\rho_0}{R_0^3H_0^2}\ . \tag{5.9}$$

The Friedmann equation can be rewritten in terms of these parameters as

$$c^2K(t)=\frac{8}{3}\pi G\rho(t)+\frac{\Lambda}{3}-H^2(t)\ . \tag{5.10}$$

Introducing the dimensionless time,

$$t_D=H_0t\ , \tag{5.11}$$

and substituting Eqs. (5.8)–(5.11) into Eq. (5.7), gives

$$\left[\frac{1}{R}\frac{dR}{dt_D}\right]^2=\Omega_0\frac{(1-R)}{R^3}+\frac{1}{R^2}+\frac{1}{3}\left[\frac{\Lambda}{H_0^2}\right]\frac{(R^2-1)}{R^2}\ . \tag{5.12}$$

A comparison of the parametrized and non-parametrized results in the UR case is made by first writing down separate density parameters for Juttner and Horwitz *et al.*,

$$\Omega_{0J}=\frac{8}{3}\pi G\frac{\rho_{0J}}{R_{0J}^3H_{0J}^2} \tag{5.13}$$

and

$$\Omega_{0H}=\frac{8}{3}\pi G\frac{\rho_{0H}}{R_{0H}^3H_{0H}^2}\ . \tag{5.14}$$

Their ratio is

$$\frac{\Omega_{0J}}{\Omega_{0H}}=\frac{\rho_{0J}}{\rho_{0H}}\frac{R_{0H}^3H_{0H}^2}{R_{0J}^3H_{0J}^2}\ . \tag{5.15}$$

Neither the scale factor nor the Hubble constant is parameter dependent. It is reasonable to treat the factors as theory independent. In this case, Eq. (5.15) simplifies to the form

$$\frac{\Omega_{0J}}{\Omega_{0H}}=\frac{\rho_{0J}}{\rho_{0H}}=\frac{E_J}{E_H}\ , \tag{5.16}$$

where we have included the relation between the ratios of free energy and mass density. Equation (5.16) shows that the density parameters are significantly different during the UR regime. The difference in theoretical density parameters leads to substantial differences in such calculated quantities as the age of Friedmann universes.

TABLE I. Age of Friedmann universes.

Free parameter Λ/H_0^2	Density parameter Ω_0	Hubble time range (Units of $1/H_0$)	Hubble time ratio
−10	0.15	0.5575–0.5600	
	0.10	0.5650–0.5675	0.987
0	0.15	0.8700–0.8725	
	0.10	0.8975–0.9000	0.969
3	0.15	1.2625–1.2650	
	0.10	1.3750–1.3775	0.918
−10	1.5	0.4700–0.4725	
	1.0	0.4925–0.4950	0.954
0	1.5	0.6100–0.6125	
	1.0	0.6650–0.6675	0.917
3	1.5	0.6950–0.6975	
	1.0	0.7850–0.7875	0.885

Several model ages have been calculated from Eq. (5.12) using the numerical procedure described by Felten and Isaacman.[34] They are presented in Table I. Each model age is shown as a range extending over the numerical step size (0.0025) of the dimensionless time t_D. Accompanying each pair of density parameter values for a given free parameter (Λ/H_0^2) value is the ratio of model ages corresponding to Eq. (5.16). Equation (5.16) does not lead to a definitive observational test because of uncertainties associated with important cosmological parameters, e.g., the Hubble constant (50–100 km sec/Mpc) and the cosmological constant. For example, the value of a Hubble time unit ($1/H_0$) varies from $(10-20)\times10^9$ yr depending on the value of Hubble's constant.[34] Thus, even though theoretical differences are calculable, the free-energy difference first noted by Horwitz *et al.*[1] cannot be used at present as the basis for an observational test of PRCM. Our results do show, however, that cosmology may eventually become a fruitful testing ground for parametrized relativistic theories.

[1]L. P. Horwitz, W. C. Schieve, and C. Piron, Ann. Phys. (N.Y.) **137**, 306 (1981).
[2]F. Juttner, Ann. Phys. (Leipzig) **34**, 856 (1911).
[3]W. Pauli, *Theory of Relativity* (Dover, New York, 1958).
[4]V. A. Fock, Phys. Z. Sowjetunion **12**, 404 (1937).
[5]E. C. G. Stückelberg, Helv. Phys. Acta **14**, 322 (1944); **14**, 588 (1944).
[6]I. J. R. Aitchison and A. J. G. Hey, *Gauge Theories in Particle Physics* (Hilger, London, 1983); M. Chaichian and N. F. Nelipa, *Introduction to Gauge Field Theories* (Springer-Verlag, New York, 1984); and several other related references.
[7]J. R. Fanchi, Phys. Rev. D **20**, 3108 (1979).
[8]J. R. Fanchi, Phys. Rev. A **34**, 1677 (1986).
[9]W. C. Davidon, Phys. Rev. **97**, 1131 (1955); **97**, 1139 (1955).
[10]J. L. Cook, Aust. J. Phys. **25**, 117 (1972).
[11]P. M. Pearle, Phys. Rev. **168**, 1429 (1968).

[12]F. Reuse, Found. Phys. **9**, 865 (1979).
[13]F. Reuse, Helv. Phys. Acta **53**, 416 (1980); **53**, 552 (1980).
[14]L. P. Horwitz and Y. Lavie, Phys. Rev. D **26**, 819 (1982).
[15]L. P. Horwitz, Found. Phys. **14**, 1027 (1984).
[16]J. R. Fanchi and W. J. Wilson, Found. Phys. **13**, 571 (1983).
[17]H. Grelland, Int. J. Quantum Chem. **19**, 873 (1981).
[18]L. P. Horwitz, S. Shashoua, and W. C. Schieve, Tel Aviv University Report No. TAUP-1408-85, 1985 (unpublished).
[19]J. R. Fanchi, Phys. Rev. A **34**, 4859 (1987).
[20]J. R. Fanchi and R. E. Collins, Found. Phys. **8**, 851 (1978).
[21]J. R. Fanchi, Found. Phys. **11**, 493 (1981).
[22]B. Thaller, Lett. Nuovo Cimento **31**, 439 (1981).
[23]L. Hostler, J. Math. Phys. **21**, 2461 (1980); **22**, 2307 (1981).
[24]J. R. Fanchi, Am. J. Phys. **49**, 850 (1981).
[25]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[26]G. Arfken, *Mathematical Methods for Physicists* (Academic, New York, 1970).
[27]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
[28]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[29]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, London, 1958).
[30]A. Konigl, Phys. Fluids **23**, 1083 (1980).
[31]L. S. Solov'ev, Dokl. Akad. Nauk SSSR **29**, 848 (1984) [Sov. Phys.—Dokl. **29**, 286 (1984)].
[32]W. A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).
[33]R. H. Brandenberger, Rev. Mod. Phys. **57**, 1 (1985).
[34]J. E. Felten and R. Isaacman, Rev. Mod. Phys. **58**, 689 (1986).
[35]B. Carnahan, H. A. Luther, and J. O. Wilkes, *Applied Numerical Methods* (Wiley, New York, 1969).
[36]N. N. Lebedev, *Special Functions and Their Applications* (Prentice-Hall, Englewood Cliffs, NJ, 1965).
[37]J. L. Kuester and J. H. Mize, *Optimization Techniques with Fortran* (McGraw-Hill, New York, 1973).
[38]R. C. Haymes, *Introduction to Space Science* (Wiley, New York, 1971).
[39]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, MA, 1962).
[40]J. L. Cook, Aust. J. Phys. **25**, 141 (1972).
[41]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1965).
[42]H. Flanders, *Differential Forms with Applications to the Physical Sciences* (Academic, New York, 1963).